

# Extension dimension and quasi-finite CW-complexes

Alex Karasev<sup>1</sup>, Vesko Valov<sup>\*,2</sup>

*Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, PO Box 5002, North Bay, ON, P1B 8L7, Canada*

Received 17 November 2004; received in revised form 24 August 2005; accepted 24 August 2005

---

## Abstract

We extend the definition of quasi-finite complexes from countable complexes to arbitrary ones and provide a characterization of quasi-finite complexes in terms of  $L$ -invertible maps and dimensional properties of compactifications. Several results related to the class of quasi-finite complexes are established, such as completion of metrizable spaces, existence of universal spaces and a version of the factorization theorem. Furthermore, we define  $UV(L)$ -spaces in the realm of metrizable spaces and show that some properties of  $UV(n)$ -spaces and  $UV(n)$ -maps remain valid for  $UV(L)$ -spaces and  $UV(L)$ -maps, respectively.

© 2006 Elsevier B.V. All rights reserved.

*MSC:* primary 54F45; secondary 55M10, 54C65

*Keywords:* Extension dimension; Cohomological dimension; Absolute extensor; Universal space; Quasi-finite complex

---

## 1. Introduction

Extension theory introduced by Dranishnikov [14,15] unifies the covering dimension and the cohomological dimension. There are two classes of maps which play an important role in extension theory. For a given complex  $L$ , these are  $L$ -invertible and  $L$ -soft maps. It should be mentioned that universal spaces in dimension  $L$  as well as absolute extensors in dimension  $L$  are obtained as preimages of Hilbert cube or Hilbert space under maps from the above classes [10]. For a countable complex  $L$ , the existence of an  $L$ -invertible mapping from certain  $L$ -dimensional compactum onto the Hilbert cube is closely connected with the dimensional properties of compactifications of spaces with extension dimension not greater than  $L$  [9]. It turned out that the existence of such  $L$ -invertible mappings can be characterized in terms of “extensional” properties of a complex. This inspired the concept of quasi-finite countable complexes [20].

In the present paper we extend the definition of quasi-finite complexes by considering not necessarily countable complexes. We also provide a characterization of quasi-finite complexes in terms of  $L$ -invertible maps and dimensional properties of compactifications. The main idea of the paper is to generalize results valid for finite or countable complexes to the class of quasi-finite CW-complexes. In particular, quasi-finite complexes possess the  $L$ -soft map

---

\* Corresponding author.

*E-mail addresses:* [alexandk@nipissingu.ca](mailto:alexandk@nipissingu.ca) (A. Karasev), [veskov@nipissingu.ca](mailto:veskov@nipissingu.ca) (V. Valov).

<sup>1</sup> The first author was partially supported by his NSERC Grant RGPIN 257231-04.

<sup>2</sup> The second author was partially supported by his NSERC Grant RGPIN 261914-03.

property and every metrizable space of extension dimension  $\leq L$  has a completion with the same extensional dimension. We also prove a version of the factorization theorem, and construct universal spaces. Finally, in case  $L$  being quasi-finite it is possible to define  $UV(L)$ -property for non-compact spaces. We show that this property does not depend on the embedding of a space into absolute neighborhood extensor in dimension  $L$  and obtain some results about  $UV(L)$ -maps and  $UV(L)$ -spaces which were known for  $UV(n)$ -maps and  $UV(n)$ -spaces, respectively.

## 2. Quasi-finite CW-complexes

Throughout this paper we assume that spaces are Tychonov and maps are continuous. Let  $X$  and  $Y$  be two spaces,  $A \subset X$  and  $g: A \rightarrow Y$  a map. We write  $Y \in ANE(g, A, X)$  if  $g$  has a continuous extension  $\bar{g}: U \rightarrow Y$ , where  $U$  is a neighborhood of  $A$  in  $X$  which has the following property: there exists a function  $h: X \rightarrow [0, 1]$  such that  $h^{-1}((0, 1]) = U$  and  $h(A) = 1$ . If, in the above definition,  $U = X$ , we write  $Y \in AE(g, A, X)$ . Let us note that, by [16, Lemma 2.8],  $Y \in ANE(g, A, X)$  if and only if  $g$  extends to a map  $\bar{g}: X \rightarrow Cone(Y)$ . Here, the cone  $Cone(Y)$  is the quotient set  $Y \times [0, 1]/Y \times \{1\}$  with the following topology:  $U$  is open in  $Cone(Y)$  iff  $U \cap Y \times [0, 1)$  is open in the product topology and, if the vertex  $v$  belongs to  $U$ , then  $Y \times (a, 1) \subset U$  for some  $0 < a < 1$ .

In what follows  $L$  always denotes a CW-complex.

We say that  $L$  is an absolute extensor of  $X$ , notation  $L \in AE(X)$ , if  $L \in AE(g, A, X)$  for every closed  $A \subset X$  and every map  $g: A \rightarrow L$  with  $L \in ANE(g, A, X)$ . We say also that the extension dimension of  $X$  is not greater than  $L$ , notation  $e\text{-dim } X \leq L$ , if  $L \in AE(X)$ . Using Dydak's version of the Homotopy Extension Theorem [16, Theorem 13.7] one can show that if  $L_1$  is homotopy equivalent to  $L_2$ , then  $e\text{-dim } X \leq L_1$  is equivalent to  $e\text{-dim } X \leq L_2$  for any space  $X$ . Moreover, our definition of  $e\text{-dim}$  coincides with that one of Chigogidze [8] in case  $L$  is countable and with the original definition of Dranishnikov [14,15] when compact spaces are considered.

A pair of spaces  $K \subset P$  is called  $L$ -connected if whenever  $A \subset X$  is a closed subset of a space  $X$  with  $e\text{-dim } X \leq L$ , then every map  $g: A \rightarrow K$  has an extension  $\bar{g}: X \rightarrow P$  provided  $A$  is normally placed in  $X$  with respect to  $(g, P)$ . The notion of a normally placed set was introduced in [8] under different notation and means that for every continuous function  $h$  on  $P$  the function  $h \circ g$  can be continuously extended over  $X$ . Obviously, this condition is satisfied for every normal space  $X$  and every map  $g: A \rightarrow K$  with  $A \subset X$  closed. We sometimes say that a pair  $K \subset P$  is  $L$ -connected with respect to a given class of spaces  $\mathcal{B}$  if the additional requirement  $X \in \mathcal{B}$  is imposed in the above definition.

Quasi-finite CW-complexes were introduced in [20] as countable complexes  $L$  satisfying the following condition: every finite subcomplex  $K$  of  $L$  is contained in a finite subcomplex  $P \subset L$  such that the pair  $K \subset P$  is  $L$ -connected with respect to Polish spaces. It was also shown in [20] that there exists a countable quasi-finite complex  $M$  extension type  $[M]$  of which does not contain a finitely dominated complex (see [10] for more information on extension types). In this note we extend the above definition by considering not necessarily countable complexes. Here is our revised definition: a CW-complex  $L$  is quasi-finite if every finite subcomplex  $K$  of  $L$  is contained in a finite subcomplex  $P \subset L$  such that the pair  $K \subset P$  is  $L$ -connected. It is easy to verify that this definition coincides with the definition given in [20] in case  $L$  is countable.

We say that a map  $f: X \rightarrow Y$  is  $L$ -invertible if for any map  $g: Z \rightarrow Y$  with  $e\text{-dim } Z \leq L$  there is a map  $h: Z \rightarrow X$  such that  $g = f \circ h$ . If, in addition,  $Z$  is required to be from a given class of spaces  $\mathcal{B}$ , then we say that the map  $f$  is  $L$ -invertible with respect to the class  $\mathcal{B}$ . Everywhere below  $w(X)$  denotes the weight of the space  $X$  and  $\mathbb{I}^\tau$  denotes Tychonov cube of weight  $\tau$ .

**Theorem 2.1.** *The following conditions are equivalent for any CW-complex  $L$  and an infinite cardinal  $\tau$ :*

- (1)  $L$  is quasi-finite.
- (2)  $e\text{-dim } \beta X \leq L$  whenever  $X$  is a space with  $e\text{-dim } X \leq L$ .
- (3) There exists an  $L$ -invertible map  $f: Y_\tau \rightarrow \mathbb{I}^\tau$  such that  $Y_\tau$  is a compact space of weight  $\leq \tau$  and  $e\text{-dim } Y_\tau \leq L$ .
- (4) For every  $L$ -connected pair  $K \subset M$ , where  $K$  is a compactum of weight  $\leq \tau$  and  $M$  an arbitrary space, there exists a compactum  $P \subset M$  containing  $K$  such that  $w(P) \leq \tau$  and the pair  $K \subset P$  is  $L$ -connected.

**Proof.** (1)  $\Rightarrow$  (2) Suppose  $e\text{-dim } X \leq L$  and let  $f: A \rightarrow L$ , where  $A$  is a closed subset of  $\beta X$ . It is well known that every CW-complex is an absolute neighborhood extensor for the class of compact spaces, so  $L \in ANE(f, A, \beta X)$  and there exists a closed neighborhood  $B$  of  $A$  in  $\beta X$  and a map  $g: B \rightarrow L$  extending  $f$ . Because  $g(B)$  is compact, it is

contained in a finite subcomplex  $K$  of  $L$ . Since  $L$  is quasi-finite, there exists a finite subcomplex  $P$  of  $L$  such that the pair  $K \subset P$  is  $L$ -connected. We can assume that  $B$  is a zero-set in  $\beta X$ . Then  $B \cap X$ , being a non-empty zero-set in  $X$ , is normally placed in  $X$  with respect to  $(g, P)$ . Therefore, the map  $g : B \cap X \rightarrow K$  extends to a map  $h : X \rightarrow P$  because  $e\text{-dim } X \leq L$  and the pair  $K \subset P$  is  $L$ -connected. Finally, let  $\bar{h} : \beta X \rightarrow P$  be the unique extension of  $h$ . Then  $\bar{h}$  extends  $f$ , so  $e\text{-dim } \beta X \leq L$ .

(2)  $\Rightarrow$  (3) We consider the family of all maps  $\{h_\alpha : X_\alpha \rightarrow \mathbb{I}^\tau\}_{\alpha \in A}$  such that each  $X_\alpha$  is a closed subset of  $\mathbb{I}^\tau$  with  $e\text{-dim } X_\alpha \leq L$ . Let  $X$  be the disjoint sum of all  $X_\alpha$  and the map  $h : X \rightarrow \mathbb{I}^\tau$  coincides with  $h_\alpha$  on every  $X_\alpha$ . Clearly,  $e\text{-dim } X \leq L$ . Therefore,  $e\text{-dim } \beta X \leq L$ . Consider the extension  $\bar{h} : \beta X \rightarrow \mathbb{I}^\tau$ . Then, by the factorization theorem from [24], there exists a compact space  $Y_\tau$  of weight  $\leq \tau$  and maps  $r : \beta X \rightarrow Y_\tau$  and  $f : Y_\tau \rightarrow \mathbb{I}^\tau$  such that  $e\text{-dim } Y_\tau \leq L$  and  $f \circ r = \bar{h}$ .

Let us show that  $f$  is  $L$ -invertible. Take a space  $Z$  with  $e\text{-dim } Z \leq L$  and a map  $g : Z \rightarrow \mathbb{I}^\tau$ . Considering  $\beta Z$  and the extension  $\bar{g} : \beta Z \rightarrow \mathbb{I}^\tau$  of  $g$ , we can assume that  $Z$  is always compact. We also can assume that the weight of  $Z$  is  $\leq \tau$  (otherwise we apply again the factorization theorem from [24] to find a compact space  $T$  of weight  $\leq \tau$  and maps  $g_1 : Z \rightarrow T$  and  $g_2 : T \rightarrow \mathbb{I}^\tau$  with  $e\text{-dim } T \leq L$  and  $g_2 \circ g_1 = g$ , and then consider the space  $T$  and the map  $g_2$  instead, respectively, of  $Z$  and  $g$ ). Therefore, without losing generality, we can assume that  $Z$  is a closed subset of  $\mathbb{I}^\tau$ . According to the definition of  $X$  and the map  $h$ , there is an index  $\alpha \in A$  such that  $Z = X_\alpha$  and  $g = h_\alpha$ . The restriction  $r|Z : Z \rightarrow Y_\tau$  is a lifting of  $g$ , i.e.  $f \circ (r|Z) = g$ .

(3)  $\Rightarrow$  (4) Suppose that  $K$  is a compact subset of the space  $M$  with  $w(K) \leq \tau$  and  $K \subset M$  being  $L$ -connected. We embed  $K$  in  $\mathbb{I}^\tau$  and consider an  $L$ -invertible mapping  $f : Y_\tau \rightarrow \mathbb{I}^\tau$  such that  $Y_\tau$  is compact and  $e\text{-dim } Y_\tau \leq L$ . Let  $\tilde{K} = f^{-1}(K)$  and  $h = f|_{\tilde{K}}$ . Obviously,  $\tilde{K}$  is normally placed in  $Y_\tau$  with respect to  $(h, M)$ . Consequently,  $h$  extends to a map  $\bar{h} : Y_\tau \rightarrow M$  and let  $P = \bar{h}(Y_\tau)$ . Obviously,  $w(P) \leq \tau$ , so that it remains only to show that  $K \subset P$  is  $L$ -connected. For this end, let  $g : A \rightarrow K$ , where  $A \subset X$  is a closed normally placed subset of  $X$  with respect to  $(g, P)$  and  $e\text{-dim } X \leq L$ . This implies that  $A$  is normally placed in  $X$  with respect to  $(g, \mathbb{I}^\tau)$ . Since  $\mathbb{I}^\tau$  is an absolute extensor, there exists an extension  $g_1 : X \rightarrow \mathbb{I}^\tau$  of  $g$ . Next, we lift  $g_1$  to a map  $g_2 : X \rightarrow Y_\tau$  such that  $f \circ g_2 = g_1$  (recall that  $f$  is  $L$ -invertible) and let  $\bar{g} = \bar{h} \circ g_2$ . Clearly,  $\bar{g}$  is a map from  $X$  into  $P$  extending  $g$ . Therefore,  $K \subset P$  is  $L$ -connected.

(4)  $\Rightarrow$  (1) Take a finite subcomplex  $K$  of  $L$ . Let us first show that the pair  $K \subset L$  is  $L$ -connected. Suppose  $Z$  is a space with  $e\text{-dim } Z \leq L$ ,  $A \subset Z$  closed and  $g : A \rightarrow K$  a map such that  $A$  is normally placed in  $Z$  with respect to  $(g, L)$ . Since  $K$  is  $C$ -embedded in  $L$ ,  $A$  is normally placed in  $Z$  with respect to  $(g, K)$ . The last condition together with the fact that  $K$  is an absolute neighborhood extensor for all separable metric spaces implies that  $K \in ANE(g, A, Z)$ . Indeed, we embed  $K$  in  $\mathbb{R}^\omega$  and fix a retraction  $r : U \rightarrow K$ , where  $U$  is a neighborhood of  $K$  in  $\mathbb{R}^\omega$ . Since  $A$  is normally placed in  $Z$  with respect to  $(g, K)$ , we can find a map  $h : Z \rightarrow \mathbb{R}^\omega$  extending  $g$ . Then  $h^{-1}(U)$  is a co-zero neighborhood of  $A$  in  $Z$  which contains the zero-set  $h^{-1}(K)$  and  $r \circ h : h^{-1}(U) \rightarrow K$  extends  $g$ . Hence,  $K \in ANE(g, A, Z)$  which yields  $L \in ANE(g, A, Z)$ . Since  $e\text{-dim } Z \leq L$ ,  $g$  can be extended to a map  $\bar{g} : Z \rightarrow L$ . Thus,  $K \subset L$  is an  $L$ -connected pair. Therefore there exists a compact set  $H \subset L$  containing  $K$  such that the pair  $K \subset H$  is  $L$ -connected. Finally, we take a finite subcomplex  $P$  of  $L$  which contains  $H$  and observe that the pair  $K \subset P$  is also  $L$ -connected. Hence,  $L$  is quasi-finite.  $\square$

**Corollary 2.2.** *None of the Eilenberg–MacLane complexes  $K(G, n)$ ,  $n \geq 2$  and  $G$  an Abelian group, is quasi-finite.*

**Proof.** This follows from Theorem 2.1(2) and the following statement (see [22, Theorem 1.4]): there exists a separable metric space  $X$  with  $\dim_G X \leq 2$  and  $e\text{-dim } \beta X > L$  for every Abelian group  $G$  and every non-contractible CW-complex  $L$ . Here  $\dim_G X$  denotes the cohomological dimension of  $X$  with respect to the group  $G$ .  $\square$

Let us also observe that for every quasi-finite complex  $L$  there exists a compact metrizable space which is universal for the class of all separable metric spaces of  $e\text{-dim } \leq L$ , in particular every space from this class has a compactification of  $e\text{-dim } \leq L$ . Indeed, let  $Y_\omega$  be the space from Theorem 2.1(3). Then, for every  $X$  from the above class we take an embedding  $i : X \rightarrow \mathbb{I}^\omega$  and lift  $i$  to a map  $j : X \rightarrow Y_\omega$ . The required compactification of  $X$  is the closure of  $j(X)$  in  $Y_\omega$ . Next corollary provides a characterization of quasi-finite countable complexes in terms of compactifications.

**Corollary 2.3.** *For a countable complex  $L$  the following conditions are equivalent:*

- (a)  $L$  is quasi-finite.  
 (b) For every separable metrizable space  $X$  with  $\text{e-dim } X \leq L$  and its metrizable compactification  $c(X)$  there exists a metrizable compactification  $c^*(X)$  such that  $\text{e-dim } c^*(X) \leq L$  and  $c^*(X) \geq c(X)$  (i.e., there is a map from  $c^*(X)$  onto  $c(X)$  which is the identity on  $X$ ).

**Proof.** (a)  $\Rightarrow$  (b) Let  $L$  be quasi-finite and  $X$  a separable metric space with  $\text{e-dim } X \leq L$ . We take a metric compactification  $c(X)$  of  $X$  and a map  $f: \beta X \rightarrow c(X)$  such that  $f(x) = x$  for every  $x \in X$ . Since, by Theorem 2.1,  $\text{e-dim } \beta X \leq L$ ,  $f$  can be factored through a metrizable compactum  $Z$  with  $\text{e-dim } Z \leq L$ . Clearly,  $Z$  is a compactification of  $X$  which is  $\geq c(X)$ .

(b)  $\Rightarrow$  (a) According to [17, Corollary 3.4], there exists a metrizable compactum  $Y$  with  $\text{e-dim } Y \leq L$  and a surjective map  $f: Y \rightarrow \mathbb{I}^\omega$  such that for any map  $g: X \rightarrow \mathbb{I}^\omega$ ,  $X$  being separable metrizable with  $\text{e-dim } X \leq L$ , there exists an embedding  $i: X \rightarrow Y$  lifting  $g$ , i.e.  $f \circ i = g$ . Hence,  $f$  is  $L$ -invertible with respect to separable metric spaces. By Theorem 2.1(3), it suffices to show that  $f$  is  $L$ -invertible. Consider  $g: Z \rightarrow \mathbb{I}^\omega$  where  $\text{e-dim } Z \leq L$ . According to [8, Proposition 4.9], there exist a Polish space  $P$  with  $\text{e-dim } P \leq L$  and maps  $h: Z \rightarrow P$  and  $q: P \rightarrow \mathbb{I}^\omega$  with  $g = q \circ h$ . We lift  $q$  to a map  $\tilde{q}: P \rightarrow Y$  such that  $f \circ \tilde{q} = q$ . Then  $\tilde{q} \circ h$  is the required lifting of  $g$ .  $\square$

Here is another property of quasi-finite complexes:

**Proposition 2.4.** Every quasi-finite complex  $L$  has the following connected-pairs property:

- (CP) For any metrizable compactum  $K$  with  $\text{e-dim } K \leq L$  there exists a metrizable compactum  $P$  containing  $K$  such that  $\text{e-dim } P \leq L$  and the pair  $K \subset P$  is  $L$ -connected.

**Proof.** Suppose  $K$  is a metrizable compactum with  $\text{e-dim } K \leq L$ . We embed  $K$  into the Hilbert cube  $\mathbb{I}^\omega$  and take an  $L$ -invertible map  $f: Y \rightarrow \mathbb{I}^\omega$  such that  $Y$  is a metrizable compactum with  $\text{e-dim } Y \leq L$  (see Theorem 2.1(3)). Consider the adjunction space  $Y \cup_f K$ , i.e. the disjoint union of  $Y - f^{-1}(K)$  and  $K$  with the topology consisting of the usual open subsets of  $Y - f^{-1}(K)$  together with sets of the form  $f^{-1}(U - K) \cup (U \cap K)$  for open subsets  $U$  of  $\mathbb{I}^\omega$ . There are two associated maps  $p_K: Y \rightarrow Y \cup_f K$  and  $f_K: Y \cup_f K \rightarrow \mathbb{I}^\omega$  such that  $f = f_K \circ p_K$ . Since  $f$  is  $L$ -invertible, so is  $f_K$ . Moreover,  $Y - f^{-1}(K)$ , being open in  $Y$ , is the union of countably many compact sets each with  $\text{e-dim} \leq L$ . Hence, by the countable sum theorem,  $\text{e-dim } Y \cup_f K \leq L$ .

We need only to show that the pair  $K \subset Y \cup_f K$  is  $L$ -connected. Let  $g: A \rightarrow K$  be a map from a closed subset  $A \subset Z$  such that  $\text{e-dim } Z \leq L$  and  $A$  is normally placed in  $Z$  with respect to  $(g, Y \cup_f K)$ . Then, considering  $g$  as a map from  $A$  into  $K \subset \mathbb{I}^\omega$ , we obviously have that  $A$  is normally placed in  $Z$  with respect to  $(g, \mathbb{I}^\omega)$ . Since  $\mathbb{I}^\omega$  is an absolute extensor, there exists a map  $\tilde{g}: Z \rightarrow \mathbb{I}^\omega$  extending  $g$ . Finally, since  $f_K$  is  $L$ -invertible, we lift  $\tilde{g}$  to a map  $h: Z \rightarrow Y \cup_f K$  with  $f_K \circ h = \tilde{g}$ . Clearly,  $h$  extends  $g$ .  $\square$

**Proposition 2.5.** For every  $n \geq 2$  there is no  $K(\mathbb{Z}, n)$ -connected pair  $K \subset P$  of compact sets such that  $K$  is homeomorphic to the  $n$ -dimensional sphere  $S^n$  and  $\dim_{\mathbb{Z}} P \leq n$ .

**Proof.** We use the arguments from the proof of [17, Theorem 3.5]. Suppose for some  $n \geq 2$  there is a  $K(\mathbb{Z}, n)$ -connected compact pair  $S^n \subset P$  with  $\dim_{\mathbb{Z}} P \leq n$ . We choose a complex  $L$  of type  $K(\mathbb{Z}, n)$  and having finite skeleta. It was shown in [18] that there exist metrizable compacta  $X_k$ ,  $k \geq 1$ , such that:

- $\dim_{\mathbb{Z}} X_k \leq n$  for each  $k$ ;
- each  $X_k$  contains a copy of  $S^n$ ;
- the inclusion  $i: S^n \hookrightarrow L$  cannot be extended over  $X_k$  so that the image of the extension is contained in the  $k$ -skeleton  $L^{(k)}$  of  $L$ .

We take an extension  $h: P \rightarrow L$  of the inclusion  $i: S^n \hookrightarrow L$ , and  $m$  such that  $h(P) \subset L^{(m)}$ . This means that the inclusion  $j: S^n \hookrightarrow P$  cannot be extended to a map from  $X_m$  into  $P$  which contradicts the fact that  $S^n \subset P$  is  $L$ -connected.  $\square$

The problem [27] whether, for any fixed  $n \geq 2$  there is a universal space in the class of all metrizable compacta  $X$  with  $\dim_{\mathbb{Z}} \leq n$  is still unsolved. Zarichnyi [28] observed that each of the above classes does not have an universal element which is an absolute extensor for the same class. Proposition 2.5 yields a little bit stronger observation.

**Corollary 2.6.** *None of the complexes  $K(\mathbb{Z}, n)$ ,  $n \geq 2$ , have the (CP)-property from Proposition 2.4.*

Recall that a map  $f : X \rightarrow Y$  between metrizable spaces is called uniformly 0-dimensional [21] if there exists a metric on  $X$  generating its topology such that for every  $\varepsilon > 0$  every point of  $f(X)$  has a neighborhood  $U$  in  $Y$  with  $f^{-1}(U)$  being the union of disjoint open subsets of  $X$  each of diameter  $< \varepsilon$ . It is well known that every metric space admits uniformly 0-dimensional map into  $l_2$ .

**Proposition 2.7.** *Let  $L$  be a quasi-finite CW-complex. Then for every  $\tau \geq \omega$  there exists a perfect  $L$ -invertible surjection  $f_{(L,\tau)} : Y_{(L,\tau)} \rightarrow l_2(\tau)$  such that:*

- (a)  $Y_{(L,\tau)}$  is a completely metrizable space of weight  $\tau$  with  $e\text{-dim } Y_{(L,\tau)} \leq L$ .
- (b) Every (completely) metrizable space of weight  $\leq \tau$  and extension dimension  $\leq L$  can be embedded as a (closed) subspace of  $Y_{(L,\tau)}$ .

**Proof.** By Theorem 2.1(3), there exists an  $L$ -invertible map  $f : Y \rightarrow \mathbb{I}^\omega$ , where  $Y$  is a metrizable compactum with  $e\text{-dim } Y \leq L$ . We embed  $l_2$  in  $\mathbb{I}^\omega$  and let  $Y_{(L,\omega)} = f^{-1}(l_2)$  and  $f_{(L,\omega)} = f|_{Y_{(L,\omega)}}$ . Then  $e\text{-dim } Y_{(L,\omega)} \leq L$  and since  $f$  is  $L$ -invertible, so is  $f_{(L,\omega)}$ .

If  $\tau > \omega$ , we take a metric  $d_1$  on  $l_2(\tau)$  and a uniformly 0-dimensional map  $g : l_2(\tau) \rightarrow l_2$  with respect to  $d_1$ . Denote by  $Y_{(L,\tau)}$  the fibered product of  $l_2(\tau)$  and  $Y_{(L,\omega)}$  with respect to the maps  $g$  and  $f_{(L,\omega)}$ . We also consider the projections  $f_{(L,\tau)} : Y_{(L,\tau)} \rightarrow l_2(\tau)$  and  $h : Y_{(L,\tau)} \rightarrow Y_{(L,\omega)}$ . Since  $f_{(L,\omega)}$  is a perfect and  $L$ -invertible surjection, so is  $f_{(L,\tau)}$ . If  $d_2$  is any metric on  $Y_{(L,\omega)}$ , then  $h$  is uniformly 0-dimensional with respect to the metric  $d = \sqrt{d_1^2 + d_2^2}$  on  $Y_{(L,\tau)}$  (see [4]). Thus  $Y_{(L,\tau)}$  admits a uniformly 0-dimensional map into the space  $Y_{(L,\omega)}$  having extension dimension  $\leq L$ . Hence, by [23, Theorem 1.2],  $e\text{-dim } Y_{(L,\tau)} \leq L$ . Observe that  $Y_{(L,\tau)}$  is completely metrizable as a perfect preimage of the completely metrizable space  $l_2(\tau)$ .

To prove the second item, suppose  $M$  is a metrizable space of weight  $\leq \tau$  and  $e\text{-dim } M \leq L$ . We consider  $M$  as a subset of  $l_2(\tau)$  and use the  $L$ -invertibility of  $f_{(L,\tau)}$  to lift the identity map on  $M$ . Obviously this lifting is an embedding of  $M$  into  $Y_{(L,\tau)}$ . Moreover, if  $M$  is completely metrizable, then we can embed it in  $l_2(\tau)$  as a closed subspace. This implies that the corresponding embedding of  $M$  in  $Y_{(L,\tau)}$  is also closed.  $\square$

A completion theorem for  $L$ -dimensional metric spaces, where  $L$  is any countable CW-complex, was established in [26]. It follows from Proposition 2.7 that this is also true for quasi-finite (not necessarily countable) complexes  $L$ .

**Corollary 2.8.** *Let  $L$  be a quasi-finite complex. Then every metrizable space  $X$  with  $e\text{-dim } X \leq L$  has a completion with extension dimension  $\leq L$ .*

**Corollary 2.9.** *Let  $L$  be a quasi-finite complex and  $X$  a metrizable space. Then  $e\text{-dim } X \leq L$  if and only if  $X$  admits a uniformly 0-dimensional map into a separable metrizable space of extension dimension  $\leq L$ .*

**Proof.** In one direction (sufficiency) this follows from the mentioned above result of Levin [23, Theorem 1.2]. Suppose  $X$  is a metrizable space of weight  $\tau$  with  $e\text{-dim } X \leq L$ . By Proposition 2.7,  $X$  can be embedded in the space  $Y_{(L,\tau)}$ . It follows from the construction of  $Y_{(L,\tau)}$  that the map  $h : Y_{(L,\tau)} \rightarrow Y_{(L,\omega)}$  is uniformly 0-dimensional. Then the restriction  $h|_X$  is also uniformly 0-dimensional which completes the proof.  $\square$

A general factorization theorem for  $L$ -dimensional compact spaces, where  $L$  is an arbitrary complex, was proved in [24]. We provide here a factorization theorem for  $L$ -dimensional metrizable spaces with  $L$  being quasi-finite (see [23, Theorem 1.5] for similar result with  $L$  countable).

**Proposition 2.10.** *Let  $L$  be a quasi-finite complex and let  $f : X \rightarrow Y$  be a map with  $Y$  metrizable. If  $\text{e-dim } X \leq L$ , then  $f$  factors through a metrizable space  $Z$  such that  $\text{e-dim } Z \leq L$  and  $w(Z) \leq w(Y)$ .*

**Proof.** Let us first show how to reduce this proposition to the case  $Y$  is separable. This reduction is well known (see, for example, [4]), but we present it here for the reader's convenience. Suppose the result holds when the range space is separable and metrizable. We take a uniformly 0-dimensional map  $g : Y \rightarrow l_2$  and apply the “separable factorization theorem” to the map  $g \circ f : X \rightarrow l_2$  to obtain a separable metrizable space  $M$  and maps  $q : X \rightarrow M$  and  $h : M \rightarrow l_2$  with  $\text{e-dim } M \leq L$  and  $h \circ q = g \circ f$ . Let  $p_M : Z \rightarrow M$  and  $p_Y : Z \rightarrow Y$  be the pullbacks of  $g$  and  $h$  respectively. Clearly,  $Z$  is a metrizable space of weight  $w(Z) \leq w(Y)$ . Since  $g$  is uniformly 0-dimensional, so is  $p_M$ . Then, by [23, Theorem 1.2],  $\text{e-dim } Z \leq L$ .

Now we prove the “separable case”. Let  $\tilde{Y}$  be a metrizable compactification of  $Y$  and  $\tilde{f} : \beta X \rightarrow \tilde{Y}$  be the Čech–Stone extension of  $f$ . Since  $L$  is quasi-finite,  $\text{e-dim } \beta X \leq L$ . Therefore we can apply the factorization theorem of Levin–Rubin–Schapiro [24] to obtain a metrizable compactum  $\tilde{Z}$  and maps  $\tilde{f}_1 : \beta X \rightarrow \tilde{Z}$  and  $\tilde{f}_2 : \tilde{Z} \rightarrow \tilde{Y}$  such that  $\tilde{f}_2 \circ \tilde{f}_1 = \tilde{f}$  and  $\text{e-dim } \tilde{Z} \leq L$ . Then the space  $Z = \tilde{f}_1(X)$  and the maps  $f_1 = \tilde{f}_1|_X$  and  $f_2 = \tilde{f}_2|_Z$  form the required factorization.  $\square$

We say that a map  $f : X \rightarrow Y$  is  $L$ -soft, where  $L$  is a CW-complex, if for any space  $Z$  with  $\text{e-dim } Z \leq L$ , any closed set  $A \subset Z$  and any two maps  $h : Z \rightarrow Y$  and  $g : A \rightarrow X$ , where  $A$  is normally placed in  $Z$  with respect to  $(g, X)$  and  $f \circ g = h|_A$ , there exists a map  $\bar{g} : Z \rightarrow X$  extending  $g$  such that  $f \circ \bar{g} = h$ . If, in the above definition, we additionally require  $Z$  to be from a given class of spaces  $\mathcal{A}$ , then we say that  $f$  is  $L$ -soft with respect to the class  $\mathcal{A}$ . It was established in [11] that for every countable complex  $L$  and every metric space  $Y$  there exists an  $L$ -soft map  $f : X \rightarrow Y$  such that  $X$  is a metric space of extension dimension  $\leq L$  and  $w(X) = w(Y)$ . We are going to show that quasi-finite complexes also have this property.

**Proposition 2.11.** *Let  $L$  be a quasi-finite CW-complex. Then for every  $\tau \geq \omega$  there exists an  $L$ -soft map  $p_{(L,\tau)} : X_{(L,\tau)} \rightarrow l_2(\tau)$  such that:*

- $X_{(L,\tau)}$  is a completely metrizable space of weight  $\tau$  with  $\text{e-dim } X_{(L,\tau)} \leq L$ .
- $X_{(L,\tau)}$  is an absolute extensor for all metrizable spaces of  $\text{e-dim} \leq L$ .
- $p_{(L,\tau)}$  is a strongly  $(L, \tau)$ -universal map, i.e. for any open cover  $\mathcal{U}$  of  $X_{(L,\tau)}$ , any (complete) metrizable space  $Z$  of weight  $\leq \tau$  with  $\text{e-dim } Z \leq L$  and any map  $g : Z \rightarrow X_{(L,\tau)}$  there exists a (closed) embedding  $h : Z \rightarrow X_{(L,\tau)}$  which is  $\mathcal{U}$ -close to  $g$  and  $p_{(L,\tau)} \circ g = p_{(L,\tau)} \circ h$ .

**Proof.** Using Proposition 2.10 and following Zarichnyi's idea from [28] (see also [8]) that invertibility generates softness, we can show the existence of a complete separable metrizable space  $X$  with  $\text{e-dim } X \leq L$  and an  $L$ -soft map  $f : X \rightarrow l_2$ . Then, as in [11], we construct the space  $X_{(L,\tau)}$  and the map  $p_{(L,\tau)} : X_{(L,\tau)} \rightarrow l_2(\tau)$  possessing the desired properties.  $\square$

### 3. Additional properties of quasi-finite complexes

In this section, all spaces and all CW-complexes, unless stated otherwise, are, respectively, metrizable and quasi-finite. We are going to show that some properties of finitely dominated complexes remain valid for quasi-finite complexes. We say that a space  $X$  is an absolute (neighborhood) extensor in dimension  $L$  (notation  $X \in A(N)E(L)$ ) if for every space  $Z$  of extension dimension  $\leq L$  and every map  $g : A \rightarrow X$ , where  $A$  is a closed subset of  $Z$ , there exists an extension of  $g$  over  $Z$  (respectively, over a neighborhood of  $A$  in  $Z$ ).

Everywhere below  $\text{cov}(X)$  denotes the family of all open covers of  $X$ . Two maps  $f_0, f_1 : X \rightarrow Y$  are  $L$ -homotopic [10] if for any map  $h : Z \rightarrow X \times [0, 1]$ , where  $Z$  is a space with  $\text{e-dim } Z \leq L$ , the composition  $(f_0 \oplus f_1) \circ h|(h^{-1}(X \times \{0, 1\})) : h^{-1}(X \times \{0, 1\}) \rightarrow Y$  admits an extension  $H : Z \rightarrow Y$ . If  $\mathcal{U} \in \text{cov}(Y)$  and the extension  $H$  in the above definition can be chosen such that the collection  $\{H(h^{-1}(\{x\} \times [0, 1])) : x \in X\}$  refines  $\mathcal{U}$ , then  $f_0$  and  $f_1$  are called  $(\mathcal{U}, L)$ -homotopic.

The following three propositions were given in [10] for finitely dominated countable complexes  $L$  and Polish  $ANE(L)$ -spaces  $X$ . Because of Proposition 2.7, one can show they also hold for quasi-finite complexes  $L$  and arbitrary (not necessarily Polish)  $ANE(L)$ -spaces.

**Proposition 3.1.** *Let  $X$  be an  $ANE(L)$ -space and  $\mathcal{U} \in \text{cov}(X)$ . Then there exists a cover  $\mathcal{V} \in \text{cov}(X)$  such that any two  $\mathcal{V}$ -close maps of any space into  $X$  are  $(\mathcal{U}, L)$ -homotopic.*

**Proposition 3.2.** *Let  $X \in ANE(L)$  and  $\mathcal{U} \in \text{cov}(X)$ . Then there exists a cover  $\mathcal{V} \in \text{cov}(X)$  refining  $\mathcal{U}$ , such that the following condition holds:*

(H) *For any space  $Z$  with  $e\text{-dim } Z \leq L$ , any closed  $A \subset Z$ , and any two  $\mathcal{V}$ -close maps  $f, g : A \rightarrow X$  such that  $f$  has an extension  $F : Z \rightarrow X$ , it follows that  $g$  also can be extended to a map  $G : Z \rightarrow X$  which is  $(\mathcal{U}, L)$ -homotopic to  $F$ .*

**Proposition 3.3.** *Let  $X \in ANE(L)$ ,  $Z$  be a space with  $e\text{-dim } Z \leq L$  and  $A \subset Z$  closed. If  $f, g : A \rightarrow X$  are  $L$ -homotopic and  $f$  admits an extension  $F : Z \rightarrow X$ , then  $g$  also admits an extension  $G : Z \rightarrow X$ , and we may assume that  $F$  and  $G$  are  $L$ -homotopic.*

A pair of closed subsets  $X_0 \subset X_1$  of a space  $X$  is called  $UV(L)$ -connected in  $X$  if every neighborhood  $U$  of  $X_1$  in  $X$  contains a neighborhood  $V$  of  $X_0$  such that  $V \subset U$  is  $L$ -connected with respect to metrizable spaces, i.e. any map  $g : A \rightarrow V$ , where  $A$  is a closed subset of a space  $Z$  with  $e\text{-dim } \leq L$ , admits an extension  $\tilde{g} : Z \rightarrow U$ . When  $X_0 \subset X_1$  is  $UV(L)$ -connected in  $X$ , we say that  $X_0$  is  $UV(L)$  in  $X$ . If in the above definition all pairs under consideration are  $L$ -connected with respect to a given class  $\mathcal{A}$ , we obtain the notion of  $UV(L)$ -sets with respect to  $\mathcal{A}$ . If instead of  $L$ -connectedness of the pair  $V \subset U$  we require the inclusion  $V \subset U$  to be  $L$ -homotopic to a constant map in  $U$  then the pair  $X_0 \subset X_1$  (respectively the set  $X_0$ ) is called  $UV(L)$ -homotopic in  $X$ . Obviously, every  $UV(L)$ -connected pair is  $UV(L)$ -homotopic. Next corollary, which follows from Proposition 3.3, shows that these two properties are equivalent in case  $X \in ANE(L)$ .

**Corollary 3.4.** *Let  $X$  be an  $ANE(L)$ -space. A pair  $X_0 \subset X_1$  of closed subsets of  $X$  is  $UV(L)$ -connected in  $X$  if and only if it is  $UV(L)$ -homotopic in  $X$ .*

**Lemma 3.5.** *Let  $X_0 \subset X_1 \subset X \subset E$ , where both  $X$  and  $E$  are  $ANE(L)$ -spaces and  $X \subset E$  is closed. Then the pair  $X_0 \subset X_1$  is  $UV(L)$ -connected in  $X$  if and only if it is  $UV(L)$ -connected in  $E$ .*

**Proof.** By Proposition 2.7, there exists a perfect  $L$ -invertible surjection  $f : \tilde{E} \rightarrow E$  with  $e\text{-dim } \tilde{E} \leq L$ , and let  $\tilde{X} = f^{-1}(X)$ . Since  $X \in ANE(L)$ , we can extend  $f|_{\tilde{X}}$  to a map  $g : W \rightarrow X$  with  $W$  being a neighborhood of  $\tilde{X}$  in  $\tilde{E}$ . Since  $f$  is closed, we may assume that  $W = f^{-1}(G)$  for some neighborhood  $G$  of  $X$  in  $E$ . The claim below follows from our constructions.

**Claim.** *For every open  $O \subset X$  the set  $O^* = G - f(g^{-1}(X - O))$  is open in  $G$  and has the following two properties:  $O^* \cap X = O$  and  $g(f^{-1}(O^*)) = O$ .*

Suppose  $X_0 \subset X_1$  is  $UV(L)$ -connected in  $X$ . We are going to show that this pair is  $UV(L)$ -connected in  $E$ . To this end, let  $U \subset G$  be a neighborhood of  $X_1$  in  $E$ . Then there is a neighborhood  $O$  of  $X_0$  in  $X$  such that  $O \subset U \cap X$  is  $L$ -connected. Since  $U$  is an  $ANE(L)$  (as an open subset of  $E$ ), we can apply Proposition 3.2 for the space  $U$  and the one-element cover  $\mathcal{U} = \{U\}$  to find an open cover  $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$  of  $U$  satisfying the condition (H). For every  $\alpha$  let  $G_\alpha = V_\alpha \cap (V_\alpha \cap X)^* \cap O^*$  and  $V = \bigcup \{G_\alpha : \alpha \in \Lambda\}$ . Obviously,  $V \subset U$  is open and contains  $X_0$ . The pair  $V \subset U$  is  $L$ -connected. Indeed, let  $h : A \rightarrow V$  be a map, where  $A \subset Z$  is closed and  $e\text{-dim } Z \leq L$ . Since  $f$  is  $L$ -invertible,  $h$  admits a lifting  $h_1 : A \rightarrow f^{-1}(V)$ , i.e.  $h = f \circ h_1$ . According to the Claim,  $g(f^{-1}(G_\alpha)) \subset V_\alpha \cap X$ ,  $\alpha \in \Lambda$ , and  $V \cap X \subset O$ . This implies that  $h$  and the map  $h_2 = g \circ h_1 : A \rightarrow V \cap X$  are  $\mathcal{V}$ -close. Since the pair  $O \subset U \cap X$  is  $L$ -connected,  $h_2$  can be extended to a map from  $Z$  into  $U \cap X$ . This yields, according to Proposition 3.2, that  $h$  also can be extended to a map from  $Z$  into  $U$ .

Now, suppose the pair  $X_0 \subset X_1$  is  $UV(L)$ -connected in  $E$ . To show this pair is  $UV(L)$ -connected in  $X$ , let  $U$  be a neighborhood of  $X_1$  in  $X$ . Then  $U^* \subset G$  is open in  $E$ , and we can find a neighborhood  $V$  of  $X_0$  in  $E$  such that  $V \subset U^*$  is  $L$ -connected. The pair  $V \cap X \subset U$  is  $L$ -connected. Indeed, any map  $h: A \rightarrow V \cap X$ , where  $A \subset Z$  is closed and  $\text{e-dim } Z \leq L$ , admits an extension  $h_1: Z \rightarrow U^*$ . Then the map  $\tilde{h} = g \circ h_2: Z \rightarrow U$ , where  $h_2: Z \rightarrow f^{-1}(U^*)$  is a lifting of  $h_1$ , extends  $h$ .  $\square$

**Theorem 3.6.** *Suppose  $X$  is an  $ANE(L)$ -space and the pair  $X_0 \subset X_1$  is  $UV(L)$ -connected in  $X$ . Then it is  $UV(L)$ -connected in any  $ANE(L)$ -space in which  $X_1$  is embeddable as a closed subspace.*

**Proof.** Let  $i: X_1 \rightarrow Y$  be a closed embedding, where  $Y \in ANE(L)$ , and  $M$  be the space obtained from the disjoint union  $X \uplus Y$  by identifying all pairs of points  $x \in X_1 \subset X$  and  $i(x) \in Y$ . The space  $M$  is metrizable and if  $p: X \uplus Y \rightarrow M$  is the quotient map, then  $p(X)$ ,  $p(Y)$  and  $p(X_1)$  are closed sets in  $M$  homeomorphic, respectively, to  $X$ ,  $Y$  and  $X_1$ . Moreover,  $p(X_1)$  is the common part of  $p(X)$  and  $p(Y)$ . We embed  $M$  in a normed space  $E$  as a closed subspace. Every normed space is an absolute extensor for the class of metrizable spaces, so  $E \in ANE(L)$ . Since the pair  $p(X_0) \subset p(X_1)$  is  $UV(L)$ -connected in  $p(X)$ , by Lemma 3.5 it is also  $UV(L)$ -connected in  $E$ . This implies, again by Lemma 3.5, that  $p(X_0) \subset p(X_1)$  is  $UV(L)$ -connected in  $p(Y)$ .  $\square$

**Corollary 3.7.** *If a space  $X$  is  $UV(L)$  in a given  $ANE(L)$ -space, then  $X$  is  $UV(L)$  in any  $ANE(L)$ -space in which  $X$  is embeddable as a closed subset.*

In the existing literature, the  $UV^n$ -property, and more general, the  $UV(L)$ -property, is defined for compact spaces, see [10] and [6]. We extend this definition to arbitrary (metrizable) spaces:  $X$  is a  $UV(L)$ -space if it is  $UV(L)$  in some  $ANE(L)$ -space containing  $X$  as a closed subspace. According to Corollary 3.7, the  $UV(L)$ -property does not depend on the embeddings in  $ANE(L)$ -spaces (for compact spaces and finite complexes  $L$  this was done in [6]). It follows from Corollary 3.4 that  $X$  is a  $UV(L)$ -space if and only if  $X$  is  $UV(L)$ -homotopic in every space  $Y \in ANE(L)$  containing  $X$  as a closed subset.

Recall that a normal space  $X$  is a  $C$ -space [1] if for any sequence  $\{\omega_n\}$  of open covers of  $X$  there exists a sequence  $\{\gamma_n\}$  of open disjoint families such that each  $\gamma_n$  refines  $\omega_n$  and  $\bigcup \gamma_n$  covers  $X$ . Every finite-dimensional paracompactum, as well as every countable-dimensional metrizable space has property  $C$  [19].

We say that a complex  $L$  (not necessarily quasi-finite) possesses the *soft map property* if for every space  $X$  there exists a space  $Y$  with  $\text{e-dim } Y \leq L$  and an  $L$ -soft map from  $Y$  onto  $X$ . Every countable complex has the soft map property (see [11]), as well as every quasi-finite complex (by Proposition 2.11).

A pair of spaces  $\tilde{V} \subset \tilde{U}$  is called an  $L$ -extension of the pair  $V \subset U$  [7] if  $\tilde{U} \in AE(L)$  and there exists a map  $q: \tilde{U} \rightarrow U$  such that the restriction  $q|_{\tilde{V}}$  is an  $L$ -soft map onto  $V$ . The following property of  $L$ -extension pairs was established in [7].

**Lemma 3.8.** *Let  $L$  be a complex (not necessarily quasi-finite) with the soft map property and  $\tilde{V} \subset \tilde{U}$  an  $L$ -extension of the pair  $V \subset U$ . Let also  $A \subset B$  be a pair of closed subsets of a space  $X$  with  $\text{e-dim } X \leq L$ . Suppose we have maps  $f: B \rightarrow U$  and  $g: A \rightarrow \tilde{U}$  such that  $q \circ g = f|_A$  and  $f(B \setminus A) \subset V$ . Then there exists a map  $h: X \rightarrow \tilde{U}$  such that  $q \circ (h|_B) = f$ .*

**Lemma 3.9.** *Let  $L$  be a complex (not necessarily quasi-finite). Every  $L$ -connected pair  $V \subset U$  of spaces admits an  $L$ -extension provided  $L$  has the soft map property.*

**Proof.** We take a normed space  $E$  containing  $V$  as a closed subspace and an  $L$ -soft surjection  $g: \tilde{U} \rightarrow E$  such that  $\tilde{U}$  is a space of  $\text{e-dim} \leq L$ . Since  $V \subset U$  is  $L$ -connected, there exists a map  $q: \tilde{U} \rightarrow U$  extending the map  $g|_{\tilde{V}}$ , where  $\tilde{V} = g^{-1}(V)$ . Moreover,  $\tilde{U} \in AE(L)$  because  $E$  is an absolute extensor for the class of metrizable spaces and  $g$  is  $L$ -soft. Therefore,  $\tilde{V} \subset \tilde{U}$  is an  $L$ -extension of  $V \subset U$ .  $\square$

If  $A$  is a subset of a space  $X$  we denote the star of  $A$  with respect to a cover  $\omega \in \text{cov}(X)$  by  $\text{St}(A, \omega)$ . We say that  $\nu \in \text{cov}(X)$  is a strong star-refinement of  $\omega \in \text{cov}(X)$  if for each  $V \in \nu$  there exists  $W \in \omega$  such that  $\text{St}(V, \nu) \subset W$ .

**Auxiliary Construction.** Suppose we are given the spaces  $X, Z$  and the map  $g : A \rightarrow X$ , where  $A \subset Z$  is closed. Let  $\alpha_n = \{U_n(x) : x \in X\}$ ,  $\beta_n = \{V_n(x) : x \in X\}$ ,  $n \geq 0$ , be two sequences of open covers of  $X$  and  $\mu_n^*$ ,  $n \geq 1$ , be a sequence of disjoint open families in  $A$  such that:

- (1)  $\alpha_n$  is a strong star refinement of  $\beta_{n-1}$  for any  $n \geq 1$ .
- (2) each  $\mu_n^*$ ,  $n \geq 1$ , refines  $g^{-1}(\beta_n)$  and  $\bigcup\{\mu_n^* : n \geq 1\}$  is a locally finite cover of  $A$ .

We are going first to construct open and disjoint families  $\mu_n$ ,  $n \geq 1$ , in  $Z$  satisfying the following condition:

- (3)  $\mu = \bigcup\{\mu_n : n \geq 1\}$  is locally finite in  $Z$  and the restriction of each  $\mu_n$  on  $A$  is  $\mu_n^*$ .

To this end, we choose an upper semi-continuous (br., u.s.c.) set-valued map  $r : Z \rightarrow A$  such that each  $r(z)$  is a finite set and  $r(z) = \{z\}$  for  $z \in A$  (see [25] for the existence of such  $r$ ). Recall that  $r$  is upper semi-continuous if  $r^\sharp(T) = \{z \in Z : r(z) \subset T\}$  is open in  $Z$  whenever  $T$  is open in  $A$ . Obviously,  $r^\sharp(T) \cap A = T$  and  $r^\sharp(T_1) \cap r^\sharp(T_2) \neq \emptyset$  if and only if  $T_1 \cap T_2 \neq \emptyset$  for any open subsets  $T, T_1$  and  $T_2$  of  $A$ . Therefore all families  $\mu_n = \{r^\sharp(T) : T \in \mu_n^*\}$ ,  $n \geq 1$ , are open and disjoint in  $Z$ . Since  $\mu^*$  is locally finite in  $A$  and  $r$  is finite-valued, the family  $\mu = \bigcup\{\mu_n : n \geq 1\}$  is locally finite in  $Z$ .

The second part of our construction is to find points  $x_W \in X$  such that

- (4)  $St(g(W \cap A), \alpha_n) \subset V_{n-1}(x_W)$  for every  $W \in \mu_n$  and  $n \geq 1$ .

This can be done as follows. Since  $\alpha_n$  is a strong star refinement of  $\beta_{n-1}$  and  $\mu_n$  refines  $g^{-1}(\beta_n)$ , for every  $n \geq 1$  and  $W \in \mu_n$  there exist  $S \in \beta_n$  and a point  $x_W \in X$  such that  $St(g(W \cap A), \alpha_n) \subset St(S, \alpha_n) \subset V_{n-1}(x_W)$ . The auxiliary construction is completed.

**Lemma 3.10.** *Let  $L$  be a complex (not necessarily quasi-finite) with the soft map property and  $f : M \rightarrow X$  be a surjection with the following property:*

- (UV) *for every  $x \in X$  and its neighborhood  $U(x)$  in  $X$  there exists a smaller neighborhood  $V(x)$  of  $x$  such that the pair  $\tilde{V}(x) = f^{-1}(V(x)) \subset \tilde{U}(x) = f^{-1}(U(x))$  is  $L$ -connected with respect to the class of metrizable spaces.*

*Suppose  $p : Y \rightarrow Z$  is a surjective map with  $e\text{-dim } Y \leq L$ . Then, for any  $\omega \in \text{cov}(X)$  and any map  $g : A \rightarrow X$ , where  $A$  is a closed subset of  $Z$  such that either  $A$  or  $g(A)$  is a  $C$ -space, there is a neighborhood  $G$  of  $A$  in  $Z$  and a map  $h : p^{-1}(G) \rightarrow M$  with  $(f \circ h)|_{p^{-1}(A)}$  being  $\omega$ -close to  $g \circ p$ .*

**Proof.** For every  $x \in X$  and  $n = 0, 1, 2, \dots$  we choose a point  $P(x) \in f^{-1}(x)$  and neighborhoods  $U_n(x)$  and  $V_n(x)$  of  $x$  in  $X$  such that the cover  $\alpha_0 = \{U_0(x) : x \in X\}$  refines  $\omega$ , each pair  $\tilde{V}_n(x) \subset \tilde{U}_n(x)$  is  $L$ -connected with respect to all metrizable spaces and the covers  $\alpha_n = \{U_n(x) : x \in X\}$ ,  $\beta_n = \{V_n(x) : x \in X\}$  satisfy condition (1) from the auxiliary construction. Since either  $A$  or  $g(A)$  is a  $C$ -space, there exists a sequence of disjoint open families  $\{\mu_n^* : n \geq 1\}$  in  $A$  satisfying condition (2) above. Therefore, according to the auxiliary construction, we can extend each  $\mu_n^*$  to a disjoint open family  $\mu_n$  in  $Z$  such that  $\mu = \bigcup\{\mu_n : n \geq 1\}$  is locally finite in  $Z$  and let  $G$  be the union of all elements of  $\mu$ .

We introduce the following notations:  $B = p^{-1}(A)$ ,  $\tilde{g} = g \circ (p|_B)$ ,  $\Omega = p^{-1}(G)$ , and  $\nu_n = p^{-1}(\mu_n)$ . Obviously, each  $\nu_n$  is a disjoint open family in  $Y$  and  $\nu = \bigcup\{\nu_n : n \geq 1\}$  is a locally finite cover of  $\Omega$ . Let us also consider the open covers  $\tilde{\omega} = f^{-1}(\omega)$ ,  $\tilde{\alpha}_n = \{\tilde{U}_n(x) : x \in X\}$  and  $\tilde{\beta}_n = \{\tilde{V}_n(x) : x \in X\}$  of  $M$  corresponding, respectively, to  $\omega, \alpha_n$  and  $\beta_n$ . According to Lemma 3.9, every pair  $\tilde{V}_n(x) \subset \tilde{U}_n(x)$  has an  $L$ -extension  $\tilde{\tilde{V}}_n(x) \subset \tilde{\tilde{U}}_n(x)$  with a corresponding map  $q_{n,x} : \tilde{\tilde{U}}_n(x) \rightarrow \tilde{\tilde{V}}_n(x)$  such that  $(q_{n,x})|_{\tilde{\tilde{V}}_n(x)}$  is an  $L$ -soft surjection onto  $\tilde{\tilde{V}}_n(x)$ .

Consider the nerve  $\mathfrak{N}$  of  $\nu$  and a barycentric map  $\theta : \Omega \rightarrow |\mathfrak{N}|$ . Any simplex  $\sigma = \langle W_0, W_1, \dots, W_k \rangle$  from  $\mathfrak{N}$ , where  $W_i \in \nu_{n(i)}$ , can be ordered such that  $n(0) < n(1) < \dots < n(k)$ . This is possible because  $\bigcap\{W_i : i = 0, 1, \dots, k\} \neq \emptyset$ , so the numbers  $n(i)$  are different. It is easily seen that, for fixed  $k \geq 1$  and  $W \in \nu_k$ , condition (4) from the auxiliary construction implies the following one

- (5)  $St(\tilde{g}(W \cap B), \alpha_k) \subset V_{k-1}(x_W)$ , and therefore  $St(f^{-1}(\tilde{g}(W \cap B)), \tilde{\alpha}_k) \subset \tilde{\tilde{V}}_{k-1}(x_W)$ .

Let  $\Sigma(\sigma)$ ,  $\sigma \in \mathfrak{R}$ , be the closed subset  $\theta^{-1}(\sigma)$  of  $\Omega$  and  $\Sigma^k = \theta^{-1}(\mathfrak{R}^k)$ , where  $\mathfrak{R}^k$  denotes the  $k$ th skeleton of  $\mathfrak{R}$ . For every  $k \geq 0$  and  $\sigma = \langle W_0, W_1, \dots, W_k \rangle \in \mathfrak{R}^k$  with  $W_0 \in \nu_{n(0)}$ , we define by induction maps  $h_k : \Sigma^k \rightarrow M$  and  $h_\sigma : \Sigma(\sigma) \rightarrow \tilde{U}_{n(0)-1}(x_{W_0})$  such that

- (6)  $h_k|_{\Sigma^{k-1}} = h_{k-1}$  for  $k \geq 1$  and  $h_k|_{\Sigma(\sigma)} = q_{n(0)-1, x_{W_0}} \circ (h_\sigma|_{\Sigma(\sigma)})$  for  $k \geq 0$  and
- (7)  $f^{-1}(\bar{g}(W_0 \cap B)) \cup h_k(\Sigma(\sigma)) \subset \tilde{U}_{n(0)-1}(x_{W_0})$ ,  $k \geq 0$ .

We also require that

- (8)  $h_{\sigma_1}|_{(\Sigma(\sigma_1) \cap \Sigma(\sigma_2))} = h_{\sigma_2}|_{(\Sigma(\sigma_1) \cap \Sigma(\sigma_2))}$  for any  $\sigma_1$  and  $\sigma_2$  from  $\mathfrak{R}^k$  having the same first vertex.

For  $k = 0$  we define  $h_0 : \Sigma^0 \rightarrow M$  and  $h_{\langle W \rangle} : \Sigma(\langle W \rangle) \rightarrow \tilde{U}_{n-1}(x_W)$  by  $h_0(\Sigma(\langle W \rangle)) = P(x_W)$  and  $h_{\langle W \rangle}(\Sigma(\langle W \rangle)) = Q(x_W)$ , where  $W \in \nu_n$  and  $Q(x_W)$  is a point from  $\tilde{V}_{n-1}(x_W)$  with  $q_{0, x_W}(Q(x_W)) = P(x_W)$ . Obviously,  $h_0$  restricted on every set  $W \cap \Sigma^0$  is constant, so it is continuous. Moreover, every  $h_{\langle W \rangle}$  is also constant satisfying condition (6), and, by (5),  $h_0$  satisfies also (7). Note that condition (8) holds for  $k = 0$ .

Suppose that for some  $k \geq 1$  maps  $h_{k-1} : \Sigma^{k-1} \rightarrow M$  and  $h_\sigma : \Sigma(\sigma) \rightarrow \tilde{U}_{n-1}(x_W)$  satisfying conditions (6), (7) and (8) have already been defined. Here  $\sigma \in \mathfrak{R}^{k-1}$  and  $W \in \nu_m$  is the first vertex of the simplex  $\sigma$ .

Now, let  $\sigma = \langle W_0, W_1, \dots, W_k \rangle \in \mathfrak{R}^k$  with  $W_i \in \nu_{n(i)}$ ,  $i = 0, 1, \dots, k$ . Then  $\sigma \cap \mathfrak{R}^{k-1}$  consists of the simplexes  $\sigma_i = \langle W_0, \dots, W_{i-1}, W_{i+1}, \dots, W_k \rangle$ ,  $i = 1, 2, \dots, k$  and the simplex  $\sigma_0 = \langle W_1, W_2, \dots, W_k \rangle$ .

**Claim.**  $f^{-1}(\bar{g}(W_0 \cap B)) \cup h_{k-1}(\Sigma(\sigma_0)) \subset \tilde{V}_{n(0)-1}(x_{W_0})$  and  $f^{-1}(\bar{g}(W_0 \cap B)) \cup h_{k-1}(\Sigma(\sigma_i)) \subset \tilde{U}_{n(0)-1}(x_{W_0})$  for every  $i = 1, \dots, k$ .

Indeed, by (7) we have  $f^{-1}(\bar{g}(W_1 \cap B)) \cup h_{k-1}(\Sigma(\sigma_0)) \subset \tilde{U}_{n(1)-1}(x_{W_1})$ . But  $\bar{g}(W_1 \cap B) \cap \bar{g}(W_0 \cap B) \neq \emptyset$ , and hence  $f^{-1}(\bar{g}(W_1 \cap B)) \cup h_{k-1}(\Sigma(\sigma_0))$  is contained in  $\text{St}(f^{-1}(\bar{g}(W_0 \cap B)), \tilde{\alpha}_{n(1)-1})$ . Since  $n(0) \leq n(1) - 1$ ,  $\tilde{\alpha}_{n(1)-1}$  refines  $\tilde{\alpha}_{n(0)}$ . This fact and the inclusion  $\text{St}(f^{-1}(\bar{g}(W_0 \cap B)), \tilde{\alpha}_{n(0)}) \subset \tilde{V}_{n(0)-1}(x_{W_0})$ , which follows from (5), complete the proof of the claim for  $i = 0$ . Since  $W_0$  is a vertex of each  $\sigma_i$ ,  $i = 1, 2, \dots, k$ , the other inclusions from the claim follow directly from (7).

Consider the “boundary”  $\partial \Sigma(\sigma) = \bigcup_{i=0}^{i=k} \Sigma(\sigma_i)$  of  $\Sigma(\sigma)$ . According to the claim,  $h_{k-1}(\partial \Sigma(\sigma)) \subset \tilde{U}_{n(0)-1}(x_{W_0})$  and  $h_{k-1}(\overline{\partial \Sigma(\sigma) \setminus \Sigma_0}) \subset \tilde{V}_{n(0)-1}(x_{W_0})$ , where  $\Sigma_0 = \bigcup_{i=1}^{i=k} \Sigma(\sigma_i)$ . Since the maps  $h_{\sigma_i} : \Sigma(\sigma_i) \rightarrow \tilde{U}_{n(0)-1}(x_{W_0})$ ,  $i = 1, \dots, k$ , satisfy condition (8), they determine a map  $h_\Sigma : \Sigma_0 \rightarrow \tilde{U}_{n(0)-1}(x_{W_0})$  such that  $h_{\sigma_i}|_{\Sigma(\sigma_i)} = h_\Sigma|_{\Sigma(\sigma_i)}$  for each  $i$ . Moreover, by (6),  $q_{n(0)-1, x_{W_0}} \circ h_\Sigma = h_{k-1}|_{\Sigma_0}$ . Therefore, we can apply Lemma 3.8 for the pair  $\tilde{V}_{n(0)-1}(x_{W_0}) \subset \tilde{U}_{n(0)-1}(x_{W_0})$ , its  $L$ -extension  $\tilde{\tilde{V}}_{n(0)-1}(x_{W_0}) \subset \tilde{\tilde{U}}_{n(0)-1}(x_{W_0})$ , the sets  $\Sigma_0 \subset \partial \Sigma(\sigma) \subset \Sigma(\sigma)$  and the maps  $h_\Sigma$  and  $h_{k-1}|_{\partial \Sigma(\sigma)}$ . In this way we obtain a map  $h_\sigma : \Sigma(\sigma) \rightarrow \tilde{\tilde{U}}_{n(0)-1}(x_{W_0})$  such that  $q_{n(0)-1, x_{W_0}} \circ h_\sigma|_{\partial \Sigma(\sigma)} = h_{k-1}|_{\partial \Sigma(\sigma)}$ . Now we define  $h_k : \Sigma^k \rightarrow M$  by  $h_k|_{\Sigma(\sigma)} = q_{n(0)-1, x_{W_0}} \circ h_\sigma$ . Obviously,  $h_k$  is continuous on every “simplex”  $\Sigma(\sigma)$ ,  $\sigma \in \mathfrak{R}^k$ , and, since the family  $\nu$  is locally finite in  $\Omega$ ,  $h_k$  is continuous. Moreover,  $h_k$  and  $h_\sigma$  satisfy conditions (6), (7) and (8), and the induction is completed.

Finally, we define  $h : \Omega \rightarrow M$  letting  $h|_{\Sigma^k} = h_k$  for each  $k$ . Continuity of  $h$  follows from continuity of each  $h_k$  and the fact that  $\nu$  is locally finite. Observe also that  $(f \circ h)|_{p^{-1}(A)}$  is  $\omega$ -close to  $g \circ p$  because of condition (7).  $\square$

**Proposition 3.11.** *Let  $L$  be a complex (not necessarily quasi-finite) with the soft map property and  $f_0 : M \rightarrow X$  be a closed map such that each fiber  $f_0^{-1}(x)$ ,  $x \in X$ , is an UV( $L$ )-set in  $M$ . Then for every map  $g_0 : A \rightarrow X$ , where  $A$  is a closed subset of a space  $Z$  with  $\text{e-dim } Z \leq L$  such that either  $A$  or  $g_0(A)$  is a  $C$ -space, there exists a neighborhood  $Q$  of  $A$  in  $Z$  and an u.s.c map  $\Psi : Q \rightarrow M$  such that  $\Psi$  is single-valued on  $Q \setminus A$  and  $f_0 \circ \Psi$  is a continuous single-valued map extending  $g_0$ .*

**Proof.** Our proof is based on some ideas from [2, proof of Theorem 3.1]. Let  $f_0$  and  $g_0$  be as in the proposition. We take sequences  $\{\omega_n\} \subset \text{cov}(X)$  and  $\{\gamma_n\} \subset \text{cov}(A)$ , and open intervals  $\{\Delta_n\}$  covering the interval  $J = [0, 1)$ , with  $0 \in \Delta_1$ , such that:

- $\omega_{n+1}$  is a strong star-refinement of  $\omega_n$  and  $\gamma_{n+1}$  is a strong star-refinement of  $\gamma_n$ ,  $n = 1, 2, 3, \dots$
- $\lim \text{mesh}(\omega_n) = \lim \text{mesh}(\gamma_n) = 0$ .
- $\Delta_n \cap \Delta_m \neq \emptyset$  if and only if  $n$  and  $m$  are consecutive integers.

Then  $\omega = \{\omega_n \times \Delta_n : n = 1, 2, \dots\}$  and  $\gamma = \{\gamma_n \times \Delta_n : n = 1, 2, \dots\}$  are open covers, respectively, of  $X \times J$  and  $A \times J$ , satisfying the following conditions:

- (9<sub>i</sub>) For every point  $(x, 1) \in X \times I$  and its neighborhood  $U$  in  $X \times I$  there exists another neighborhood  $V$  such that  $\text{St}(V, \omega) \subset U$ .
- (9<sub>ii</sub>) For every point  $(a, 1) \in A \times I$  and its neighborhood  $U$  in  $A \times I$  there exists another neighborhood  $V$  such that  $\text{St}(V, \gamma) \subset U$ .

Since  $f_0$  is a closed map all fibers of which are  $UV(L)$ -sets in  $M$ , the map  $f = f_0 \times \text{id} : M \times J \rightarrow X \times J$  has the property (UV) from Lemma 3.10. Further, let  $g$  denote the map  $g_0 \times \text{id} : A \times J \rightarrow X \times J$  and consider an  $L$ -soft surjection  $p : Y \rightarrow Z \times I$ ,  $I = [0, 1]$ , such that  $Y$  is a space of  $e\text{-dim } Y \leq L$ . We have the following diagram:

$$\begin{array}{ccc}
 Y & & M \times J \\
 \downarrow p(L\text{-soft}) & & \downarrow f = f_0 \times \text{id} \\
 Z \times I \supset A \times J & \xrightarrow{g = g_0 \times \text{id}} & X \times J
 \end{array}$$

Since the product of any metrizable  $C$ -space and  $J$  is also a  $C$ -space, either  $A \times J$  or  $g_0(A) \times J$  is a  $C$ -space. Following the notations from Lemma 3.10, we can apply construction of this lemma by considering the spaces  $M \times J$ ,  $X \times J$ ,  $Z \times J$ ,  $A \times J$  and  $p^{-1}(Z \times J)$  instead of the spaces  $M$ ,  $X$ ,  $Z$ ,  $A$  and  $Y$ , respectively. Let us also note that in our situation we take  $\alpha_n$  and  $\beta_n$ ,  $n \geq 0$ , to be open covers of  $X \times J$  satisfying condition (1) from the auxiliary construction with  $\alpha_0$  refining  $\omega$ . We also require  $\mu_n^*$  to be disjoint open families in  $A \times J$  satisfying condition (2) such that  $\mu^* = \bigcup_{n=1}^\infty \mu_n^*$  is a locally finite open cover of  $A \times J$  which, in addition, refines  $\gamma$ . Then, as in the auxiliary construction, we can extend  $\mu_n^*$  to disjoint open families  $\mu_n$  in  $Z \times J$  by choosing an u.s.c. retraction  $r : Z \times I \rightarrow A \times I$  such that  $r(z, t) \subset A \times \{t\}$  for every  $t \in I$ . This can be achieved by taking an u.s.c. finite-valued retraction  $r_1 : Z \rightarrow A$  and letting  $r(z, t) = r_1(z) \times \{t\}$ . Observe that this special choice of  $r$  implies that  $r^\sharp(T)$  is open in  $Z \times I$  for every open  $T \subset A \times I$  and  $r^\sharp(T)$  is contained in  $Z \times J$  provided  $T \subset A \times J$ . We also pick the points  $x_W \in X \times J$ ,  $W \in \mu$ , satisfying condition (4).

According to Lemma 3.10, there exists a map  $h : p^{-1}(G) \rightarrow M \times J$ , where  $G = \bigcup \{A : A \in \mu\}$ , such that each  $h_k = h|_{\Sigma^k}$  satisfies condition (7) and  $(f \circ h)|(p^{-1}(A \times J))$  is  $\omega$ -close to  $g \circ p$ . Now, let  $H = p^{-1}(G \cup (A \times \{1\}))$  and define the set-valued map  $\psi : H \rightarrow M \times I$  letting  $\psi(y) = h(y)$  if  $y \in p^{-1}(G)$  and  $\psi(y) = (f_0^{-1}(g_0(p(y))), 1)$  if  $y \in p^{-1}(A \times \{1\})$ . Let also  $\psi_1 = \pi \circ \psi : H \rightarrow M$ , where  $\pi : M \times I \rightarrow M$  is the projection.

**Claim.** *The map  $\psi_1$  is u.s.c.*

Since  $\pi$  is continuous, it suffices to prove that  $\psi$  is u.s.c. To this end, observe that  $p^{-1}(G)$  is open in  $H$  and  $\psi$  is single-valued and continuous on  $p^{-1}(G)$ , so we need to show only that  $\psi$  is u.s.c. at the points of  $p^{-1}(A \times \{1\})$ . Let  $\{y_i\} \subset H$  be a sequence converging to a point  $y_0 \in p^{-1}(A \times \{1\})$  and  $U_0 = V_0 \times (t, 1]$  be a neighborhood of  $\psi(y_0) = (f_0^{-1}(g_0(p(y_0))), 1)$  in  $M \times I$ . We are going to show that  $\psi(y_i) \subset U_0$  for almost all  $i$  which will complete the proof of the claim. Since  $f_0$  is a closed map,  $\psi$  is u.s.c. on  $p^{-1}(A \times \{1\})$ . Therefore we can assume that  $\{y_i\} \subset p^{-1}(G)$ , hence  $\psi(y_i) = h(y_i)$  for all  $i$ . Thus  $p(y_0) = (a, 1) \in A \times \{1\}$  and  $p(y_i) \in G$ . Since  $f_0$  is closed, we can find a neighborhood  $V$  of  $g_0(p(y_0))$  in  $X$  with  $f_0^{-1}(V) \subset V_0$ . By (9<sub>i</sub>), there exists a neighborhood  $U_1 = V_1 \times (q, 1]$  of  $(g_0(p(y_0)), 1)$  in  $X \times I$  such that  $\text{St}(U_1, \omega) \subset U = V \times (t, 1]$ . Choose a neighborhood  $T(a)$  of  $a$  in  $A$  with  $g_0(T(a)) \subset V_1$  and apply (9<sub>ii</sub>) to find a neighborhood  $S = T_1(a) \times (q^*, 1]$  of  $(a, 1)$  in  $A \times I$  such that  $\text{St}(S, \gamma) \subset T(a) \times (q, 1]$ . Then  $r^\sharp(S)$  is a neighborhood of  $(a, 1)$  in  $Z \times I$ . Since  $\{p(y_i)\}$  converges to  $(a, 1)$ , we can assume that  $\{p(y_i)\} \subset r^\sharp(S)$ . It suffices to show that  $f(h(y_i)) \in U$  for all  $i$ . To this end, fix  $i$  and  $\Lambda_0 \in \mu_{k(0)}$  containing  $p(y_i)$ , where  $k(0)$  is the minimal  $k$  such that  $p(y_i)$  is contained in some element of  $\mu_k$ . Then  $\Lambda_0 = r^\sharp(\Lambda_0^*)$  for some  $\Lambda_0^* \in \mu^*$  and therefore  $p(y_i) \in r^\sharp(\Lambda_0^*) \cap r^\sharp(S)$ . Consequently,  $S$  meets  $\Lambda_0^*$  and let  $p(y_i^*) \in \Lambda_0^* \cap S$ , where  $y_i^* \in p^{-1}(\Lambda_0^*)$ . On the other

hand, there exists  $\Gamma \in \gamma$  containing  $\Lambda_0^*$  (recall that  $\mu^*$  refines  $\gamma$ ). Therefore,  $p(y_i^*) \in \text{St}(S, \gamma) \subset T(a) \times (q, 1]$ . Since  $g(p(y_i^*)) = (g_0 \times \text{id})(p(y_i^*))$ , according to the choice of  $T(a) \times (q, 1]$  we have

$$(10) \quad g(p(y_i^*)) \in U_1 = V_1 \times (q, 1].$$

Since  $k(0)$  is the minimal  $k$  such that  $y_i$  is contained in some  $W \in \nu_k$ , according to the definition of the maps  $h_k$  and condition (7) from Lemma 3.10, we have  $h(y_i) \in \tilde{U}_{k(0)-1}(x_{W_0})$ , where  $W_0 = p^{-1}(A_0)$ . The last inclusion implies  $f(h(y_i)) \in U_{k(0)-1}(x_{W_0})$ . Also, condition (5) from Lemma 3.10 yields that

$$(11) \quad g(p(y_i^*)) \in g(p(W_0 \cap p^{-1}(A \times J))) \subset V_{k(0)-1}(x_{W_0}).$$

Hence, both  $g(p(y_i^*))$  and  $f(h(y_i))$  are points from  $U_{k(0)-1}(x_{W_0})$ . But the cover  $\alpha_{k(0)-1}$  refines  $\omega$ , and hence  $U_{k(0)-1}(x_{W_0})$  is contained in an element  $O$  of  $\omega$ . Therefore,  $O$  contains  $g(p(y_i^*))$  and  $f(h(y_i))$ . This means, according to (10), that  $f(h(y_i)) \in \text{St}(U_1, \omega)$ . Finally, since  $\text{St}(U_1, \omega) \subset U$ , we obtain  $f(h(y_i)) \in U$  which completes the proof of the claim.

Now we can finish the proof. There exists a decreasing sequence  $\{Q_i\}$  of open subsets of  $Z$  and an increasing sequence of real numbers  $0 = t_0 < t_1 < \dots < 1$  such that  $\bigcap_{i=1}^{\infty} Q_i = A$ ,  $\lim t_i = 1$ ,  $\overline{Q_{i+1}} \subset Q_i$  and  $Q_i \times [0, t_i] \subset G$  for all  $i$ . Let  $\varphi_i: Z \rightarrow [t_{i-1}, t_i]$ ,  $i \geq 1$ , be continuous functions such that  $\varphi_i(Z \setminus Q_i) = t_{i-1}$  and  $\varphi_i(z) = t_i$  for  $z \in \overline{Q_{i+1}}$ . Then  $\varphi: Z \rightarrow [0, 1]$  defined by  $\varphi(z) = \varphi_i(z)$  for  $z \in Q_i \setminus Q_{i+1}$ ,  $\varphi(Z \setminus Q_1) = 0$ , and  $\varphi(A) = 1$ , is continuous. Consequently, the map  $\theta: Q_1 \rightarrow G \cup (A \times \{1\})$ ,  $\theta(z) = (z, \varphi(z))$ , is well defined and continuous. Moreover,  $\theta(z) = (z, 1)$  for all  $z \in A$ . Since  $p$  is  $L$ -invertible and  $\text{e-dim } Q_1 \leq L$  (as an open subset of  $Z$ ), we can lift  $\theta$  to a map  $\bar{\theta}: Q_1 \rightarrow H$ . Then  $\Psi = \psi_1 \circ \bar{\theta}: Q \rightarrow M$ , where  $Q = Q_1$ , is the required map.  $\square$

Theorem 3.12 below is a generalization of the well-known result that if  $G$  is an u.s.c. decomposition of a metrizable space  $X$  such that each element of  $G$  is  $UV^n$  in  $X$ , then  $X/G$  is  $LC^n$  [13, Theorem 11]. The result from Theorem 3.12 was also established in [6, Corollary 7.5] for finite complexes  $L$  and proper  $UV(L)$ -maps between Polish spaces ( $UV(L)$ -maps are maps with all fibers being  $UV(L)$ -spaces). The version of Theorem 3.12 when  $L$  is a point is a generalization of the well known result of Ancel [3, Theorem C.5.9]. This version was also established in [12, Proposition 3.5].

**Theorem 3.12.** *Let  $L$  be quasi-finite and  $f: X \rightarrow Y$  be a closed map with all fibers being  $UV(L)$ -sets in  $X$ . Then  $Y$  is an  $ANE(L)$  with respect to  $C$ -spaces. If, in addition,  $X$  is  $C^L$  (i.e., every map into  $X$  is  $L$ -homotopic to a constant map in  $X$ ), then  $Y \in AE(L)$  with respect to  $C$ -spaces.*

**Proof.** Let  $g: A \rightarrow Y$  be an arbitrary map, where  $A$  is a closed subspace of a space  $Z$  with  $\text{e-dim } Z \leq L$ , such that  $A$  is a  $C$ -space. Since  $L$  is quasi-finite, it has the soft mapping property. Therefore we can apply Proposition 3.11 to obtain a neighborhood  $U$  of  $A$  in  $Z$  and an u.s.c. map  $\Psi: U \rightarrow X$  such that  $\Psi$  is single-valued outside  $A$  and  $f \circ \Psi$  is a single-valued extension of  $g$ . Hence,  $Y \in ANE(L)$  with respect to  $C$ -spaces (actually we proved that  $Y \in ANE(g, A, Z)$  for arbitrary  $g: A \rightarrow Y$ , where  $A$  is a closed subspace of  $Z$  such that  $\text{e-dim } Z \leq L$  and  $A$  is a  $C$ -space).

Suppose now that  $X$  is  $C^L$  and let  $A \subset Z$  and  $g: A \rightarrow Y$  be as above. To show that  $Y \in AE(L)$  with respect to  $C$ -spaces, we need to extend  $g$  over  $Z$ . Embedding  $Z$  as a closed subset of an  $AE(L)$ -space with  $\text{e-dim } \leq L$ , we can assume that  $Z \in AE(L)$ . Then, as before, there exists a neighborhood  $U$  of  $A$  in  $Z$  and an u.s.c. map  $\Psi: U \rightarrow X$  such that  $\Psi$  is single-valued outside  $A$  and  $f \circ \Psi$  extends  $g$ . Take neighborhoods  $V_1$  and  $V_2$  of  $A$  in  $Z$  such that  $\overline{V_1} \subset V_2 \subset \overline{V_2} \subset U$ . Let  $W = Z \setminus \overline{V_1}$  and  $F = W \cap \overline{V_2}$ . Since  $W \cap U$  is open in the  $AE(L)$ -space  $Z$ , the cone  $\text{Cone}(W \cap U)$  is an  $AE(L)$ . So, the inclusion  $F \subset W \cap U$  can be extended to a map  $\varphi: W \rightarrow \text{Cone}(W \cap U)$  because  $F$  is closed in  $W$  and  $\text{e-dim } W \leq L$ . On the other hand, since  $X \in C^L$ ,  $\Psi|_{(W \cap U)}$  is  $L$ -homotopic to a constant map in  $X$ . Consequently, the map  $\Psi|_F$  can be extended to a map  $h: W \rightarrow X$ . Finally, we define the set-valued map  $\theta: Z \rightarrow X$  by  $\theta(z) = h(z)$  if  $z \in Z \setminus V_2$  and  $\theta(z) = \Psi(z)$  otherwise. Obviously,  $\theta$  is u.s.c. and single-valued outside  $A$ . Moreover,  $f \circ \theta$  is the required extension of  $g$ .  $\square$

We say that a space  $X$  is locally  $ANE(L)$  if every point from  $X$  is  $UV(L)$  in  $X$ . Let us mention the following corollary from Theorem 3.12.

**Corollary 3.13.** *Let  $Y$  be locally  $ANE(L)$ , where  $L$  is quasi-finite. Then  $Y \in ANE(L)$  with respect to  $C$ -spaces. If, in addition,  $Y \in C^L$ , then  $Y \in AE(L)$  with respect to  $C$ -spaces.*

**Remark.** We can show that if, in Corollary 3.13, the property of  $X$  to be locally  $ANE(L)$  is replaced by the weaker one  $X$  to be  $LC^L$  (every  $x \in X$  is  $UV(L)$ -homotopic in  $X$  [10]), then  $X$  is an  $ANE(L)$  with respect to finite-dimensional spaces (see also [6, Theorem 4.1] for a similar result).

We know that the domain and the range of a  $UV^n$ -map between compacta are simultaneously  $UV^n$  (see, for example [5]). Here is a generalization of this result for a subclass of quasi-finite complexes. We say that a CW complex  $L$  is a  $C$ -complex if every space of  $e\text{-dim} \leq L$  is a  $C$ -space. Each complex  $L$  with  $L \leq S^n$  for some  $n$  (this means that  $e\text{-dim } Z \leq L$  implies  $\dim Z \leq n$  for any space  $Z$ ) is a  $C$ -complex, in particular every sphere  $S^k$  is such a complex. Observe that Lemma 3.10 and Proposition 3.11 remain valid for  $C$ -complexes  $L$  having the soft map property without the requirements either  $A$  or  $g(A)$  (respectively,  $g_0(A)$ ) to be  $C$ -spaces. This yields that, if in Theorem 3.12 and Corollary 3.13  $L$  is a quasi-finite  $C$ -complex, then  $Y$  is an  $A(N)E(L)$ .

**Theorem 3.14.** *Let  $L$  be a quasi-finite  $C$ -complex and  $f : X \rightarrow Y$  a closed map with  $UV(L)$ -fibers. Then  $X$  is  $UV(L)$  if and only if  $Y$  is.*

**Proof.** Let  $E_X$  be a normed space containing  $X$  as a strong  $Z$ -set. This means that  $X \subset E_X$  is closed and for every  $\omega \in \text{cov}(E_X)$  and every map  $g : Q \rightarrow E_X$ , where  $Q$  is an arbitrary space, there is another map  $h : Q \rightarrow E_X$  which is  $\omega$ -close to  $g$  and  $\overline{h(Q)} \cap X = \emptyset$  (such space  $E_X$  can be constructed as follows: embed  $X$  as a closed subset of a normed space  $F$  and let  $E_X$  be the product  $F \times l_2(\tau)$ , where  $w(X) \leq \tau$ ; then  $X \times \{0\}$  is a copy of  $X$  which is a strong  $Z$ -set in  $E_X$ ). Identifying each fiber of  $f$  with a point, we obtain space  $E_Y$  (equipped with the quotient topology) and let  $p : E_X \rightarrow E_Y$  be the natural quotient map. Obviously,  $p(X) \subset E_Y$  is closed and, since  $f$  is a closed map,  $p(X)$  is homeomorphic to  $Y$ . Everywhere below we write  $Y$  instead of  $p(X)$ . Moreover,  $p$  is a closed map and  $E_Y$  is metrizable. Any fiber of  $p$  is either a point or  $f^{-1}(y)$  for some  $y \in Y$ . Hence,  $p$  is an  $UV(L)$ -map. Since  $E_X$  is an absolute extensor for metrizable spaces, the fibers of  $p$  are  $UV(L)$ -sets in  $E_X$ . Consequently, by the modified version of Theorem 3.12 for  $C$ -complexes,  $E_Y \in AE(L)$ .

$X \in UV(L) \implies Y \in UV(L)$ . To prove this implication, by Corollary 3.7, it suffices to show that  $Y$  is  $UV(L)$  in  $E_Y$ . Let  $U$  be a neighborhood of  $Y$  in  $E_Y$ . Since  $X$  is  $UV(L)$  in  $E_X$  (recall that  $E_X$  is an absolute extensor) and  $p$  is closed, there exists a neighborhood  $V$  of  $Y$  in  $E_Y$  such that the pair  $p^{-1}(V) \subset p^{-1}(U)$  is  $L$ -connected. We choose a neighborhood  $V_1$  of  $Y$  in  $E_Y$  with  $\overline{V_1} \subset V$  and show that the pair  $V_1 \subset U$  is  $L$ -connected. To this end, take a space  $Z$  with  $e\text{-dim } Z \leq L$  and a map  $h : A \rightarrow V_1$  with  $A \subset Z$  being closed. Since  $U$  is an  $ANE(L)$ , there exists  $\omega \in \text{cov}(U)$  satisfying condition (H) from Proposition 3.2. Further, let  $\beta \in \text{cov}(E_Y)$  be the cover  $\{G \cap V : G \in \omega\} \cup \{E_Y \setminus \overline{V_1}\}$ . By Lemma 3.10, there exists a map  $h_1 : A \rightarrow E_X$  such that  $p \circ h_1$  is  $\beta$ -close to  $h$ . Obviously,  $h_1(A) \subset p^{-1}(V)$  and hence there exists an extension  $h_2 : Z \rightarrow p^{-1}(U)$  of  $h_1$ . Then  $p \circ h_2$  is a map from  $Z$  into  $U$  such that  $(p \circ h_2)|_A$  is  $\omega$ -close to  $h$ . Finally, according to the choice of  $\omega$ ,  $h$  admits an extension  $\bar{h} : Z \rightarrow U$ .

$Y \in UV(L) \implies X \in UV(L)$ . As in the previous implication, it suffices to show that  $X$  is  $UV(L)$  in  $E_X$ . To this end, let  $U$  be a neighborhood of  $X$  in  $E_X$ . We can assume that  $U \equiv p^{-1}(U_0)$  for some neighborhood  $U_0$  of  $Y$  in  $E_Y$ . Choose neighborhoods  $V_0, G_0$  and  $W_0$  of  $Y$  such that  $V_0 \subset \overline{V_0} \subset G_0 \subset \overline{G_0} \subset W_0 \subset \overline{W_0} \subset U_0$  and the pair  $G_0 \subset W_0$  is  $L$ -connected. Denote by  $V, G$  and  $W$ , respectively, the preimages  $p^{-1}(V_0), p^{-1}(G_0)$  and  $p^{-1}(W_0)$ . We claim that the pair  $V \subset U$  is  $L$ -connected. Indeed, consider a map  $g_V : A \rightarrow V$ , where  $A$  is a closed subset of a space  $Z$  with  $e\text{-dim } Z \leq L$ . Let  $\alpha \in \text{cov}(U)$  satisfy condition (H) from Proposition 3.2 and  $\alpha_1 = \{T \cap G : T \in \alpha\} \cup \{E_X \setminus \overline{V}\} \in \text{cov}(E_X)$ . Since  $X$  is a strong  $Z$ -set in  $E_X$ , we can find a map  $g_G : A \rightarrow E_X$  which is  $\alpha_1$ -close to  $g_V$  and  $\overline{g_G(A)} \cap X = \emptyset$ . It is easily seen that  $g_G(A) \subset G$  and  $g_G$  is  $\alpha$ -close to  $g_V$ . The last yields (because of the choice of  $\alpha$ ) that  $g_V$  can be extended to a map from  $Z$  into  $U$  if and only if  $g_G$  has such an extension. Hence, our proof is reduced to show that  $g_G$  admits an extension from  $Z$  into  $U$ . Obviously,  $g_G$  can be considered as a map from  $A$  into  $G_0$  such that the closure  $\overline{g_G(A)}$  (this is a closure in  $E_Y$ ) does not meet  $Y$ . Since  $G_0 \subset W_0$  is  $L$ -connected,  $g_G$  can be extended to a map  $g_W : Z \rightarrow W_0$ . Finally, consider the cover  $\gamma \in \text{cov}(E_Y)$  defined by  $\gamma = \{p(T \setminus X) : T \in \alpha\} \cup \{E_Y \setminus \overline{g_G(A)}\} \cup \{E_Y \setminus \overline{W_0}\}$ . According to Lemma 3.10, there exists a map  $g_U : Z \rightarrow E_X$  such that  $p \circ g_U$  is  $\gamma$ -close to  $g_W$ . It is easily seen that  $g_U(Z) \subset U$  and  $g_U|_A$  is  $\alpha$ -close to  $g_G$ . The last condition implies that  $g_G$  admits an extension from  $Z$  into  $U$  which completes our proof.  $\square$

## References

- [1] D. Addis, J. Gresham, A class of infinite-dimensional spaces. Part I: Dimension theory and Alexandroff's problem, *Fund. Math.* 101 (1978) 195–205.
- [2] S. Ageev, D. Repovš, A method of approximate extension of maps in theory of extensors, *Sibirsk. Mat. Z.* 43 (4) (2002) 739–756 (in Russian).
- [3] F. Ancel, The role of countable dimensionality in the theory of cell-like relations, *Trans. Amer. Math. Soc.* 287 (1) (1985) 1–40.
- [4] P. Alexandrov, B. Pasynkov, *Introduction to Dimension Theory*, Nauka, Moscow, 1973.
- [5] M. Bestvina, Characterizing  $k$ -dimensional universal Menger compacta, *Mem. Amer. Math. Soc.* 71 (380) (1988).
- [6] N. Brodsky, A. Chigogidze, A. Karasev, Approximations and selections of multivalued mappings of finite-dimensional spaces, *JP J. Geom. Topology* 2 (1) (2002) 29–73.
- [7] N. Brodsky, A. Chigogidze, Extension dimensional approximation theorem, *Topology Appl.* 125 (3) (2002) 385–391.
- [8] A. Chigogidze, Cohomological dimension of Tychonov spaces, *Topology Appl.* 79 (3) (1997) 197–228.
- [9] A. Chigogidze, Compactifications and universal spaces in extension theory, *Proc. Amer. Math. Soc.* 128 (2000) 2187–2190.
- [10] A. Chigogidze, Infinite dimensional topology and shape theory, in: R.J. Daverman, R.B. Sher (Eds.), *Handbook of Geometric Topology*, Elsevier, Amsterdam, 2002, pp. 307–371.
- [11] A. Chigogidze, V. Valov, Universal metric spaces and extension dimension, *Topology Appl.* 113 (2001) 23–27.
- [12] A. Chigogidze, V. Valov, Extension dimension and  $C$ -spaces, *Bull. London Math. Soc.* 34 (2002) 708–716.
- [13] R. Daverman, *Decomposition of Manifolds*, Academic Press, New York, 1986.
- [14] A.N. Dranishnikov, The Eilenberg–Borsuk theorem for mappings into an arbitrary complex, *Russian Acad. Sci. Sb.* 81 (1995) 467–475.
- [15] A.N. Dranishnikov, J. Dydak, Extension dimension and extension types, *Proc. Steklov Inst. Math.* 212 (1) (1996) 55–88.
- [16] J. Dydak, Extension theory: The interface between set-theoretic and algebraic topology, *Topology Appl.* 74 (1–3) (1996) 225–258.
- [17] J. Dydak, J. Mogilski, Universal cell-like maps, *Proc. Amer. Math. Soc.* 122 (3) (1994) 943–948.
- [18] J. Dydak, J. Walsh, Spaces without cohomological dimension preserving compactifications, *Proc. Amer. Math. Soc.* 113 (1991) 1155–1162.
- [19] R. Engelking, *Theory of Dimensions: Finite and Infinite*, Heldermann, Lemgo, 1995.
- [20] A. Karasev, On two problems in extension theory, *Proc. Amer. Math. Soc.*, in press.
- [21] M. Katetov, On the dimension of non-separable spaces. I, *Czech. Math. J.* 2 (1952) 333–368 (in Russian).
- [22] M. Levin, Some examples in cohomological dimension theory, *Pacific J. Math.* 202 (2) (2002) 371–378.
- [23] M. Levin, On extensional dimension of metrizable spaces, Preprint.
- [24] M. Levin, L. Rubin, P. Schapiro, The Mardešić factorization theorem for extension theory and  $C$ -separation, *Proc. Amer. Math. Soc.* 128 (2000) 3099–3106.
- [25] S. Nedev, Selections and factorization theorems for set-valued mappings, *Serdica* 6 (1980) 291–317.
- [26] W. Olszewski, Completion theorem for cohomological dimensions, *Proc. Amer. Math. Soc.* 123 (1995) 2261–2264.
- [27] J. West, Open problems in infinite-dimensional topology, in: *Open Problems in Topology*, North-Holland, Amsterdam, 1990.
- [28] M. Zarichnyi, Universal spaces and absolute extensors for integral cohomological dimension, *Topology Appl.*, submitted for publication.