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Roberts-type embeddings and conversion of transversal Tverberg's theorem

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Abstract. Central in the paper are two results on the existence of 'economical' embeddings in a Euclidean space. The first result (Corollary 1.4) states the existence of an embedding with image intersecting the large-dimensional planes in sets of 'controllable' dimension. The second result (Corollary 1.6) proves the existence of maps such that each small-dimensional plane contains 'controllably' many points of the image.

Well known results of Nöbeling-Pontryagin, Roberts, Hurewicz, Boltyanskiĭ, and Goodsell can be obtained as consequences of these results. Their infinite-dimensional version concerning an embedding in a Hilbert space is also established (Theorem 1.8).

Bibliography: 31 titles.

§ 1. Introduction

All the maps considered in our paper are assumed to be continuous and all the spaces are at least completely regular. Throughout, $\Pi^d \subset \mathbb{R}^m$ is a d-dimensional, not necessarily coordinate plane (a d-plane) in $\mathbb{R}^m$. Unless otherwise explicitly stated, all function spaces in the paper are equipped with the source limitation topology [1].

Our aim consists in the proof of Theorem 1.1 stated below and in establishing several applications of this result.

Theorem 1.1. *Let $A_{i,j}$, $i = 1, 2, \dots, q$, $j = 1, 2, \dots, n_i + 1$, be points in $\mathbb{R}^m$ such that the set of their coordinates is algebraically independent. Assume that $0 \leqslant t \leqslant d \leqslant T \leqslant m$ and let Π^d be a d-plane in $\mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$. If either $d - t + 1 \leqslant q$ and $n_1 + n_2 + \dots + n_q + 1 \leqslant (m-d)(q-1)-(T-d)(d-t)$, or $q \leqslant d-t+1$ and $n_1+n_2+\dots+n_q+1 \leqslant (m-T)(q-1)$, then there exists an index $i \in \{1, 2, \dots, q\}$ such that Π^d is disjoint from the linear hull $\Pi(M_i)$ of the set $M_i = \{A_{i,1}, \dots, A_{i,n_i+1}\}$.*

The first part of Theorem 1.1 for $t = 0$, $T = m$, and $d - t + 1 \leqslant q$ was stated as a conjecture in [2], Conjecture 2 and [3], Conjecture 4.2.

Recall that a real number v is said to be *algebraically dependent* on real numbers $u_1, \dots, u_k$ if v satisfies an equation $p_0(u) + p_1(u)v + \dots + p_n(u)v^n = 0$, where

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$p_0(u), \dots, p_n(u)$ are polynomials in $u_1, \dots, u_k$ with rational coefficients, not all of them equal to zero. A finite set of real numbers is *algebraically independent* if none of them depends algebraically on the other numbers.

Corollary 1.2. *Let K be a finite simplicial complex, $\theta\colon K \to \mathbb{R}^m$ a semilinear map and assume that $\varepsilon > 0$. Then there exists a semilinear map $g\colon K \to \mathbb{R}^m$ such that $d(g(v), \theta(v)) < \varepsilon$ for each vertex v of K, and for arbitrary integers n, d, t, T such that $0 \leqslant t \leqslant d \leqslant T \leqslant m$ and $d \leqslant m - n - 1$ and each d-plane $\Pi^d \subset \mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$, the number q of pairwise disjoint simplexes of K of dimension $\leqslant n$ with g-images intersecting Π^d satisfies the inequalities*

$$q \leqslant d + 1 - t + \frac{n + (n + T - m)(d - t)}{m - n - d} \qquad \textit{for} \ \ n \geqslant (m - n - T)(d - t),$$

$$q \leqslant 1 + \frac{n}{m - n - T} \qquad \textit{for} \ \ n \leqslant (m - n - T)(d - t).$$

The idea of using algebraically independent sets for the proof of general-position results similar to Corollary 1.2 goes back to Roberts [4]. This idea was also used by Berkowitz and Roy in [5], where they suggested a version of Proposition 1.2 for $t = 0$ and $T = m$. A proof of the Berkowitz–Roy theorem was carried out by Goodsell in [6], Theorem A.1 (see also [7] for other applications of the Berkowitz–Roy theorem).

Here are several applications of Corollary 1.2.

Theorem 1.3. *Let $f\colon X \to Y$ be a perfect map of paracompact spaces such that $\dim f \leqslant n$ and $\dim Y = 0$. Then for each $m \geqslant n + 1$ the space $C^*(X, \mathbb{R}^m)$ of bounded continuous maps from X into $\mathbb{R}^n$ contains a dense G_δ-subset $\mathcal{H}$ of maps g such that $\dim g(f^{-1}(y)) \cap \Pi^d \leqslant n + d - m$ for each $y \in Y$ and each d-plane $\Pi^d \subset \mathbb{R}^m$ with $m - n \leqslant d \leqslant m$.*

Corollary 1.4. *Let X be a normal space with $\dim X \leqslant n$ and $m \geqslant n + 1$. Then the space $C^*(X, \mathbb{R}^m)$ equipped with the uniform-convergence topology contains a dense G_δ-subset $\mathcal{H}$ of maps g such that $\overline{g(X)} \cap \Pi^d$ is at most $(n + d - m)$-dimensional for each d-plane $\Pi^d \subset \mathbb{R}^m$ with $m - n \leqslant d \leqslant m$.*

We shall show how Corollary 1.4 follows from Theorem 1.3. By considering the Stone–Čech compactification βX of X and the function space $C(\beta X, \mathbb{R}^m)$ in place of X and $C^*(X, \mathbb{R}^m)$, respectively, we can assume that X is compact. Then we apply Theorem 1.3 to a constant map f on X.

As Roberts proved in [4], Theorem 1.2, if X is a compact metrizable space of dimension $\leqslant n$ and $n + 1 \leqslant d \leqslant 2n + 1$, then $C(X, \mathbb{R}^{2n+1})$ equipped with the topology of uniform convergence contains a dense G_δ-subset consisting of maps g such that $\dim g(X) \cap \Pi^d \leqslant d - n - 1$ for each d-plane $\Pi^d \subset \mathbb{R}^{2n+1}$. From this result and Hurewicz's theorem on dimension-preserving metrizable compactifications Roberts deduced the existence of such embeddings for separable metrizable spaces of dimension $\leqslant n$. Obviously, Roberts's results follow from the combination of Corollary 1.4 and the Nöbeling-Pontryagin embedding theorem.

Theorem 1.3 does not work for $d \leqslant m - n - 1$. However, as the next result shows, in that case we can prove slightly more: we can find a residual subset of $C^*(X, \mathbb{R}^m)$ consisting of maps g such that $g(f^{-1}(y)) \cap \Pi^d$ is finite for each $y \in Y$ and each d-plane in $\mathbb{R}^m$.

Theorem 1.5. *Let* $f\colon X \to Y$ *be a perfect map between metrizable spaces such that* $\dim f \leqslant n$ *and* $\dim Y \leqslant 0$. *Then* $C^*(X,\mathbb{R}^m)$ *contains a dense* G_δ*-subset* $\mathcal{K}$ *of maps* g *such that for arbitrary integers* d, t, T *with* $0 \leqslant t \leqslant d \leqslant T \leqslant m$ *and* $d \leqslant m-n-1$ *and for each* d*-plane* $\Pi^d \subset \mathbb{R}^m$ *parallel to some coordinate planes* $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$ *each set* $f^{-1}(y)\cap g^{-1}(\Pi^d)$, $y \in Y$, *contains at most* q *points*, *where*

$$\begin{aligned} q &= d+1-t+\frac{n+(n+T-m)(d-t)}{m-n-d} && \textit{for } \ n \geqslant (m-n-T)(d-t),\\ q &= 1+\frac{n}{m-n-T} && \textit{otherwise.} \end{aligned}$$

Corollary 1.6. *Let* X *be a metrizable compactum with* $\dim X \leqslant n$ *and assume that* $m \geqslant n+1$. *Then* $C(X,\mathbb{R}^m)$ *contains a dense* G_δ*-subset of maps* g *such that for arbitrary integers* d, t, T *with* $0 \leqslant t \leqslant d \leqslant T \leqslant m$ *and* $d \leqslant m-n-1$ *and each* d*-plane* $\Pi^d \subset \mathbb{R}^m$ *parallel to some coordinate planes* $\Pi^t \subset \Pi^T$ *in* $\mathbb{R}^m$ *the inverse image* $g^{-1}(\Pi^d)$ *contains at most* q *points*, *where*

$$\begin{aligned} q &= d+1-t+\frac{n+(n+T-m)(d-t)}{m-n-d} && \textit{for } \ n \geqslant (m-n-T)(d-t),\\ q &= 1+\frac{n}{m-n-T} && \textit{otherwise.} \end{aligned}$$

Taking Y in Theorem 1.5 to be a point, and taking $m=n+2$, $d=1$, $t=0$, and $T=r$, we obtain another consequence.

Corollary 1.7. *Let* X *be a metrizable compactum with* $\dim X \leqslant n$. *Then the space* $C(X,\mathbb{R}^{n+2})$ *contains a dense* G_δ*-subset of maps* g *such that for each positive integer* $r \leqslant n+2$ *and each line* $\Pi^1 \subset \mathbb{R}^m$ *parallel to some coordinate* r*-plane* Π^r *in* $\mathbb{R}^m$ *the inverse image* $g^{-1}(\Pi^1)$ *contains at most* $n+r$ *points.*

The maps obtained in this paper are never semilinear since each line contains finitely many points from their ranges. However, for $m \geqslant 2n+1$ we obtain embeddings which are moreover tame ([8], Theorem 5.1).

Theorem 1.8 below is an infinite-dimensional version of Theorem 1.5. For compact X and a one-point space Y Theorem 1.8 was established by Boltyanskiĭ [9] under the additional constraint $r=0$.

Theorem 1.8. *Let* $f\colon X \to Y$ *be a perfect map between metrizable spaces such that* Y *is a* C*-space. For arbitrary integers* d *and* r *let* $\mathcal{P}(d,r)$ *be the family of all* d*-planes* $\Pi^d \subset l_2$ *parallel to some coordinate planes* $\Pi^r \subset l_2$. *Then* $C^*(X,l_2)$ *contains a dense* G_δ*-subset of maps* g *such that for each* $y \in Y$ *and each* $\Pi^d \in \mathcal{P}(d,r)$ *the set* $f^{-1}(y)\cap g^{-1}(\Pi^d)$ *contains at most* $d+1-r$ *points if* $r \leqslant d$ *and at most one point if* $r \geqslant d$.

This paper is organized as follows. We present the proofs of Theorem 1.1 and Corollary 1.2 in § 2. Section 3 is devoted to the proof of Theorem 1.3. Theorems 1.5 and 1.8 are discussed in § 4. The final § 5 contains further applications of Theorem 1.1 and Corollary 1.2. We also include in the final section several conjectures.

A few words about the source limitation topology [1]. For arbitrary spaces M and K we denote by $C(K,M)$ the set of all continuous maps from K into M.

If (M,d) is a metric space and K an arbitrary space, then the source limitation topology in $C(K,M)$ is defined as follows: a subset U of $C(K,M)$ is open in $C(K,M)$ in the source limitation topology if for each $g \in U$ there exists a continuous function $\alpha\colon K \to (0,\infty)$ such that $\overline{B}(g,\alpha) \subset U$. Here $\overline{B}(g,\alpha)$ is the set

$$\{h \in C(K,M) : d(g(x),h(x)) \leqslant \alpha(x) \quad \text{for each } x \in K\}.$$

It is well known (see, for instance, [10]) that if (M,d) is a complete metric space, then $C(K,M)$ has the Baire property in this topology. Hence $C^*(K,H)$ with the source limitation topology also has the Baire property for each Banach space H.

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In conclusion we also wish to thank Prof. T. Goodsell, who provided us with his proof ([6], Appendix) of the Berkowitz–Roy theorem.

§ 2. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Assume that Π^d intersects the linear hull $\Pi(M_i)$ of each set M_i and let $Y_i \in \Pi^d \cap \Pi(M_i)$, $i = 1,\dots,q$. It is sufficient to show that under these assumptions we either have $n_1+n_2+\dots+n_q+1 > (m-d)(q-1)-(T-d)(d-t)$ for $q \geqslant d-t+1$ or $n_1+n_2+\dots+n_q+1 > (m-T)(q-1)$ for $1 \leqslant q \leqslant d-t+1$. For what follows we shall need the inequality below (see [5] and [6]), which follows from the properties of algebraically independent sets.

Proposition 2.1. *Let A be an algebraically independent subset of $\mathbb{R}$, and B a subset of $\mathbb{R}$ such that each element of A depends algebraically on B. Then the cardinality of A does not exceed the cardinality of B.*

We shall also require the following constructions due to Roberts [4]. Let $\{r_i\}$ be an infinite algebraically independent set: each finite subset of it is algebraically independent. Let $R_i = \{q + r_i : q \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers. Then each R_i is a dense subset of $\mathbb{R}$ and distinct R_i are disjoint. Moreover, each finite set M containing at most one element from each R_i, $i = 1,2,\dots$, is algebraically independent.

We can assume that Π^t and Π^T are the planes of the first t and T coordinate variables, respectively. Let π be the projection of $\mathbb{R}^m$ onto the space $\mathbb{R}^{m-t}$ of the last $m-t$ coordinates. Then $\Pi^{d-t} = \pi(\Pi^d)$ is a $(d-t)$-plane in $\mathbb{R}^{m-t}$ parallel to the coordinate plane $\Pi^{T-t} = \pi(\Pi^T)$. Moreover, the set of coordinates of all points $B_{i,j} = \pi(A_{i,j})$ is algebraically independent (as a subset of the set of coordinates of the points $A_{i,j}$). Hence by considering the space $\mathbb{R}^{m-t}$ of the last $m-t$ coordinates and the projections $B_{i,j}$, $\pi(\Pi^d)$, and $\pi(\Pi^T)$ onto it we can assume that $t = 0$.

Since $Y_i \in \Pi(M_i)$, there exist coefficients $\{\lambda_{i,j}\}_{j=1}^{n_i+1}$, $\lambda_{i,1} + \dots + \lambda_{i,n_i+1} = 1$, such that

$$\begin{aligned} Y_i &= \lambda_{i,1}A_{i,1} + \dots + \lambda_{i,n_i}A_{i,n_i} + \lambda_{i,n_i+1}A_{i,n_i+1} \\ &= \lambda_{i,1}A_{i,1} + \dots + \lambda_{i,n_i}A_{i,n_i} + \bigl(1 - \lambda_{i,1} - \dots - \lambda_{i,n_i}\bigr)A_{i,n_i+1} \qquad (1) \\ &\qquad \text{for each } i = 1,\dots,q. \end{aligned}$$

At least one of the numbers $\{\lambda_{i,j}\}_{j=1}^{n_i+1}$ is non-zero. We assume without loss of generality that $\lambda_{i,1} \neq 0$. (If some of the numbers $\{\lambda_{i,j}\}_{j=1}^{n_i+1}$ are equal to zero, then we can leave the corresponding points out of consideration and obtain a subset $\{A_{i,j}\}$ satisfying the assumptions of the theorem.) By (1) we can express $A_{i,1}$ as a linear combination of $A_{i,2}, \ldots, A_{i,n_i+1}$, Y_i. Then we obtain n_i additional coefficients, $\lambda_{i,1}, \ldots, \lambda_{i,n_i}$.

We have thus expressed the coordinates of all the points $\{A_{i,j}\}$, $i = 1, \ldots, q$, $j = 1, \ldots, n_i + 1$, in terms of the coordinates of the points $\{A_{i,j}\}$, $i = 1, \ldots, q$, $j = 2, \ldots, n_i + 1$; $Y_1, \ldots, Y_q$ and the numbers $\{\lambda_{i,j}\}$, $i = 1, \ldots, q$, $j = 1, \ldots, n_i$.

(I) Assume that $q \leqslant d+1$. Since Π^d is parallel to Π^T, the plane $Y_1 + \Pi^T$ contains all the points $Y_2, \ldots, Y_q$. Thus,

$$Y_i - Y_1 = \sum_{j=1}^{T} \alpha_{i,j} \mathbf{e}_j, \qquad i = 2, \ldots, q, \tag{2}$$

where $\mathbf{e}_j$ is the jth unit basis vector.

We have thus expressed all the coordinates of the points $\{A_{i,j}\}$, $i = 1, \ldots, q$, $j = 2, \ldots, n_i + 1$; $Y_1, \ldots, Y_q$ and the numbers $\{\lambda_{i,j}\}$, $i = 1, \ldots, q$, $j = 1, \ldots, n_i$, in terms of the coordinates of $\{A_{i,j}\}$, $i = 1, \ldots, q$, $j = 2, \ldots, n_i + 1$; Y_1, and the numbers $\{\lambda_{i,j}\}$, $i = 1, \ldots, q$, $j = 1, \ldots, n_i$; $\alpha_{i,j}$, $i = 2, \ldots, q$, $j = 1, \ldots, T$.

Hence by Proposition 2.1, $qm \leqslant m + n_1 + \cdots + n_q + T(q-1)$, that is,

$$n_1 + \cdots + n_q \geqslant (m - T)(q - 1). \tag{3_{I}}$$

We observe that (3_{I}) survives the replacement of m, d, and T by $m - t$, $d - t$, and $T - t$, respectively. Hence (3_{I}) holds for each t with $0 \leqslant t \leqslant d$.

(II) Assume that $q \geqslant d + 1$. Then there exist $d + 1$ points Y_i, for instance, $Y_1, \ldots, Y_{d+1}$, such that each Y_j, $j = d + 2, \ldots, q$, is a linear combination of the Y_i, $i = 1, \ldots, d+1$. Note that such $d+1$ points exist even if the linear hull of $Y_1, \ldots, Y_q$ has dimension $< d$. Hence

$$Y_j = \sum_{i=1}^{d} \beta_{j,i} Y_i + \left(1 - \sum_{i=1}^{d} \beta_{j,i}\right) Y_{d+1}, \qquad j = d + 2, \ldots, q,$$

for some coefficients $\{\beta_{j,i}\}$, $i = 1, \ldots, d$, $j = d + 2, \ldots, q$.

In this case equations (2) hold for $i = 2, \ldots, d + 1$. We have thus expressed all the coordinates of the points $\{A_{i,j}\}$, $i = 1, \ldots, q$, $j = 2, \ldots, n_i + 1$; $Y_1, \ldots, Y_q$, and the numbers $\{\lambda_{i,j}\}$, $i = 1, \ldots, q$, $j = 1, \ldots, n_i$, in terms of the coordinates of $\{A_{i,j}\}$, $i = 1, \ldots, q$, $j = 2, \ldots, n_i + 1$; Y_1, and the numbers $\{\lambda_{i,j}\}$, $i = 1, \ldots, q$, $j = 1, \ldots, n_i$; $\{\beta_{j,i}\}$, $j = d + 2, \ldots, q$, $i = 1, \ldots, d$; $\alpha_{i,j}$, $i = 2, \ldots, d + 1$, $j = 1, \ldots, T$.

Hence by Proposition 2.1 we obtain $qm \leqslant m + n_1 + \cdots + n_q + (q - d - 1)d + Td$, or equivalently, $n_1 + \cdots + n_q \geqslant (m - d)(q - 1) - (T - d)d$. Replacing in the last inequality m, d, and T by $m - t$, $d - t$, and $T - t$, respectively, we obtain

$$n_1 + \cdots + n_q \geqslant (m - d)(q - 1) - (T - d)(d - t). \tag{3_{II}}$$

Inequalities (3_{I}) and (3_{II}) complete the proof of Theorem 1.1.

Proof of Corollary 1.2. Let $\{v_i\}$ be the vertices of K, and R_j the sets in the above-discussed Roberts's construction carried out for some denumerable algebraically independent set. For each i we consider a point $A_i = (A_i(1), \dots, A_i(m)) \in \mathbb{R}^m$ such that $\operatorname{dist}\bigl(\theta(v_i), A_i\bigr) < \varepsilon$ and $A_i(s) \in R_{(i-1)m+s}$. Then the set of all the coordinates of $\{A_i(k)\}$ is algebraically independent. We define a map $g\colon K \to \mathbb{R}^m$ by setting $g(v_i) = A_i$ and extending g linearly to each simplex of K. Obviously, $g(v_{i_1}) \neq g(v_{i_2})$ for $i_1 \neq i_2$. Moreover, the restriction of the map g to each n-simplex of K is one-to-one. Let Π^d be a d-plane in R^m parallel to some coordinate planes $\Pi^t \subset \Pi^T$, and let q be the number of disjoint simplexes of K of dimension not exceeding n the g-images of which meet Π^d. That is, assume that K contains q at most n-dimensional disjoint simplexes $\sigma_i = \langle v_{i,j} : j = 1, \dots, n_i + 1\rangle$ such that the sets $g(\sigma_i) = \langle A_{i,j} : j = 1, \dots, n_i + 1\rangle$ intersect Π^d. We claim that

$$q \leqslant N_1 = d + 1 - t + \frac{n + (n + T - m)(d - t)}{m - n - d} \quad \text{for } n \geqslant (m - n - T)(d - t).$$

Assume that $q \geqslant d + 1 - t$. Since Π^d intersects the images

$$g(\sigma_i) \subset \Pi\bigl(\{A_{i,j} : j = 1, \dots, n_i + 1\}\bigr)$$

of all the disjoint simplexes σ_i, $i = 1, \dots, q$, it follows by Theorem 1.1 that

$$n_1 + \dots + n_q \geqslant (m - d)(q - 1) - (T - d)(d - t).$$

Consequently,

$$nq \geqslant (m - d)(q - 1) - (T - d)(d - t) \tag{4}$$

because $n_i \leqslant n$ for each i. Relation (4) is equivalent to the required inequality $q \leqslant N_1$.

If $q \leqslant d+1-t$, then we have $q \leqslant N_1$ because our assumption $n \geqslant (m-n-T)(d-t)$ means that $d + 1 - t \leqslant N_1$.

We now show that

$$q \leqslant N_2 = 1 + \frac{n}{m - n - T} \quad \text{for } n \leqslant (m - n - T)(d - t).$$

Assume that $q \leqslant d + 1 - t$. Then, as before, $n_1 + \dots + n_q \geqslant (m - T)(q - 1)$ by Theorem 1.1, and therefore

$$nq \geqslant (m - T)(q - 1) \tag{5}$$

since $n_i \leqslant n$ for all i. Relation (5) is equivalent to the required inequality $q \leqslant N_2$.

Finally, assume that $q \geqslant d - t + 1$. Then $q \leqslant N_1$ as already proved. On the other hand, by our assumption $n \leqslant (m - n - T)(d - t)$, that is, $d + 1 - t \leqslant N_1$. All this means that under our assumptions we have $q = d - t + 1 = N_1$ and $n = (m - n - T)(d - t)$, so that $q = d - t + 1 = N_1 = N_2$.

§ 3. Proof of Theorem 1.3

We prove Theorem 1.3 in the special case $d = m - n$ first. We fix spaces X, Y and a map f satisfying the assumptions of Theorem 1.3. Unless otherwise stated we use throughout this section the following notation: ϱ is the Euclidean metric in $\mathbb{R}^m$, an ε-disjoint subset of $\mathbb{R}^m$ is a subset that can be covered by a disjoint family of open subsets of $\mathbb{R}^m$ of diameter $< \varepsilon$ each. We say that a fixed subset A of $\mathbb{R}^m$ is of type (d, ε) if $\Pi^d \cap A$ is ε-disjoint for each d-plane Π^d in $\mathbb{R}^m$.

Let $\mathcal{H}_\varepsilon$, $\varepsilon > 0$, be the set of maps $g \in C^*(X, \mathbb{R}^m)$ such that $g(f^{-1}(y))$ is of type $(m - n, \varepsilon)$ for each $y \in Y$. Since $C^*(X, \mathbb{R}^m)$ equipped with the source limitation topology has the Baire property, it is sufficient to show that each $\mathcal{H}_\varepsilon$ is open and dense in $C^*(X, \mathbb{R}^m)$. Indeed, in that case $\mathcal{H} = \bigcap_{k=1}^{\infty} \mathcal{H}_{1/k}$ is dense and G_δ in $C^*(X, \mathbb{R}^m)$. Moreover, if $g \in \mathcal{H}$ and $y \in Y$, then $g(f^{-1}(y)) \cap \Pi$ is at most 0-dimensional for each $(n - m)$-plane $\Pi \subset \mathbb{R}^m$.

Lemma 3.1. *Let A be a compact subset of X and assume that $\varepsilon > 0$. Let $g_0(A)$ be a set of type (d, ε) for some $g_0 \in C^*(X, \mathbb{R}^m)$ and d. Then there exist a neighbourhood U of A in X and $\delta > 0$ such that the set $\overline{g(U)}$ has type (d, ε) for each map $g \in C^*(X, \mathbb{R}^m)$ such that $g|U$ is δ-close to $g_0|U$.*

Proof. Assume that the conclusion of the lemma fails. To obtain a contradiction we shall follow § 2.4 in [4]. For each $i \geqslant 1$ we consider a neighbourhood U_i of A such that $U_i \subset g_0^{-1}(W_i)$, where W_i is the $1/i$-neighbourhood of $g_0(A)$. There exist $g_i \in C^*(X, \mathbb{R}^m)$ and d-planes Π_i^d such that $g_i|U_i$ is $1/i$-close to $g_0|U_i$, but $\overline{g_i(U_i)} \cap \Pi_i^d$ is not ε-disjoint. We choose points $z_i \in \overline{g_i(U_i)} \cap \Pi_i^d$ and $x_i \in U_i$ such that $\rho\bigl(g_i(x_i), z_i\bigr) \leqslant 1/i$, $i \geqslant 1$. Obviously, $K = \{z_i\}_{i=1}^{\infty} \cup g_0(A)$ is a compactum intersecting each plane Π_i^d. Hence there exists a subsequence of $\{\Pi_i^d\}_{i=1}^{\infty}$ convergent to a d-plane Π_0^d. We shall assume that the sequence $\{\Pi_i^d\}_{i=1}^{\infty}$ converges itself to Π_0^d.

Let V be an open subset of $\mathbb{R}^m$ containing $g_0(A) \cap \Pi_0^d$ that is the union of a disjoint finite family of open subsets of $\mathbb{R}^m$ having diameter $< \varepsilon$. Because the sets $\overline{g_i(U_i)} \cap \Pi_i^d$ are not ε-disjoint, there exist points $a_i \in U_i$ and $b_i \in \overline{g_i(U_i)} \cap \Pi_i^d$ such that V does not contain the set $\{g_i(a_i), b_i\}_{i=1}^{\infty}$. We can also assume that $\varrho\bigl(b_i, g_i(a_i)\bigr) \leqslant 1/i$ for all i. This means the existence of a point $b \in g_0(A)$ and a subsequence of $\{b_i\}$ converging to b. We shall continue to write $\lim b_i = b$. Then $b \in \Pi_0^d$ because $\{\Pi_i^d\}$ converges to Π_0^d. Hence $b \in g_0(A) \cap \Pi_0^d \subset V$. Consequently, $b_i \in V$ for some i, which contradicts the choice of b_i.

Corollary 3.2. *Suppose that $g_0(f^{-1}(y_0))$ is of type $(m-n, \varepsilon)$ for some point $y_0 \in Y$ and $g_0 \in C^*(X, \mathbb{R}^m)$. Then there exist a neighbourhood V of y_0 in Y and $\delta > 0$ such that the set $\overline{g(f^{-1}(V))}$ has type $(m - n, \varepsilon)$ for each $g \in C^*(X, \mathbb{R}^m)$ such that $g|f^{-1}(V)$ is δ-close to $g_0|f^{-1}(V)$.*

Proof. Applying Lemma 3.1 to $f^{-1}(y_0)$ we obtain positive δ and a neighbourhood U of the set $f^{-1}(y_0)$ such that $\overline{g(U)}$ is of type $(m-n, \varepsilon)$ once $g \in C^*(X, \mathbb{R}^m)$ and $g|U$ is δ-close to $g_0|U$. Since f is a closed map, we can find a closed neighbourhood V of y_0 in Y such that $f^{-1}(V) \subset U$. Assume now that $g|f^{-1}(V)$ is δ-close to $g_0|f^{-1}(V)$ for some $g \in C^*(X, \mathbb{R}^m)$. We extend $g|f^{-1}(V)$ to a map $h \in C^*(X, \mathbb{R}^m)$ such that $h|U$ remains δ-close to $g_0|U$. Then by our choice of U and δ the set $\overline{h(U)}$ has type

$(m-n,\varepsilon)$. Finally, since $\overline{h(f^{-1}(V))} \subset \overline{h(U)}$ and $h|f^{-1}(V) = g|f^{-1}(V)$, the proof is complete.

Proposition 3.3. *Each set* $\mathcal{H}_\varepsilon$ *is open in* $C^*(X,\mathbb{R}^m)$.

Proof. We fix $g_0 \in \mathcal{H}_\varepsilon$. By Corollary 3.2, for each $y \in Y$ there exist a neighbourhood V_y of y in Y and $\delta_y > 0$ such that if $g \in C^*(X,\mathbb{R}^m)$ and $\varrho\bigl(g(x), g_0(x)\bigr) \leqslant \delta_y$ for all $x \in f^{-1}(V_y)$, then $\overline{g(f^{-1}(V_y))}$ has type $(m-n,\varepsilon)$. Consider a locally finite open cover ω of Y refining $\{V_y : y \in Y\}$, and for each $W \in \omega$ fix a point $y(W) \in Y$ such that $W \subset V_{y(W)}$. We define a set-valued map $\phi\colon Y \to (0,\infty)$ by the formula $\phi(y) = \bigcup\{(0,\delta_{y(W)}] : y \in W\}$. Obviously, ϕ is convex-valued and lower semicontinuous. By [11], Theorem 6.2 the map ϕ admits a continuous selection $\beta\colon Y \to (0,\infty)$; let $\alpha = \beta \circ f$. It is sufficient to show that if $g \in C^*(X,\mathbb{R}^m)$ and $\varrho\bigl(g_0(x), g(x)\bigr) < \alpha(x)$ for each $x \in X$, then $g \in \mathcal{H}_\varepsilon$. In fact, let $y \in Y$ and consider $W \in \omega$ containing y such that $\alpha(x) \leqslant \delta_{y(W)}$ for each $x \in f^{-1}(y)$. Consider a function $h_y \in C^*(X,\mathbb{R}^m)$ coinciding with g on the set $f^{-1}(y)$ and satisfying the inequality $\varrho\bigl(h_y(x), g_0(x)\bigr) \leqslant \delta_{y(W)}$ for all $x \in X$. In accordance with our choice of $V_{y(W)}$, the set $\overline{h_y(f^{-1}(V_{y(W)}))}$ is of type $(m-n,\varepsilon)$, so that $\overline{g(f^{-1}(y))}$ has the same type. Hence each map $g \in C^*(X,\mathbb{R}^m)$ that is α-close to g_0 belongs to $\mathcal{H}_\varepsilon$, and therefore $\mathcal{H}_\varepsilon$ is open in $C^*(X,\mathbb{R}^m)$.

For an arbitrary space M and $\varepsilon>0$ let $C_{(n,\varepsilon)}(M,\mathbb{R}^m)$ be the set of $g \in C^*(M,\mathbb{R}^m)$ such that $g(M)$ is of type $(m-n,\varepsilon)$ with $m \geqslant n+1$.

Lemma 3.4. *Let* M *be an* n*-dimensional compactum and assume that* $m \geqslant n+1$. *Then* $C_{(n,\varepsilon)}(M,\mathbb{R}^m)$ *is dense in* $C(M,\mathbb{R}^m)$ *for each* $\varepsilon > 0$.

Proof. Let $g_0 \in C(M,\mathbb{R}^m)$ and assume that $\delta > 0$. Representing g_0 as the composite of two maps $q_1\colon M \to Z$ and $q_2\colon Z \to \mathbb{R}^m$, where Z is a metrizable compactum of dimension $\leqslant n$, and considering Z and q_2 instead of M and g_0 we reduce our proof to the case when M is a metrizable compactum. Consider a positive number η satisfying the following conditions:

$$5\eta < \frac{\delta}{2} \quad \text{and} \quad 9\eta(r+1) < \varepsilon, \quad \text{where} \quad r = n(m+1-n). \tag{6}$$

Since $\dim M \leqslant n$, we can use a standard procedure to find a finite n-dimensional complex K and maps $h\colon M \to K$, $\theta\colon K \to \mathbb{R}^m$ such that $\theta \circ h$ is $\delta/2$-close to g_0. Moreover, we can assume that

$$\operatorname{diam}\bigl(\theta(\sigma)\bigr) < \eta \quad \text{for each simplex } \sigma \in K. \tag{7}$$

It is sufficient to find a map $g\colon K \to \mathbb{R}^m$ in the $\delta/2$-neighbourhood of θ such that $g \circ h \in C_{(n,\varepsilon)}(M,\mathbb{R}^m)$. To this end we use Corollary 1.2 (with $d = m-n$, $t = 0$, $T = m$, $\varepsilon = \eta$, and n replaced by $n-1$) to obtain a semilinear map $g\colon K \to \mathbb{R}^m$ such that $\varrho\bigl(g(v), \theta(v)\bigr) \leqslant \eta$ for all vertices v of K and, for each $(m-n)$-plane $\Pi \subset \mathbb{R}^m$, the number q of disjoint, at most $(n-1)$-dimensional simplexes K with g-images intersecting Π is at most $r = n(m+1-n)$. We can choose g such that, in addition to the above, $g(v_i) \neq g(v_j)$ for each pair of distinct vertices v_i and v_j of K.

Let v_i and v_j be vertices of the same simplex $\Delta \in K$. Then by (7) and our choice of g we obtain

$$\varrho\big(g(v_i), g(v_j)\big) \leqslant \varrho\big(g(v_i), \theta(v_i)\big) + \varrho\big(\theta(v_i), \theta(v_j)\big) + \varrho\big(\theta(v_j), g(v_j)\big) < 3\eta.$$

Consequently,

$$g(\Delta) \text{ has diameter } < 3\eta \text{ for each simplex } \Delta \in K. \tag{8}$$

Condition (8) means that $\varrho\big(g(y), \theta(y)\big) < 5\eta$ for all $y \in K$. Hence by (6) the maps g and θ are $\delta/2$-close.

It merely remains to show that $g \circ h \in C_{(n,\varepsilon)}(M, \mathbb{R}^m)$ or, equivalently, that $g(K)$ has type $(m-n, \varepsilon)$. To this end we use an idea from [4] (proof of 2.3, p. 568). We fix an $(m-n)$-plane $\Pi \subset \mathbb{R}^m$. It is sufficient to show that each component of $g(K) \cap \Pi$ has a diameter $\leqslant 9(r+1)\eta$, because $9(r+1)\eta < \varepsilon$ by (6). Assume that $\varrho(a,b) > 9(r+1)\eta$ for some component P of $g(K) \cap \Pi$ and some points $a \in P$, $b \in P$. Consider an arc ab in P. By (8) each subarc of diameter $\geqslant 3\eta$ must contain at least one point of the boundary of a simplex of $g(K)$, which therefore belongs to a simplex of $g(K)$ of dimension $\leqslant n-1$. Next, we consider points $a_i \in ab$, $i = 1, \dots, r+1$, such that $3(3i-2)\eta < \varrho(a, a_i) \leqslant 3(3i-1)\eta$, $i = 1, \dots, r+1$, and each a_i belongs to a simplex $g(\sigma_i) \in g(K)$ of dimension $\leqslant n-1$. Then $\varrho(a_i, a_j) > 6\eta$ for $i \neq j$, which means in accordance with (8) that $g(\sigma_i) \cap g(\sigma_j) = \varnothing$. We have thus obtained $r+1$ disjoint simplexes $\sigma_i \in K$, $i = 1, \dots, r+1$, of dimensions $\leqslant n-1$ with g-images intersecting Π. This is a contradiction because by our choice of g the number of disjoint simplexes of K of dimension not exceeding $(n-1)$ with g-images intersecting Π is at most $n(m-n+1) = r$.

We consider below the set-valued map $\psi_\varepsilon \colon Y \to C^*(X, \mathbb{R}^m)$ defined by the formula $\psi_\varepsilon(y) = C^*(X, \mathbb{R}^m) \setminus \mathcal{H}_\varepsilon(y)$, where $\mathcal{H}_\varepsilon(y)$ is the set of maps $g \in C^*(X, \mathbb{R}^m)$ such that $g(f^{-1}(y))$ has type $(m-n, \varepsilon)$.

The next statement in combination with Proposition 3.3 completes the proof of Theorem 1.3 for $k = m-n$.

Proposition 3.5. *The set $\mathcal{H}_\varepsilon$ is dense in $C^*(X, \mathbb{R}^m)$.*

Proof. We show first that the graph G of ψ_ε is closed in $Y \times C^*(X, \mathbb{R}^m)$, provided that $C^*(X, \mathbb{R}^m)$ is equipped with the topology of uniform convergence induced by the metric ϱ. Let $(y_0, g_0) \in (Y \times C^*(X, \mathbb{R}^m)) \setminus G$. Then $g_0 \notin \psi_\varepsilon(y_0)$, therefore $g_0 \in \mathcal{H}_\varepsilon(y_0)$. By Corollary 3.2 there exist $\delta > 0$ and a neighbourhood V of y_0 in Y such that $\overline{g(f^{-1}(V))}$ has type $(m-n, \varepsilon)$ for each map $g \in C^*(X, \mathbb{R}^m)$ such that $g|f^{-1}(V)$ is δ-close to $g_0|f^{-1}(V)$. Let $W \subset C^*(X, \mathbb{R}^m)$ be the set of all maps g that are δ-close to g_0. Obviously, W is a neighbourhood of g_0 in $C^*(X, \mathbb{R}^m)$ with respect to the topology of uniform convergence, so that $V \times W$ is a neighbourhood of (y_0, g_0) in $Y \times C^*(X, \mathbb{R}^m)$ disjoint from G. Hence G is closed in $Y \times C^*(X, \mathbb{R}^m)$.

Claim. *$\overline{B}(g_0, \alpha) \setminus \psi_\varepsilon(y) \neq \varnothing$ for all $y \in Y$, $\alpha \colon X \to (0, \infty)$, and $g_0 \in C^*(X, \mathbb{R}^m)$.*

Proof. We must show that $\overline{B}(g_0, \alpha) \cap \mathcal{H}_\varepsilon(y) \neq \varnothing$ for fixed $y \in Y$, $\alpha \in C(X, (0, \infty))$, and $g_0 \in C^*(X, \mathbb{R}^m)$. Let $\delta > 0$ be the minimum value of α on $f^{-1}(y)$. Since $\dim f^{-1}(y) \leqslant n$, it follows by Lemma 3.4 that there exists a map $h \in C(f^{-1}(y), \mathbb{R}^m)$

δ-close to $g_0|f^{-1}(y)$ such that $h(f^{-1}(y))$ has type $(m-n,\varepsilon)$. For all $x\in f^{-1}(y)$ we obviously have $\varrho\bigl(h(x),g_0(x)\bigr)\leqslant\alpha(x)$ Thus, each extension $g\in C^*(X,\mathbb{R}^m)$ of h belongs to $\mathcal{H}_\varepsilon(y)$. Hence we have reduced the proof to finding an extension of h that also belongs to $\overline{B}(g_0,\alpha)$. To this end we consider the set-valued map $\Phi\colon X\to\mathbb{R}^m$ defined by the formula $\Phi(x)=h(x)$ for $x\in f^{-1}(y)$ and $\Phi(x)=B\bigl(g_0(x),\alpha(x)\bigr)$ for $x\notin f^{-1}(y)$. Here $B(g_0(x),\alpha(x))$ is the closed ball in $\mathbb{R}^m$ with centre at $g_0(x)$ and of radius $\alpha(x)$. This is a lower semicontinuous map with closed and convex values in $\mathbb{R}^m$. Hence by Michael's convex-valued selection theorem Φ admits a continuous selection g, which extends h. Moreover, we can assume that α is a bounded function, which means that g is also bounded. Thus, $g\in\overline{B}(g_0,\alpha)$, which completes the proof of the claim.

We are now in a position to complete the proof of the proposition. We fix $g_0\in C^*(X,\mathbb{R}^m)$ and $\alpha\in C^*(X,(0,1))$ and consider the constant set-valued map $\varphi\colon Y\to C^*(X,\mathbb{R}^m)$, $\varphi(y)=\overline{B}(g_0,\alpha)$, where $C^*(X,\mathbb{R}^m)$ is equipped with the topology of uniform convergence. We require the following result of Michael ([12], Theorem 5.3):

Let Y be a paracompact space with $\dim Y=0$, *M a completely metrizable space, and $\phi\colon Y\to M$ a lower semicontinuous closed-valued map. If $\psi\colon Y\to M$ is a set-valued map with closed graph such that $\phi(y)\setminus\psi(y)\neq\varnothing$ for each $y\in Y$, then ϕ has a selection avoiding ψ.*

In our case φ and ψ_ε satisfy the hypotheses of Michael's theorem, therefore there exists a map $\theta\colon Y\to C^*(X,\mathbb{R}^m)$ such that $\theta(y)\in\overline{B}(g_0,\alpha)\setminus\psi_\varepsilon(y)$ for all $y\in Y$. Let $g\in C^*(X,\mathbb{R}^m)$ be the map defined by the formula $g(x)=\theta(f(x))$, $x\in X$. Then $g\in\overline{B}(g_0,\alpha)\cap\mathcal{H}_\varepsilon(y)$ for each $y\in Y$. Consequently, $g\in\overline{B}(g_0,\alpha)\cap\mathcal{H}_\varepsilon$ and the proof is complete.

As already mentioned, it follows from Proposition 3.3 and Proposition 3.5 that the set $\mathcal{H}$ is dense and G_δ in $C^*(X,\mathbb{R}^m)$. This gives us a proof of Theorem 1.3 in the special case $d=m-n$. We now prove Theorem 1.3 in the general case. We shall show that each $g\in\mathcal{H}$ satisfies the requirements of the theorem, that is, has the following property: the compactum $A=g(f^{-1}(y))\cap\Pi^d$ is at most $(n+d-m)$-dimensional for each $y\in Y$ and each d-plane $\Pi^d\subset\mathbb{R}^m$ for $m-n+1\leqslant d\leqslant m$. Fix some $(m-n)$-plane $\Pi^{m-n}\subset\Pi^d$ and consider the orthogonal projection p of Π^d onto the $(n+d-m)$-plane $\Pi^{n+d-m}\subset\Pi^d$ that is the orthogonal complement of Π^{m-n} in Π^d. Then the compactum $B=p\bigl(g(f^{-1}(y))\cap\Pi^d\bigr)\subset\Pi^{n+d-m}$ has dimension $\leqslant n+d-m$. The fibres of the map $p\big|_A\colon A\to B$ are intersections of $g(f^{-1}(y))$ with some $(m-n)$-planes, therefore these fibres are zero-dimensional by our choice of g. Hence by Hurewicz's theorem on perfect zero-dimensional maps $\dim A\leqslant\dim B\leqslant n+d-m$.

§ 4. Proofs of Theorem 1.5 and Theorem 1.8

Proof of Theorem 1.5. It is sufficient to show that for arbitrary integers d, t, T such that $0\leqslant t\leqslant d\leqslant T\leqslant m$ and $d\leqslant m-n-1$, and for arbitrary coordinate planes $\Pi^t\subset\Pi^T$ in $\mathbb{R}^m$ the space $C^*(X,\mathbb{R}^m)$ contains a dense G_δ-subset of maps g such that $f^{-1}(y)\cap g^{-1}(\Pi^d)$ contains at most q points for each $y\in Y$ and each d-plane $\Pi^d\subset\mathbb{R}^m$ parallel to $\Pi^t\subset\Pi^T$, where q is the integer part of

$N_1 = d+1-t+\dfrac{n+(n+T-m)(d-t)}{m-n-d}$ for $n \geqslant (m-n-T)(d-t)$ and the integer part of $N_2 = 1+\dfrac{n}{m-n-T}$ otherwise. Thus, we fix integers d, t, T satisfying the above inequalities and coordinate planes $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$. We point out that we always have $q \geqslant 1$; and let $\mathcal{P}$ be the set of d-planes in $\mathbb{R}^m$ parallel to $\Pi^t \subset \Pi^T$. We now define the following concept, which is used in this section: a subset A of an arbitrary metric space M is said to be of *cotype* (q, ε, g), where $\varepsilon > 0$ and $g \in C^*(M, \mathbb{R}^m)$, if for each $\Pi^d \in \mathcal{P}$ the set $A \cap g^{-1}(\Pi^d)$ can be covered by at most q disjoint open subsets of M of diameter $\leqslant \varepsilon$. Let $\mathcal{K}_\varepsilon$, $\varepsilon > 0$, be the set of all maps $g \in C^*(X, \mathbb{R}^m)$ such that $f^{-1}(y)$ has cotype (q, ε, g) for each $y \in Y$. Our proof reduces to the verification that each set $\mathcal{K}_\varepsilon$ is open and dense in $C^*(X, \mathbb{R}^m)$.

Lemma 4.1. *Let A be a compact subset of X having cotype (q, ε, g_0) for some $g_0 \in C^*(X, \mathbb{R}^m)$ and $\varepsilon > 0$. Then there exist a neighbourhood U of A in X and $\delta > 0$ such that U has cotype (q, ε, g) for each map $g \in C^*(X, \mathbb{R}^m)$ such that $g|U$ is δ-close to $g_0|U$.*

Proof. Assume that the lemma fails. Then for each $i \geqslant 1$ we consider a $1/i$-neighbourhood U_i of A such that $g_0(U_i)$ lies in the interior of the $1/i$-neighbourhood of $g_0(A)$ in $\mathbb{R}^m$. Then there exist $g_i \in C^*(X, \mathbb{R}^m)$ and a d-plane $\Pi_i^d \in \mathcal{P}$ such that $g_i|U_i$ is $1/i$-close to $g_0|U_i$, but $g_i^{-1}(\Pi_i^d) \cap U_i$ is not covered by any disjoint family of $\leqslant q$ open subsets of X of diameter $\leqslant \varepsilon$. Similarly to the proof of Lemma 3.1, passing to subsequences we can assume that $\{\Pi_i^d\}_{i=1}^\infty$ converges to a d-plane Π_0^d. Since all the Π_i^d belong to $\mathcal{P}$ (that is, are parallel to $\Pi^t \subset \Pi^T$), it follows that $\Pi_0^d \in \mathcal{P}$. Hence $A \cap g_0^{-1}(\Pi_0^d)$ is covered by a disjoint family $\{V_j\}$ of $\leqslant q$ open subsets of X of diameters $\leqslant \varepsilon$. Consider points $x_i \in g_i^{-1}(\Pi_i^d) \cap (U_i \setminus V)$ and $y_i \in A$, such that $\operatorname{dist}(x_i, y_i) \leqslant 1/i$, $i = 1, 2, \ldots$, where $V = \bigcup V_j$. We can assume that the sequence $\{x_i\}$ converges to some point $x_0 \in A$ (recall that A is compact). Then $\{g_i(x_i)\}$ converges to $g_0(x_0)$ and $g_0(x_0) \in g_0(A) \cap \Pi_0^d$. Hence $x_0 \in A \cap g_0^{-1}(\Pi_0^d) \subset V$, and therefore $x_i \in V$ for almost all i, which is a contradiction.

The proof of the next result is similar to the proof of Corollary 3.2.

Corollary 4.2. *Suppose that for some point $y_0 \in Y$ and some map $g_0 \in C^*(X, \mathbb{R}^m)$ the set $f^{-1}(y_0)$ is of cotype (q, ε, g_0). Then there exist a neighbourhood V of y_0 in Y and $\delta > 0$ such that the set $f^{-1}(V)$ has cotype (q, ε, g) for each map $g \in C^*(X, \mathbb{R}^m)$ such that $g|f^{-1}(V)$ is δ-close to $g_0|f^{-1}(V)$.*

Proposition 4.3. *Each set $\mathcal{K}_\varepsilon$ is open in $C^*(X, \mathbb{R}^m)$.*

The proof follows the scheme of the proof of Proposition 3.3, but we now use Corollary 4.2 instead of Corollary 3.2.

Lemma 4.4. *Let M be a metrizable compactum of a dimension at most n. Then the set $\mathcal{K}_0(M, \mathbb{R}^m)$ of all maps $g \in C(M, \mathbb{R}^m)$ such that $g^{-1}(\Pi^d)$ contains at most q points for each plane $\Pi^d \in \mathcal{P}$ is dense in $C(M, \mathbb{R}^m)$.*

Proof. Let Ω be the system of disjoint families $\{\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_{q+1}\}$ of $q+1$ elements such that each V_j belongs to a fixed countable base for M. Also for $\Gamma \in \Omega$ let $C_\Gamma = \big\{g \in C(M, \mathbb{R}^m) : g^{-1}(\Pi^d)$ meets at most q elements of Γ for each $\Pi^d \in \mathcal{P}\big\}$.

Obviously, Ω is countable and $\mathcal{K}_0(M,\mathbb{R}^m)$ is the intersection of all sets C_Γ, $\Gamma \in \Omega$. Hence our proof reduces to the verification that each C_Γ is dense and open in $C(M,\mathbb{R}^m)$.

Claim 1. *Each set C_Γ is open in $C(M,\mathbb{R}^m)$.*

Proof. We fix $\Gamma \in \Omega$ and $g_0 \in C_\Gamma$. Assume that for each i there exists $g_i \notin C_\Gamma$ that is $1/i$-close to g_0. Then we can find a plane $\Pi_i^d \in \mathcal{P}$ such that $g_i^{-1}(\Pi_i^d)$ intersects each element of Γ. Similarly to Lemma 3.1 we can assume that the sequence $\{\Pi_i^d\}$ converges to some $\Pi_0^d \in \mathcal{P}$. Then $g_0^{-1}(\Pi_0^d)$ intersects at most q elements of Γ, for instance, the first q sets. Now, for each i we choose a point $x_i \in g_i^{-1}(\Pi_i^d) \cap \overline{V}_{q+1}$ and since M is compact, we can assume that the sequence $\{x_i\}$ converges to some point $x_0 \in \overline{V}_{q+1}$. Then $\{g_i(x_i)\}$ converges to $g_0(x_0) \in \Pi_0^d$. Thus, $x_0 \in g_0^{-1}(\Pi_0^d) \cap \overline{V}_{q+1}$, which is a contradiction.

Claim 2. *Each set C_Γ is dense in $C(M,\mathbb{R}^m)$.*

Proof. Let $\Gamma = \{\overline{V}_1, \overline{V}_2, \dots, \overline{V}_{q+1}\}$, $g_0 \in C_\Gamma$, and assume that $\delta > 0$. Then there exist an open cover ω of M with $\operatorname{mesh}(\omega) \leqslant r/3$, where $r = \min\{\operatorname{dist}(\overline{V}_i, \overline{V}_j) : i \neq j\}$, and a piecewise linear map $h\colon L \to \mathbb{R}^m$ such that $g = h \circ \pi$ is δ-close to g_0. Here L is the polyhedron underlying the nerve of ω and $\pi\colon M \to L$ is the canonical map. By Corollary 1.2 we can assume that for each plane $\Pi^d \in \mathcal{P}$ the number of disjoint at most n-dimensional simplexes $\sigma \in L$ with images $h(\sigma)$ intersecting Π^d is at most q. We can also assume that ω is of order $\leqslant n+1$, so that L is at most n-dimensional. If there exists a plane $\Pi^* \in \mathcal{P}$ such that $g^{-1}(\Pi^*)$ intersects each $\overline{V}_i$, then we choose $x_i \in g^{-1}(\Pi^*) \cap \overline{V}_i$. Let ω_i be the family of elements of ω containing the point x_i. Then each family ω_i, $i = 1, \dots, q+1$, generates a simplex $\sigma_i \in L$ of dimension $\leqslant n$ such that $h(x_i) \in h(\sigma_i) \cap \Pi^*$ and $\sigma_i \cap \sigma_j = \varnothing$ for $i \neq j$. This contradicts our choice of h.

Since $\mathcal{K}_0(M,\mathbb{R}^m) \subset \mathcal{K}_\varepsilon(M,\mathbb{R}^m)$ for each $\varepsilon > 0$, where $\mathcal{K}_\varepsilon(M,\mathbb{R}^m)$ is the set of maps $g \in C^*(M,\mathbb{R}^m)$ such that M has cotype (q,ε,g), Lemma 4.4 has the following consequence.

Corollary 4.5. *Let M be a metrizable compactum of dimension $\leqslant n$. Then each set $\mathcal{K}_\varepsilon(M,\mathbb{R}^m)$ is dense in $C(M,\mathbb{R}^m)$.*

The next result in combination with Proposition 4.3 completes the proof of Theorem 1.5.

Proposition 4.6. *Each set $\mathcal{K}_\varepsilon$ is dense in $C^*(X,\mathbb{R}^m)$.*

Proof. Let $\psi_\varepsilon\colon Y \to C^*(X,\mathbb{R}^m)$ be the set-valued map $\psi_\varepsilon(y) = C^*(X,\mathbb{R}^m) \setminus \mathcal{K}_\varepsilon(y)$, where $\mathcal{K}_\varepsilon(y)$ is the set of all $g \in C^*(X,\mathbb{R}^m)$ such that $f^{-1}(y)$ has cotype (q,ε,g). The rest of the proof follows the arguments in the proof of Proposition 3.5. To show that the graph of ψ_ε is closed we now use Corollary 4.2 instead of Corollary 3.2, and we also replace Lemma 3.4 in the proof by Corollary 4.5.

Proof of Theorem 1.8. We shall prove that $C^*(X,l_2)$ contains a dense G_δ-subset of maps g such that for $y \in Y$ and each plane $\Pi^d \in \mathcal{P}(d,r)$ the set $f^{-1}(y) \cap g^{-1}(\Pi^d)$ contains at most $d+1-r$ points if $r \leqslant d$ and at most one point otherwise. For fixed integers d and r let $\mathcal{Q}$ be the set of d-planes parallel to the r-plane of the first r

coordinate variables in l_2. Similarly to Theorem 1.5 we introduce the concept of subset of cotype (q, ε, g) of a metric space M by considering now planes $\Pi^d \in \mathcal{Q}$ and by setting $q = d + 1 - r$ for $d \geqslant r$ and $q = 1$ otherwise. Let $\mathcal{F}_\varepsilon$, $\varepsilon > 0$, be the set of maps $g \in C^*(X, l_2)$ such that $f^{-1}(y)$ has cotype (q, ε, g) for each $y \in Y$. Since l_2 contains countably many coordinate r-planes, it is sufficient to show that all the $\mathcal{F}_\varepsilon$ are open and dense in $C^*(X, l_2)$. Following the proof of Theorem 1.5 we can show that all the sets $\mathcal{F}_\varepsilon$ are open in $C^*(X, l_2)$. For the proof of the density of $\mathcal{F}_\varepsilon$ we use the following lemma.

Lemma 4.7. *Let M be a metrizable compactum and assume that $\varepsilon > 0$. Then the set $\mathcal{F}_\varepsilon(M, l_2)$ of maps $g \in C(M, l_2)$ such that M has cotype (q, ε, g) is dense in $C(M, l_2)$.*

Proof. Consider $g_0 \in C(M, l_2)$ and $\lambda > 0$. Then there exist a finite complex K and maps $\varphi\colon M \to K$ and $h\colon K \to l_2$ such that $h \circ \varphi$ is $\lambda/2$-close to g_0. Moreover, we can assume that each fibre of φ has diameter $< \varepsilon$. For each $A \subset \mathbb{N}$ we identify $\mathbb{R}^A$ with the subspace $\{y \in l_2 : y_i = 0 \text{ for all } i \notin A\}$ of the Hilbert space l_2 and denote by π_A the canonical projection $\pi_A\colon l_2 \to \mathbb{R}^A$. Let $\dim K = n$ and let A be a finite subset of $\mathbb{N}$ satisfying the following conditions:

(i) $\{1, \dots, r\} \subset A$;
(ii) $|A| \geqslant r + d + 2n + n \cdot |d - r| + 1$, where $|A|$ is the cardinality of A.

Since each projection π_A is an open map, we can show that so also is the map

$$\Lambda_A\colon C(K, l_2) \to C(K, \mathbb{R}^A), \quad \Lambda_A(g) = \pi_A \circ g.$$

Assume first of all that $r \leqslant d$. We observe that if $T = m$, then we have $(m-n-T)(d-r) = -n(d-r) \leqslant 0$, therefore $n \geqslant (m-n-T)(d-r)$. By Theorem 1.5 with $m = T = |A|$ and $t = r$ we can now find a dense G_δ-subset $\mathcal{F}_A$ of $C(K, \mathbb{R}^A)$ consisting of maps g such that $g^{-1}(\Pi^d_A)$ contains at most $1 + d - r + \dfrac{n + n(d-r)}{|A| - n - d}$ points for each d-plane $\Pi^d_A \subset \mathbb{R}^A$ parallel to the r-plane of the first r coordinates in $\mathbb{R}^A$. Since $|A| \geqslant r+d+2n+n\cdot|d-r|+1$, the inverse image of such a plane contains at most $1 + d - r$ points. The set $\Lambda_A^{-1}(\mathcal{F}_A)$ is also dense and G_δ in $C(K, l_2)$. Hence there exists a map $g \in \Lambda_A^{-1}\bigl(\mathcal{F}_A\bigr)$ that is $\lambda/2$-close to h. Then $\overline{g} = g \circ \varphi$ is λ-close to g_0. It remains to show that $\overline{g} \in \mathcal{F}_\varepsilon(M, l_2)$. To this end consider $\Pi^d \in \mathcal{Q}$. Since $\Pi^d_A = \pi_A(\Pi^d)$ is a d-plane in $\mathbb{R}^A$ parallel to the r-plane of the first r coordinates in $\mathbb{R}^A$, the set $g^{-1}\bigl(\pi_A^{-1}(\Pi^d_A)\bigr)$ contains $\leqslant 1 + d - r$ points. The plane Π^d_A can formally have dimension $< d$, but in this case the estimate on the cardinality of the inverse image is even better. It follows from the inclusion $\Pi^d \subset \pi_A^{-1}(\Pi^d_A)$ that the set $g^{-1}(\Pi^d)$ contains $\leqslant 1 + d - r$ points. Hence $(\overline{g})^{-1}(\Pi^d) = \varphi^{-1}\bigl(g^{-1}(\Pi^d)\bigr)$ consists of $\leqslant 1 + d - r$ fibres of φ. Since each fibre of φ has diameter $< \varepsilon$, it follows that $\overline{g} \in \mathcal{F}_\varepsilon(M, l_2)$.

Assume now that $d \leqslant r$. We apply Theorem 1.5 to A again, but now for $m = |A|$, $t = 0$, and $T = r$. Obviously, in this case $(m-n-T)(d-t) > n$. Hence there exists a dense G_δ-subset $\mathcal{F}_A$ of $C(K, \mathbb{R}^A)$ consisting of maps g such that $g^{-1}(\Pi^d_A)$ contains at most one point for each d-plane $\mathbb{R}^A$ parallel to the plane of the first r coordinate variables in $\mathbb{R}^A$. As before, we take a map $g \in \Lambda_A^{-1}\bigl(\mathcal{F}_A\bigr)$ that is $\lambda/2$-close to h and show that $\overline{g} = g \circ \varphi \in \mathcal{F}_\varepsilon(M, l_2)$.

Proposition 4.8. *Each set $\mathcal{F}_\varepsilon$ is dense in $C^*(X, l_2)$.*

Proof. We follow the proof of Proposition 3.5 and describe all the necessary changes. In our case $\psi_\varepsilon \colon Y \to C^*(X, l_2)$ is defined by the formula $\psi_\varepsilon(y) = C^*(X, l_2) \setminus \mathcal{F}_\varepsilon(y)$, where $\mathcal{F}_\varepsilon(y)$ is the set of all maps $g \in C^*(X, l_2)$ such that $f^{-1}(y)$ has cotype (q, ε, g) with $q = 1 + d - r$ for $r \leqslant d$ and $q = 1$ otherwise. To show that ψ_ε has a closed graph we apply to the space $C^*(X, l_2)$ an analogue of Corollary 4.2 in place of Corollary 3.2. We also require the following result.

Claim. *Let $g_0 \in C^*(X, l_2)$, $\alpha \colon X \to (0, \infty)$, and $y \in Y$. Then for each $\varepsilon > 0$, $\psi_\varepsilon(y) \cap \overline{B}(g_0, \alpha)$ is a Z-set in $\overline{B}(g_0, \alpha)$ regarded as a subset of $C^*(X, l_2)$ with the topology of uniform convergence.*

Proof. The proof follows in part the arguments from the proof of Lemma 8 in [13]. We must show that each map $h \colon Q \to \overline{B}(g_0, \alpha)$, where Q is the Hilbert cube, can be approximated by a map $h_1 \colon Q \to \overline{B}(g_0, \alpha)$ avoiding the set $\psi_\varepsilon(y) \cap \overline{B}(g_0, \alpha)$. Thus, we fix a map h and $\eta > 0$. Then h induces a map $u \colon Q \times X \to l_2$, $u(z, x) = h(z)(x)$, such that $\operatorname{dist}(u(z, x), g_0(x)) \leqslant \alpha(x)$ for each point (z, x) in $Q \times X$. We choose $\lambda \in (0, 1)$ such that $\lambda \sup\{\alpha(x) : x \in f^{-1}(y)\} < \eta/2$, and define $u_1 \in C(Q \times f^{-1}(y), l_2)$ by the formula $u_1(z, x) = (1 - \lambda)u(z, x) + \lambda g_0(x)$. For each $(z, x) \in Q \times f^{-1}(y)$ we have

$$\operatorname{dist}(u_1(z, x), g_0(x)) < \alpha(x) \quad \text{and} \quad \operatorname{dist}(u_1(z, x), u(z, x)) < \frac{\eta}{2}.$$

Assume that $\delta < \inf\{\eta/2, \alpha(x) - \operatorname{dist}(u_1(z, x), g_0(x)) : (z, x) \in Q \times f^{-1}(y)\}$. Then there exists by Lemma 4.7 a map $u_2 \colon Q \times f^{-1}(y) \to l_2$ such that $Q \times f^{-1}(y)$ has cotype (q, ε, u_2) and $\operatorname{dist}(u_1(z, x), u_2(z, x)) < \delta$ for each $(z, x) \in Q \times f^{-1}(y)$. Then for $(z, x) \in Q \times f^{-1}(y)$ we have

$$\operatorname{dist}(u_2(z, x), g_0(x)) < \alpha(x) \quad \text{and} \quad \operatorname{dist}(u_2(z, x), u(z, x)) < \eta.$$

The equality $h_2(z)(x) = u_2(z, x)$ defines a map $h_2 \colon Q \to C(f^{-1}(y), l_2)$ with $f^{-1}(y)$ of cotype $(q, \varepsilon, h_2(z))$ for each $z \in Q$. One can show that the projection map $\operatorname{pr} \colon \overline{B}(g_0, \alpha) \to C(f^{-1}(y), l_2)$, $\operatorname{pr}(g) = g|f^{-1}(y)$, is open in the topology of uniform convergence and $\operatorname{pr}\bigl(\overline{B}(g_0, \alpha)\bigr)$ contains $h_2(Q)$. Hence h_2 can be lifted to a map $w \colon Q \to \overline{B}(g_0, \alpha)$ such that w is η-close to h. We observe that w avoids the set $\psi_\varepsilon(y) \cap \overline{B}(g_0, \alpha)$ because $f^{-1}(y)$ is of cotype $(q, \varepsilon, h_2(z))$ for all $z \in Q$. This completes the proof of the claim.

We now return to the proof of Proposition 4.8. For fixed $g \in C^*(X, l_2)$ and a function $\alpha \colon X \to (0, 1)$ consider the set-valued map $\phi \colon Y \to C^*(X, l_2)$, $\phi(y) = \overline{B}(g, \alpha)$, where $C^*(X, l_2)$ carries the topology of uniform convergence. In accordance with the above claim, $\phi(y) \cap \psi_\varepsilon(y)$ is a Z-set in $\phi(y)$ for all $y \in Y$. Moreover, Y is a C-space, therefore we can apply Theorem 1.1 of [14] to find a map $\theta \colon Y \to C^*(X, l_2)$ such that $\theta(y) \in \overline{B}(g, \alpha) \setminus \psi_\varepsilon(y)$ for all $y \in Y$. Finally, we define a map $\overline{g} \in C^*(X, l_2)$ by the formula $\overline{g}(x) = \theta(f(x))$, $x \in X$. Then $\overline{g} \in \overline{B}(g, \alpha) \cap \mathcal{F}_\varepsilon(y)$ for each $y \in Y$. Hence $\overline{g} \in \overline{B}(g, \alpha) \cap \mathcal{F}_\varepsilon$, which completes the proof of Proposition 4.8.

The proof of Theorem 1.8 is now also complete.

§ 5. Appendix

We present several further applications of our results.

Theorem 5.1. *Let* $f\colon X\to Y$ *be a perfect map between paracompact spaces and let* $\dim Y=0$*. Let* $\{F_i\}$ *be a sequence of closed subsets of* X *and* $\{n_i\}$ *a sequence of integers such that* $\dim f|F_i\leqslant n_i$ *for all* i*. If* $n\geqslant n_i$, $i=1,2,\dots$*, then for each* $m\geqslant n+1$ *the space* $C^*(X,\mathbb{R}^m)$ *contains a dense* G_δ*-subset of maps* g *such that for each* i *and each* d*-plane* $\Pi^d\subset\mathbb{R}^m$*, where* $m-n_i\leqslant d\leqslant m$*, the inequality*

$$\dim g\bigl(f^{-1}(y)\cap F_i\bigr)\cap\Pi^d\leqslant n_i+d-m$$

holds for all points $y\in Y$*.*

Proof. The proof of Theorem 5.1 is based on Lemma 5.2 below. Indeed, for each i we apply Theorem 1.3 to the spaces F_i, $f(F_i)$, and the map $f|F_i$ to conclude that $C^*(F_i,\mathbb{R}^m)$ contains a dense G_δ-subset $\mathcal{H}_i$ of maps g such that for each $y\in f(F_i)$ and each d-plane $\Pi^d\subset\mathbb{R}^m$ with $m-n_i\leqslant d\leqslant m$ we have $\dim g\bigl(f^{-1}(y)\cap F_i\bigr)\cap\Pi^d\leqslant n_i+d-m$. Since each restriction map

$$\pi_i\colon C^*(X,\mathbb{R}^m)\to C^*(F_i,\mathbb{R}^m)$$

is open by Lemma 5.2, $\mathcal{K}_i=\pi_i^{-1}(\mathcal{H}_i)$ is a dense G_δ-subset of $C^*(X,\mathbb{R}^m)$. Then the intersection of all the $\mathcal{K}_i$ satisfies the requirements of Theorem 5.1.

Lemma 5.2. *Let* F *be a closed subset of the normal space* X *and assume that* $m\geqslant 1$*. Then the restriction map* $\pi\colon C^*(X,\mathbb{R}^m)\to C^*(F,\mathbb{R}^m)$ *defined by the formula* $\pi(g)=g|F$ *is open if* $C^*(X,\mathbb{R}^m)$ *and* $C^*(F,\mathbb{R}^m)$ *are both equipped with the source limitation topology or with the topology of uniform convergence.*

The next consequence of Theorem 5.1 can be established in the same way that Corollary 1.4 was deduced from Theorem 1.3.

Corollary 5.3. *Let* $\{F_i\}$ *be a sequence of closed subsets of the normal space* X *with* $\dim F_i\leqslant n_i$*. If* $m\geqslant n_i+1$ *for each* i*, then* $C^*(X,\mathbb{R}^m)$ *equipped with the topology of uniform convergence contains a dense* G_δ*-subset of maps* g *such that* $\overline{g(F_i)}\cap\Pi^d$ *is at most* (n_i+d-m)*-dimensional for each* d*-plane* $\Pi^d\subset\mathbb{R}^m$ *with* $m-n_i\leqslant d\leqslant m$, $i=1,2,\dots$*.*

Our final application is an analogue of Fox's theorem [15] on the economical extension of maps. If A is a closed subset of a space X and $h\in C^*(A,\mathbb{R}^m)$, then we denote by $C^*_h(X,\mathbb{R}^m)$ the set of maps $g\in C^*(X,\mathbb{R}^m)$ such that $g|A=h$. Throughout what follows we regard $C^*_h(X,\mathbb{R}^m)$ as a subspace of $C^*(X,\mathbb{R}^m)$ with the topology of uniform convergence.

Corollary 5.4. *Let* X *be a normal space and let* A *be a closed* G_δ*-subset of* X *with* $\dim(X\setminus A)\leqslant n$*. Then for each* $m\geqslant n+1$ *and* $h\in C^*(A,\mathbb{R}^m)$ *there exists a dense* G_δ*-subset of* $C^*_h(X,\mathbb{R}^m)$ *consisting of maps* g *such that the set* $g(X\setminus A)\cap\Pi^d$ *is at most* $(n+d-m)$*-dimensional for each* d*-plane* $\Pi^d\subset\mathbb{R}^m$ *with* $m-n\leqslant d\leqslant m$*.*

Proof. Let $\{F_i\}$ be a sequence of closed subsets of X such that $X\setminus A=\bigcup_{i=1}^{\infty}F_i$ and $h\in C^*(A,\mathbb{R}^m)$. By Corollary 1.4 each $C^*(F_i,\mathbb{R}^m)$ contains a dense G_δ-subset $\mathcal{H}_i$

of maps g such that

$$\dim \overline{g(F_i)} \cap \Pi^d \leqslant n+d-m \quad \text{for each } d\text{-plane} \quad \Pi^d \subset \mathbb{R}^m \text{ for } m-n \leqslant d \leqslant m. \tag{9}$$

Using the fact that F_i and A are disjoint closed subsets of X one can show that each projection $p_i\colon C^*_h(X,\mathbb{R}^m) \to C^*(F_i,\mathbb{R}^m)$, $p_i(g) = g|F_i$, is an open surjective map if both spaces $C^*_h(X,\mathbb{R}^m)$ and $C^*(F_i,\mathbb{R}^m)$ are equipped with the topology of uniform convergence. Hence $\mathcal{K}_i = p_i^{-1}(\mathcal{H}_i)$ is a dense G_δ-subset of $C^*_h(X,\mathbb{R}^m)$. Since $C^*_h(X,\mathbb{R}^m)$ has the Baire property, $\mathcal{K} = \bigcap_{i=1}^\infty \mathcal{K}_i$ is a dense G_δ-subset of $C^*_h(X,\mathbb{R}^m)$. We now observe that condition (9) means that $\dim g(X \setminus A) \cap \Pi^d \leqslant n+d-m$ for each $g \in \mathcal{K}$ and each d-plane $\Pi^d \subset \mathbb{R}^m$ with $m-n \leqslant d \leqslant m$.

Finally, we present several conjectures and observations.

Conjecture 1. *Let $f\colon X \to Y$ be a map of finite-dimensional metrizable compacta. Then the space $C(X,\mathbb{R}^m)$ contains a dense G_δ-subset of maps φ such that for arbitrary integers d, t, T, $0 \leqslant t \leqslant d \leqslant T \leqslant m$ and $\dim f + d + 1 \leqslant m$, and for each d-plane and $\Pi^d \subset \mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T$ in $\mathbb{R}^m$ each set $f^{-1}(y) \cap \varphi^{-1}(\Pi^d)$, $y \in Y$, contains at most*

$$1 + \frac{\dim Y + \dim f + (T-d)(d-t)}{m - \dim f - d}$$

points.

We point out that Theorem 1.5 yields Conjecture 1 in the case when Y is 0-dimensional.

Conjecture 2. *Let $f\colon X \to Y$ be a map of finite-dimensional compacta. Then $C(X,\mathbb{R}^m)$ contains a dense G_δ-subset of maps φ such that*

$$\dim\bigl(\varphi(f^{-1}(y)) \cap \Pi^d\bigr) \leqslant \dim f + d - m$$

for each d-plane $\Pi^d \subset \mathbb{R}^m$ such that $m - \dim f \leqslant d \leqslant m$ and each $y \in Y$.

Theorem 1.3 yields Conjecture 2 in the case when Y is 0-dimensional. As in Theorem 1.3, it is sufficient to establish Conjecture 2 in the special case $d = m - \dim f$. Conjecture 2 holds for $\dim f = 0$: this is Uspenskij's theorem on light mappings [16]. The case of a perfect map f between paracompact spaces with $\dim f > 0$ is considered in the Tuncali–Valov theorem [17].

Let $f\colon X \to Y$, $\varphi \in C(X,\mathbb{R}^m)$ and let t, d, and T be integers such that $0 \leqslant t \leqslant d \leqslant T \leqslant m$ and $d - t + 1 \leqslant q$. We consider below the set $B^f_{q,d,t,T}(\varphi)$ consisting of all points $(y, y_1, \dots, y_q) \in Y \times (\mathbb{R}^m)^q$ satisfying the following condition: there exist points $x_1, \dots, x_q \in f^{-1}(y)$, $x_i \neq x_j$ for $i \neq j$, such that $y_i = \varphi(x_i)$ and all the y_i, $i = 1, \dots, q$, belong to a d-plane in $\mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$.

Conjecture 3. *Let $f\colon X \to Y$ be a map of metrizable finite-dimensional compacta. Then $C(X,\mathbb{R}^m)$ contains a dense G_δ-subset $\mathcal{H}$ of maps φ such that*

$$\dim B^f_{q,d,t,T}(\varphi) \leqslant \dim Y + \dim f + (T-d)(d-t) - (q-1)(m - \dim f - d)$$

for all integers d, t, T, q such that $0 \leqslant t \leqslant d \leqslant T \leqslant m$, $\dim f + d + 1 \leqslant m$, and $d - t + 1 \leqslant q$.

If the right-hand side of the inequality in Conjecture 3 is $\leqslant -1$, then the set $B^f_{q,d,t,T}(\varphi)$ is empty. Conditions ensuring that $B^f_{q,d,t,T}(\varphi)$ is empty are discussed in Conjecture 1. If $d=0$, then the set $B^f_{q,0,0,T}(\varphi)$ is independent of T. In that case it is homeomorphic to the set $B^f_q(\varphi) = \{(y,z) \in Y \times \mathbb{R}^m : |f^{-1}(y) \cap \varphi^{-1}(z)| \geqslant q\}$. For $d=0$ and Y a point Conjecture 3 was established by Hurewicz in 1933 [18] (recall that the maps $\varphi \in \mathcal{H}$ in Hurewicz's theorem are called *regularly branched maps* [19]). A parametric version of Hurewicz's result was obtained in [20], Theorem 1.1. Using the terminology of [20] for $d=0$ the inequality of Conjecture 3 holds for all $q \geqslant 1$ if and only if φ is an f-regularly branched map. By [20], Theorem 1.1 the space $C(X,\mathbb{R}^m)$ contains a dense G_δ-subset of f-regularly branched maps, therefore Conjecture 3 holds for $d=0$. We point out that for $d \geqslant 1$, that is, in the non-parametric version, the conjecture is open even in the case when Y is a point.

For maps $f_i\colon X_i \to \mathbb{R}^m$, $i=1,\dots,q$, and integers t, d, T, m, $0 \leqslant t \leqslant d \leqslant T \leqslant m$, let $B_{d,t,T}(f_1,\dots,f_q)$ be the set of points $(y_1,\dots,y_q) \in (\mathbb{R}^m)^q$ such that the points $y_i = f(x_i)$, $i=1,\dots,q$, belong to a d-plane in $\mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T$. Also let

$$C_{d,t,T}(f_1,\dots,f_q) = \bigl\{(x_1,\dots,x_q) \in X_1 \times \dots \times X_q : \bigl(f(x_1),\dots,f(x_q)\bigr) \in B_{d,t,T}(f_1,\dots,f_q)\bigr\}.$$

Conjecture 4. *Let $n_1,\dots,n_q$, m, d, t, T be integers satisfying the inequalities $0 \leqslant t \leqslant d \leqslant T \leqslant m$, $0 \leqslant n_1,\dots,0 \leqslant n_q$, $n_1+1+d \leqslant m,\dots,n_q+1+d \leqslant m$, $d-t+1 \leqslant q$, and*

$$n_1+\dots+n_q \geqslant (m-d)(q-1)-(T-d)(d-t).$$

Then there exist $\varepsilon>0$ and maps $f_i\colon \Delta^{n_i} \to \mathbb{R}^m$, $i=1,\dots,q$, such that the set $B_{d,t,T}(g_1,\dots,g_q)$ is non-empty for arbitrary maps $g_i\colon \Delta^{n_i} \to \mathbb{R}^m$ such that g_i and f_i are ε-close for all $i=1,\dots,q$.

In the simplest case ($q=1$, $n_1=0$, $n_2=m$, $d=0$) the statement of Conjecture 4 is precisely a theorem of Alexandroff that the identity map of a ball onto itself is essential. For $d=0$ Conjecture 4 holds in the general case (see, for instance, [21], Corollary 3, and [22], Lemma 6.3 on p. 65). In the case $d=q-1$, $t=0$, and $T=m$ Conjecture 4 was proved by Boltyanskiĭ ([9], Lemma 9; the main ingredient of his proof is the following result: if for some n-dimensional polyhedron X the set of k-regular maps from X into $\mathbb{R}^m$ is dense in $C(X,\mathbb{R}^m)$, then $m \geqslant nk+n+k$).

Conjecture 5. *Let $n_1,\dots,n_q$, m, d, t, T be integers satisfying the inequalities $0 \leqslant t \leqslant d \leqslant T \leqslant m$, $0 \leqslant n_1,\dots,0 \leqslant n_q$, $n_1+1+d \leqslant m,\dots,n_q+1+d \leqslant m$, $d-t+1 \leqslant q$. Then there exist $\varepsilon>0$ and maps $f_i\colon \Delta^{n_i} \to \mathbb{R}^m$, $i=1,\dots,q$, such that*

$$\dim C_{d,t,T}(g_1,\dots,g_q) \geqslant n_1+\dots+n_q-(m-d)(q-1)+(T-d)(d-t)$$

for arbitrary maps $g_i\colon \Delta^{n_i} \to \mathbb{R}^m$ such that g_i are ε-close to f_i for all $i=1,\dots,q$.

For $d=0$ Conjecture 5 was established in [21], Corollary 3.

In conclusion we consider the following problem, in which Δ^N_n is the n-dimensional skeleton of the N-dimensional simplex Δ^N.

Problem. *Find all integers* n, m, q, d, t, T *such that* $0 \leqslant n$, $0 \leqslant t \leqslant d \leqslant T \leqslant m$, $n+1+d \leqslant m$, $d-t+1 \leqslant q$,

$$n \geqslant (m-n-d)(q-1) - (T-d)(d-t), \qquad (n, m, q, d, t, T)$$

and the following condition holds: *there exists a positive integer* N *such that for each map* $f\colon \Delta_n^N \to \mathbb{R}^m$ *one can find disjoint simplexes* $\sigma_1, \dots, \sigma_q \subset \Delta_n^N$ *with* f*-images intersecting a* d*-plane* $\Pi^d \subset \mathbb{R}^m$ *parallel to some coordinate planes* $\Pi^t \subset \Pi^T$.

We observe that if this problem has an affirmative solution for some integers n, m, q, d, t, T satisfying the assumptions of the problem, then Corollary 1.6 cannot be improved not just on the level of a dense set of maps, but also on the level of the *existence* of one map with inverse images of small cardinality, even in the class of polyhedra. Most results on the existence of such N were established for $d=0$: van Kampen and Flores [23], [24] ($q=2$ and $N=2n+2$), Sarkaria [25] (q a prime and $N=qn+2q-2$), Volovikov [26] (q a power of a prime and $N=qn+2q-2$), Bogatyĭ [21], Corollary 11 ($q=n+1$ and $N \leqslant 2n^2+5n$). Another result of Bogatyĭ [3], [27] yields such N for $d=q-1$, $t=0$, $T=m$ with odd q. Živalević [28] obtained a result on embeddings of the bichromatic graph $K_{6,6}$ in $\mathbb{R}^3$, which yields a positive solution of the problem for $n=1$, $m=3$, $t=0$, $T=3$, $q=4$, and $N=11$.

If $d=q-1$, $q=2$, $t=0$, and $T=m$ then there exists no integer N satisfying the assumptions of the above problem (see [9]).

One can state the above problem in a slightly more general form: *for a fixed family of integers* n, q, d, t, T *find the largest integer* m *such that such* N *exists.* Then the *problem of finding the smallest number* N *with this property* arises. A more complicated question is *the description in this case of such minimal subpolyhedra of* Δ_n^N. In the case $n=1$, $d=0$, $q=2$, $m=2$ Kuratowski graphs: the complete graph K_5 and the complete bichromatic graph $K_{3,3}$, are minimal subpolyhedra of this kind.

We point out that there exists a close connection between this problem and conjectures about various forms of Tverberg's theorem [3], [28], [29]. It is also well known that k-regular maps are closely related to interpolation and approximation problems [30], [31]. In this connection it would be important to *find applications of the maps described by Theorems* 1.5 *and* 1.8 *in interpolation and approximation problems.*

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