



Extraordinary dimension of maps

Alex Chigogidze^{a,*}, Vesko Valov^{b,1}

^a Department of Mathematical Sciences, University of North Carolina at Greensboro,
PO Box 26170, Greensboro, NC 27402-6170, USA

^b Department of Mathematics, Nipissing University,
100 College Drive, PO Box 5002, North Bay, ON P1B 8L7, Canada

Received 21 January 2004; accepted 6 April 2004

Abstract

We consider the extraordinary dimension \dim_L introduced recently by Shchepin [E.V. Shchepin, Arithmetic of dimension theory, Russian Math. Surveys 53 (5) (1998) 975–1069]. If L is a CW-complex and X a metrizable space, then $\dim_L X$ is the smallest number n such that $\Sigma^n L$ is an absolute extensor for X , where $\Sigma^n L$ is the n th suspension of L . We also write $\dim_L f \leq n$, where $f: X \rightarrow Y$ is a given map, provided $\dim_L f^{-1}(y) \leq n$ for every $y \in Y$. The following result is established:

Suppose $f: X \rightarrow Y$ is a perfect surjection between metrizable spaces, Y a C -space and L a countable CW-complex. Then conditions (1)–(3) below are equivalent:

- (1) $\dim_L f \leq n$;
- (2) There exists a dense and G_δ subset \mathcal{G} of $C(X, \mathbb{I}^n)$ with the source limitation topology such that $\dim_L(f \times g) = 0$ for every $g \in \mathcal{G}$;
- (3) There exists a map $g: X \rightarrow \mathbb{I}^n$ is such that $\dim_L(f \times g) = 0$;
If, in addition, X is compact, then each of the above three conditions is equivalent to the following one;
- (4) There exists an F_σ set $A \subset X$ such that $\dim_L A \leq n - 1$ and the restriction map $f|(X \setminus A)$ is of dimension $\dim f|(X \setminus A) \leq 0$.

© 2005 Elsevier B.V. All rights reserved.

* Corresponding author.

E-mail addresses: chigogidze@uncg.edu (A. Chigogidze), veskov@nipissingu.ca (V. Valov).

¹ The second named author was partially supported by his NSERC grant No 261914-03.

MSC: primary 54F45; secondary 55M10, 54C65

Keywords: Extensional dimension; Extraordinary dimension; C-space

1. Introduction

The paper deals with extensional dimension of maps, specially, with the extraordinary dimension introduced recently by Shchepin [10] and considered in more details by the first author in [1]. If L is a CW-complex and X a metrizable space, we write $\text{e-dim } X \leq L$ provided L is an absolute extensor for X (in such a case we say that the extensional dimension of X is $\leq L$, see [3,4]). The extraordinary dimension of X generated by a complex L , notation $\text{dim}_L X$, is the smallest integer n such that $\text{e-dim } X \leq \Sigma^n L$, where $\Sigma^n L$ is the n th iterated suspension of L (by $\Sigma^0 L$ we always mean L). If L is the 0-dimensional sphere S^0 , then dim_L coincides with the covering dimension dim . As usual, $\text{dim}_L f \leq n$, where $f: X \rightarrow Y$ is a map, means $\text{dim}_L f^{-1}(y) \leq n$ for every $y \in Y$. Next is our main result.

Theorem 1.1. *Let $f: X \rightarrow Y$ be a σ -perfect map of metrizable spaces and let L be a CW-complex. Consider the following properties:*

- (1) $\text{dim}_L f \leq n$;
- (2) *There exists an F_σ subset A of X such that $\text{dim}_L A \leq n - 1$ and the restriction map $f|(X \setminus A)$ is of dimension $\text{dim}_L f|(X \setminus A) = 0$;*
- (3) *There exists a dense and G_δ subset \mathcal{G} of $C(X, \mathbb{I}^n)$ with the source limitation topology such that $\text{dim}_L(f \times g) = 0$ for every $g \in \mathcal{G}$;*
- (3') *There exists a map $g: X \rightarrow \mathbb{I}^n$ is such that $\text{dim}_L(f \times g) = 0$.*

Then (3) \Rightarrow (3') \Rightarrow (1) and (3') \Rightarrow (2). Moreover, (1) \Rightarrow (3) provided Y is a C-space and L is countable.

Here, $f: X \rightarrow Y$ is σ -perfect if X is the union of countably many closed sets X_i such that $f(X_i) \subset Y$ are closed and the restriction maps $f|X_i$ are perfect.

Theorem 1.1 is inspired by the following result of Levin and Lewis [7, Theorem 1.8]: If X and Y are metrizable compacta then (3) \Rightarrow (3') \Rightarrow (1) and (3) \Rightarrow (2') \Rightarrow (1), where (2') is obtained from our condition (2) by replacing $\text{dim}_L f|(X \setminus A) = 0$ with $\text{dim } f|(X \setminus A) = 0$. Moreover, the implication (1) \Rightarrow (3) was also established in [7] for a finite-dimensional compactum Y and a countable CW-complex L .

Therefore, we have the following characterization of extraordinary dimension of perfect maps between metrizable spaces:

Corollary 1.2. *Let $f: X \rightarrow Y$ be a perfect surjection between metrizable spaces with Y being a C-space. If L is a countable CW-complex, then the following conditions are equivalent:*

- (1) $\text{dim}_L f \leq n$;

- (2) There exists a dense and G_δ subset \mathcal{G} of $C(X, \mathbb{I}^n)$ with the source limitation topology such that $\dim_L(f \times g) = 0$ for every $g \in \mathcal{G}$;
 (3) There exists a map $g: X \rightarrow \mathbb{I}^n$ is such that $\dim_L(f \times g) = 0$.

If, in addition, X is compact, then each of the above three conditions is equivalent to the following one:

- (4) There exists an F_σ set $A \subset X$ such that $\dim_L A \leq n - 1$ and the restriction map $f|(X \setminus A)$ is of dimension $\dim f|(X \setminus A) \leq 0$.

The equivalence of the first three conditions follows from Theorem 1.1. More precisely, by Theorem 1.1 we have the following implications: (2) \Rightarrow (3) \Rightarrow (1) \Rightarrow (2). When X is compact, the result of Levin–Lewis which was mentioned above yields that (2) \Rightarrow (4) \Rightarrow (1). Therefore, combining the last two chains of implications, we can obtain the compact version of Corollary 1.2.

Corollary 1.2 is a parametric version of [1, Theorem 4.9]. For the covering dimension \dim such a characterization was obtained by Pasyukov [9] and Toruńczyk [11] in the realm of finite-dimensional compact metric spaces and extended in [12] to perfect maps between metrizable C -spaces. Since the class of C -spaces contains the finite-dimensional ones as a proper subclass (see [5]), the compact version of Corollary 1.2 is more general than the Levin–Lewis result [7, Theorem 1.8]. It is interesting if all conditions (1)–(4) in Corollary 1.2 remain equivalent without the requirement of X and Y being compact.

The source limitation topology on $C(X, M)$, where (M, d) is a metric space, can be described as follows: a subset $U \subset C(X, M)$ is open if for every $g \in U$ there exists a continuous function $\alpha: X \rightarrow (0, \infty)$ such that $\bar{B}(g, \alpha) \subset U$. Here, $\bar{B}(g, \alpha)$ denotes the set $\{h \in C(X, M): d(g(x), h(x)) \leq \alpha(x) \text{ for each } x \in X\}$. The source limitation topology does not depend on the metric d if X is paracompact and $C(X, M)$ with this topology has the Baire property provided (M, d) is a complete metric space. Moreover, if X is compact, then the source limitation topology coincides with the uniform convergence topology generated by d .

All function spaces in this paper, if not explicitly stated otherwise, are equipped with the source limitation topology.

2. Some preliminary results

Throughout this section K is a closed and convex subset of a given Banach space E and $f: X \rightarrow Y$ a perfect map with X and Y paracompact spaces. Suppose for every $y \in Y$ we are given a property $\mathcal{P}(y)$ of maps $h: f^{-1}(y) \rightarrow K$ and let $\mathcal{P} = \{\mathcal{P}(y): y \in Y\}$. By $C_{\mathcal{P}}(X|H, K)$ we denote the set of all bounded maps $g: X \rightarrow K$ such that $g|f^{-1}(y)$ has the property $\mathcal{P}(y)$ for every $y \in H$, where $H \subset Y$. We also consider the set-valued map $\psi_{\mathcal{P}}: Y \rightarrow 2^{C^*(X, K)}$, defined by the formula $\psi_{\mathcal{P}}(y) = C^*(X, K) \setminus C_{\mathcal{P}}(X|\{y\}, K)$, where $C^*(X, K)$ is the space of bounded maps from X into K .

Lemma 2.1. *Suppose that $\mathcal{P} = \{\mathcal{P}(y)\}_{y \in Y}$ is a family of properties satisfying the following conditions:*

- (a) $C_{\mathcal{P}}(X|H, K)$ is open in $C^*(X, K)$ with respect to the source limitation topology for every closed $H \subset Y$;
- (b) $g \in C_{\mathcal{P}}(X|\{y\}, K)$ implies $g \in C_{\mathcal{P}}(X|U, K)$ for some neighborhood U of y in Y .

Then the map $\psi_{\mathcal{P}}$ has a closed graph provided $C^*(X, K)$ is equipped with the uniform convergence topology.

Proof. The proof of this lemma follows the arguments from the proof of [12, Lemma 2.6]. \square

Recall that a closed subset F of the metrizable space M is said to be a Z_m -set in M , if the set $C(\mathbb{I}^m, M \setminus F)$ is dense in $C(\mathbb{I}^m, M)$ with respect to the uniform convergence topology, where \mathbb{I}^m is the m -dimensional cube. If F is a Z_m -set in M for every $m \in \mathbb{N}$, we say that F is a Z -set in M .

Lemma 2.2. Suppose $y \in Y$ and $\mathcal{P}(y)$ satisfy the following condition:

- For every $m \in \mathbb{N}$ the set of all maps $h \in C(\mathbb{I}^m \times f^{-1}(y), K)$ with each $h|(\{z\} \times f^{-1}(y))$, $z \in \mathbb{I}^m$, having the property $\mathcal{P}(y)$ (as a map from $f^{-1}(y)$ into K) is dense in $C(\mathbb{I}^m \times f^{-1}(y), K)$ with respect to the uniform convergence topology.

Then, for every $\alpha : X \rightarrow (0, \infty)$ and $g \in C^*(X, K)$, $\psi_{\mathcal{P}}(y) \cap \overline{B}(g, \alpha)$ is a Z -set in $\overline{B}(g, \alpha)$ provided $\overline{B}(g, \alpha)$ is considered as a subset of $C^*(X, K)$ equipped with the uniform convergence topology and $\psi_{\mathcal{P}}(y) \subset C^*(X, K)$ is closed.

Proof. See the proof of [12, Lemma 2.8]. \square

Proposition 2.3. Let Y be a C -space and $\mathcal{P} = \{\mathcal{P}(y)\}_{y \in Y}$ such that:

- (a) the map $\psi_{\mathcal{P}}$ has a closed graph;
- (b) $\psi_{\mathcal{P}}(y) \cap \overline{B}(g, \alpha)$ is a Z -set in $\overline{B}(g, \alpha)$ for any continuous function $\alpha : X \rightarrow (0, \infty)$, $y \in Y$ and $g \in C^*(X, K)$, where $\overline{B}(g, \alpha)$ is considered as a subspace of $C^*(X, K)$ with the uniform convergence topology.

Then the set $\{g \in C^*(X, K) : g \in C_{\mathcal{P}}(X|\{y\}, K) \text{ for every } y \in Y\}$ is dense in $C^*(X, K)$ with respect to the source limitation topology.

Proof. Let $G = \{g \in C^*(X, K) : g \in C_{\mathcal{P}}(X|\{y\}, K) \text{ for every } y \in Y\}$. It suffices to show that, for fixed $g_0 \in C^*(X, K)$ and a positive continuous function $\alpha : X \rightarrow (0, \infty)$, there exists $g \in \overline{B}(g_0, \alpha) \cap G$. We equip $C^*(X, K)$ with the uniform convergence topology and consider the constant (and hence, lower semi-continuous) convex-valued map $\phi : Y \rightarrow 2^{C^*(X, K)}$, $\phi(y) = \overline{B}(g_0, \alpha_1)$, where $\alpha_1(x) = \min\{\alpha(x), 1\}$. Because of the conditions (a) and (b), we can apply the selection theorem [6, Theorem 1.1] to obtain a continuous map $h : Y \rightarrow C^*(X, K)$ such that $h(y) \in \phi(y) \setminus \psi_{\mathcal{P}}(y)$ for every $y \in Y$. Observe that h is a map from Y into $\overline{B}(g_0, \alpha_1)$ such that $h(y) \in C_{\mathcal{P}}(X|\{y\}, K)$ for every

$y \in Y$. Then $g(x) = h(f(x))(x)$, $x \in X$, defines a bounded map $g \in \overline{B}(g_0, \alpha)$ such that $g|_{f^{-1}(y)} = h(y)|_{f^{-1}(y)}$, $y \in Y$. Therefore, $g \in C\mathcal{P}(X|\{y\}, K)$ for all $y \in Y$, i.e., $g \in \overline{B}(g_0, \alpha) \cap G$. \square

3. Proof of Theorem 1.1

(1) \Rightarrow (3) Suppose L is countable and Y is a C -space. Let X_i be closed subsets of X such that each $f_i = f|_{X_i} : X_i \rightarrow Y_i = f(X_i)$ is a perfect map and $Y_i \subset Y$ closed. Then all Y_i are C -spaces, and since the restriction maps $\pi_i : C(X, \mathbb{I}^n) \rightarrow C(X_i, \mathbb{I}^n)$, $\pi_i(g) = g|_{X_i}$, are continuous and open, the proof of this implication is reduced to the case f is a perfect map. So, we assume that f is perfect.

By [13, Theorem 1.1] (see also [8]), there exists a map q from X into the Hilbert cube Q such that $f \times q : X \rightarrow Y \times Q$ is an embedding. Let $\{W_i\}_{i \in \mathbb{N}}$ be a countable finitely-additive base for Q . For every i we choose a sequence of mappings $h_{ij} : \overline{W}_i \rightarrow L$, representing all the homotopy classes of mappings from \overline{W}_i to L (this is possible because L is a countable CW-complex). Following the notations from Section 2, for fixed i, j and $y \in Y$ we say that a map $g \in C(X, \mathbb{I}^n)$ has the property $\mathcal{P}_{ij}(y)$ provided the map $h_{ij} \circ q : q^{-1}(\overline{W}_i) \rightarrow L$ can be continuously extended to a map over the set $q^{-1}(\overline{W}_i) \cup (f^{-1}(y) \cap g^{-1}(t))$ for every $t \in g(f^{-1}(y))$.

Let $\mathcal{P}_{ij} = \{\mathcal{P}_{ij}(y) : y \in Y\}$ and for every $H \subset Y$ we denote $C_{\mathcal{P}_{ij}}(X|H, \mathbb{I}^n)$ by $C_{ij}(X|H, \mathbb{I}^n)$. Hence, $C_{ij}(X|H, \mathbb{I}^n)$ consists of all $g \in C(X, \mathbb{I}^n)$ having the property $\mathcal{P}_{ij}(y)$ for every $y \in H$. Let $\psi_{ij} : Y \rightarrow 2^{C(X, \mathbb{I}^n)}$ be the set-valued map $\psi_{ij}(y) = C(X, \mathbb{I}^n) \setminus C_{ij}(X|\{y\}, \mathbb{I}^n)$.

Lemma 3.1. *Let $g \in C_{ij}(X|\{y\}, \mathbb{I}^n)$. Then, there exist a neighborhood U_y of y in Y and a neighborhood $V_t \subset \mathbb{I}^n$ of each $t \in g(f^{-1}(y))$ such that $h_{ij} \circ q$ can be extended to a map from $q^{-1}(\overline{W}_i) \cup (f^{-1}(U_y) \cap g^{-1}(V_t))$ into L .*

Proof. Since $g \in C_{ij}(X|\{y\}, \mathbb{I}^n)$, $h_{ij} \circ q$ can be extended to a map from $q^{-1}(\overline{W}_i) \cup (f^{-1}(y) \cap g^{-1}(t))$ into L for every $t \in g(f^{-1}(y))$. Because L is an absolute neighborhood extensor for X , there exists an open set $G_t \subset X$ containing $f^{-1}(y) \cap g^{-1}(t)$ and a map $h_t : q^{-1}(\overline{W}_i) \cup G_t \rightarrow L$ extending $h_{ij} \circ q$. Using that $f \times g$ is a closed map, we can find a neighborhood $U_y^t \times V_t$ of (y, t) in $Y \times \mathbb{I}^n$ such that $S_t = (f \times g)^{-1}(U_y^t \times V_t) \subset G_t$. Next, choose finitely many points $t(k)$, $k = 1, 2, \dots, m$, with $f^{-1}(y) \subset \bigcup_{k=1}^{k=m} S_{t(k)}$ and a neighborhood U_y of y in Y such that $U_y \subset \bigcap_{k=1}^{k=m} U_y^{t(k)}$ and $f^{-1}(U_y) \subset \bigcup_{k=1}^{k=m} S_{t(k)}$ (this can be done since f is perfect). If $t \in g(f^{-1}(y))$, then $t \in V_{t(k)}$ for some k and $f^{-1}(U_y) \cap g^{-1}(V_{t(k)}) \subset S_{t(k)}$. Since, $S_{t(k)} \subset G_{t(k)}$, the map $h_{t(k)}$ is an extension of $h_{ij} \circ q$ over the set $q^{-1}(\overline{W}_i) \cup (f^{-1}(U_y) \cap g^{-1}(V_{t(k)}))$. \square

Lemma 3.2. *The set $C_{ij}(X|H, \mathbb{I}^n)$ is open in $C(X, \mathbb{I}^n)$ for any i, j and closed $H \subset Y$.*

Proof. We follow the proof of [12, Lemma 2.5]. For a fixed $g_0 \in C_{ij}(X|H, \mathbb{I}^n)$ we are going to find a function $\alpha : X \rightarrow (0, \infty)$ such that $\overline{B}(g_0, \alpha) \subset C_{ij}(X|H, \mathbb{I}^n)$. By Lemma 3.1,

for every $z = (y, t) \in (f \times g_0)(f^{-1}(H))$ there exists a neighborhood U_z in $Y \times \mathbb{I}^n$ such that

(1) $h_{ij} \circ q$ can be extended to a map from $q^{-1}(\overline{W}_i) \cup (f \times g_0)^{-1}(U_z)$ into L .

Obviously, $K = (f \times g_0)(f^{-1}(H))$ is closed in $Y \times \mathbb{I}^n$, so there exists open $G \subset Y \times \mathbb{I}^n$ with $K \subset G \subset \overline{G} \subset U = \bigcup \{U_z : z \in K\}$. Then $\nu = \{U_z : z \in K\} \cup \{(Y \times \mathbb{I}^n) \setminus \overline{G}\}$ is an open cover of $Y \times \mathbb{I}^n$. Let γ be an open locally finite cover of $Y \times \mathbb{I}^n$ such that the family

(2) $\{St(W, \gamma) : W \in \gamma\}$ refines ν and $St(W, \gamma) \subset G$ provided $W \cap K \neq \emptyset$.

Consider the metric $\rho = d + d_1$ on $Y \times \mathbb{I}^n$, where d is a metric on Y and d_1 the usual metric on \mathbb{I}^n , and define the function $\alpha : X \rightarrow (0, \infty)$ by $\alpha(x) = 2^{-1} \sup\{\rho((f \times g_0)(x), (Y \times \mathbb{I}^n) \setminus W) : W \in \gamma\}$. Let show that $\overline{B}(g_0, \alpha) \subset C_{ij}(X|H, \mathbb{I}^n)$. Take $g \in \overline{B}(g_0, \alpha)$, $y \in H$ and $t \in g(f^{-1}(y))$. Then, $(y, t) = (f \times g)(x)$ for some $x \in f^{-1}(y)$. Since g is α -close to g_0 , there exists $W \in \gamma$ such that $W \cap K \neq \emptyset$ and W contains both $(f \times g)(x)$ and $(f \times g_0)(x)$. It follows from (2) that $(f \times g)^{-1}(W) \subset (f \times g_0)^{-1}(U_z)$ for some $z \in K$. In particular, $f^{-1}(y) \cap g^{-1}(t) \subset (f \times g_0)^{-1}(U_z)$. Consequently, by (1), $h_{ij} \circ q$ is extendable to a map from $q^{-1}(\overline{W}_i) \cup (f^{-1}(y) \cap g^{-1}(t))$ into L . Therefore, $\overline{B}(g_0, \alpha) \subset C_{ij}(X|\{y\}, \mathbb{I}^n)$ for every $y \in H$ which completes the proof. \square

Because of Lemmas 3.1 and 3.2, we can apply Lemma 2.1 to obtain the following corollary.

Corollary 3.3. *For any i and j the map ψ_{ij} has a closed graph.*

Lemma 3.4. *Let $g \in C(X, \mathbb{I}^n)$, $\alpha : X \rightarrow (0, \infty)$ and $y \in Y$. Then, for any i, j , $\psi_{ij}(y) \cap \overline{B}(g, \alpha)$ is a Z -set in $\overline{B}(g, \alpha)$ provided $\overline{B}(g, \alpha)$ is considered as a subset of $C(X, \mathbb{I}^n)$ with the uniform convergence topology.*

Proof. It follows from [7, Theorem 1.8, (1) \Rightarrow (3)] that if $m \in \mathbb{N}$, then all maps $g : \mathbb{I}^m \times f^{-1}(y) \rightarrow \mathbb{I}^n$ such that $\text{e-dim}(\{z\} \times f^{-1}(y) \cap g^{-1}(t)) \leq L$ for every $z \in \mathbb{I}^m$ and $t \in \mathbb{I}^n$, form a dense subset G of $C(\mathbb{I}^m \times f^{-1}(y), \mathbb{I}^n)$ with the uniform convergence topology. It is clear that, for every $g \in G$ and $z \in \mathbb{I}^m$, the restriction $g|(\{z\} \times f^{-1}(y))$, considered as a map on $f^{-1}(y)$, has the following property: $h_{ij} \circ q$ can be extended to a map from $q^{-1}(\overline{W}_i) \cup (f^{-1}(y) \cap g^{-1}(t))$ into L for any $t \in \mathbb{I}^n$. Hence, we can apply Lemma 2.2 to conclude that $\psi_{ij}(y) \cap \overline{B}(g, \alpha)$ is a Z -set in $\overline{B}(g, \alpha)$. \square

Now, we can finish the proof of this implication. Because of Corollary 3.3 and Lemma 3.4, we can apply Proposition 2.3 to obtain that the set $C_{ij} = C_{ij}(X|Y, \mathbb{I}^n)$ is dense in $C(X, \mathbb{I}^n)$ for every i, j . Since, by Lemma 3.2, all C_{ij} are also open, their intersection \mathcal{G} is dense and G_δ in $C(X, \mathbb{I}^n)$. Let show that $\dim_L(f \times g) = 0$ for every $g \in \mathcal{G}$, i.e., $\text{e-dim}(f \times g) \leq L$. We fix $y \in Y$ and $t \in \mathbb{I}^n$ and consider the fiber $(f \times g)^{-1}(y, t) = f^{-1}(y) \cap g^{-1}(t)$. Take a closed set $A \subset f^{-1}(y) \cap g^{-1}(t)$ and a map $h : A \rightarrow L$. Because the map $q_y = q|f^{-1}(y)$ is a homeomorphism, $h' = h \circ q_y^{-1} : q(A) \rightarrow L$ is well defined.

Next, extend h' to a map from a neighborhood W of $q(A)$ (in Q) into L and find W_i with $q(A) \subset W_i \subset \overline{W}_i \subset W$. Therefore, there exists a map $h'' : \overline{W}_i \rightarrow L$ extending h' . Then h'' is homotopy equivalent to some h_{ij} , so are $h'' \circ q$ and $h_{ij} \circ q$ (considered as maps from $q^{-1}(\overline{W}_i)$ into L). Since $h_{ij} \circ q$ can be extended to a map from $q^{-1}(\overline{W}_i) \cup (f^{-1}(y) \cap g^{-1}(t))$ into L , by the Homotopy Extension Theorem, there exists a map $\bar{h} : q^{-1}(\overline{W}_i) \cup (f^{-1}(y) \cap g^{-1}(t)) \rightarrow L$ extending $h'' \circ q$. Obviously, $\bar{h}|_{(f^{-1}(y) \cap g^{-1}(t))}$ extends h . Hence, $e\text{-dim}(f^{-1}(y) \cap g^{-1}(t)) \leq L$.

(3) \Rightarrow (3') \Rightarrow (1) The implication (3) \Rightarrow (3') is trivial. It is easily seen that in the proof of (3') \Rightarrow (1) we can assume f is perfect. Let $g : X \rightarrow \mathbb{I}^n$ be such that $\dim_L(f \times g) = 0$ and $y \in Y$. Then the restriction $g|_{f^{-1}(y)} : f^{-1}(y) \rightarrow \mathbb{I}^n$ is a perfect map with all of its fibers having extensional dimension $e\text{-dim} \leq L$. Hence, by [2, Corollary], $e\text{-dim } f^{-1}(y) \leq \Sigma^n L$, i.e., $\dim_L f \leq n$.

(3') \Rightarrow (2) Because of the Countable Union Theorem, we can suppose that f is perfect. We fix a map $g : X \rightarrow \mathbb{I}^n$ such that $\dim_L(f \times g) = 0$. According to [12, Lemma 4.1], there exists an F_σ subset $B \subset Y \times \mathbb{I}^n$ such that $\dim B \leq n - 1$ and $\dim(\{y\} \times \mathbb{I}^n) \setminus B \leq 0$ for every $y \in Y$. Then, applying again [2, Corollary 2.7], we conclude that the set $A = (f \times g)^{-1}(B)$ is as required.

References

- [1] A. Chigogidze, Extraordinary dimension theories generated by complexes, *Topology Appl.* 138 (2004) 1–20.
- [2] A. Chigogidze, V. Valov, Extension dimension and C -spaces, *Bull. London. Math. Soc.* 34 (6) (2002) 708–716.
- [3] A. Dranishnikov, The Eilenberg–Borsuk theorem for mappings into an arbitrary complex, *Russian Acad. Sci. Sb.* 81 (1995) 467–475.
- [4] A. Dranishnikov, J. Dydak, Extension dimension and extension types, *Proc. Steklov Inst. Math.* 212 (1996) 55–88.
- [5] R. Engelking, *Theory of Dimensions: Finite and Infinite*, Heldermann, Lemgo, 1995.
- [6] V. Gutev, V. Valov, Continuous selections and C -spaces, *Proc. Amer. Math. Soc.* 130 (1) (2002) 233–242.
- [7] M. Levin, W. Lewis, Some mapping theorems for extensional dimension, *Israel J. Math.* 133 (2003) 61–76.
- [8] B. Pasynkov, On geometry of continuous maps of countable functional weight, *Fund. Prikl. Mat.* 4 (1) (1998) 155–164 (in Russian).
- [9] B. Pasynkov, On geometry of continuous maps of finite-dimensional compact metric spaces, *Trudy Steklov Mat. Inst.* 212 (1996) 147–172 (in Russian).
- [10] E.V. Shchepin, Arithmetic of dimension theory, *Russian Math. Surveys* 53 (5) (1998) 975–1069.
- [11] H. Toruńczyk, Finite-to-one restrictions of continuous functions, *Fund. Math.* 125 (3) (1985) 237–249.
- [12] M. Tuncali, V. Valov, On dimensionally restricted maps, *Fund. Math.* 175 (1) (2002) 35–52.
- [13] M. Tuncali, V. Valov, On finite-dimensional maps, *Tsukuba J. Math.* 28 (2004) 155–167.