Topology in North Bay: some problems in continuum theory, dimension theory and selections

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Dedicated to Ted Chase

1. Introduction

This article reports on some of the research activities in Topology at Nipissing University. Although our research areas encompass geometric topology, dimension theory, general topology, topological algebras, functional analysis, continuum theory and topological dynamics,in this article we only concentrate on some problems in dimension theory, selections and continuum theory. Section 2 is devoted to the problems on extension dimension. In the third section, problems concerning selections and C-spaces are discussed. The fourth section discusses questions concerning the parametric version of disjoint disks property. The last section is devoted to locally connected Hausdorff continua and rim-metrizablity.

Historically, since 1994, there have been regular Topology workshops in the month of May. Initially these workshops were organized by Tuncali. In 2000, Vesko Valov joined Nipissing University, and in 2003, Alexandre Karasev joined this group. The group has been organizing ongoing seminars and workshops at Nipissing. Since 1994, many topologists have visited Nipissing and participated in workshops and seminars. Among them, Nikolay Brodskiy, Dale Daniel, John C. Mayer, Jacek Nikiel and E.D. Tymchatyn have been visiting Nipissing regularly. During the last three years, Jacek Nikiel (2003–04), Taras Banakh (2004–2005) and Andriy Zagorodnyuk (2005–06) have visited Nipissing for entire academic years. Thus, this article is about some of the research interests of the topology group at Nipissing as well as some of the ongoing research programs of seminar/workshop participants.

2. Problems in dimension theory

All spaces in this section are assumed to be metrizable and separable, if not stated otherwise. Let G be an Abelian group. As usual, $K(G,n)$ is the *Eilenberg–Mac Lane complex*, i.e., a CW complex such that $\pi_n(K(G,n)) \approx G$ and $\pi_i(K(G,n)) \approx 0$ for all $i \neq n$. The *cohomological dimension* of a space X with respect to the coefficient group G is denoted by $\dim_G X$. Recall that a space Y is an *absolute (neighborhood) extensor* for X [notation: $Y \in A(N)E(X)$] if any map

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to Y, defined on an arbitrary closed subspace A of X, can be extended to a map of the whole X to Y (resp., to a map of some open neighborhood of A to Y).

Following Dranishnikov [**27**] we say that *extension dimension* of a space X does not exceed a complex L, notation $\text{e-dim}\, X \leq L$, if $L \in AE(X)$. Note that $\dim X \leq n \Leftrightarrow \text{e-dim}\, X \leq S^n$ and $\dim_G X \leq n \Leftrightarrow \text{e-dim}\, X \leq K(G,n)$. Further, we say [**27**] that $L \leq K$ if for each space X the condition $L \in AE(X)$ implies the condition $K \in AE(X)$. The relation $L \leq K$ leads to a preorder relation on complexes and generates an equivalence relation. Obviously, this equivalence relation can be described in terms of extension of maps: a complex L is equivalent to a complex K if $L \in AE(X) \Leftrightarrow K \in AE(X)$ for any space X. The equivalence class of complex L is called the extension type of L and is denoted by $[L]$. Due to the homotopy extension property of ANR-spaces, if L is homotopy equivalent to K, then $[L] = [K]$. The converse is not the case. For example, $[S^n] = [S^n \vee S^m]$, if $n \leq m$. It should be emphasized that there are many incomparable complexes. For instance, results of [**29**] imply that $\mathbb{R}P^2$ is not comparable with any sphere S^n, $n \geq 2$. It can be shown that for complexes L and K the minimum of their extension types is given by the extension type $[L \vee K]$. We refer the reader to the papers [**12**, **28**, **34**] for more information about extension dimension and extension types.

Extension and cohomological dimensions are related as follows.

Theorem 2.1 (Dranishnikov [**26**])**.** *Let X be a metrizable compactum and L be a complex. If* $\text{e-dim}\, X \leq L$ *then* $\dim_{H_n(L)} \leq n$ *for all positive integers n. If* $\dim X < \infty$ *and L is a simply connected complex then the following three conditions are equivalent:* (1) $\text{e-dim}\, X \leq L$*;* (2) $\dim_{H_n(L)} X \leq n$ *for all $n > 0$;* (3) $\dim_{\pi_n(L)} X \leq n$ *for all $n > 0$.*

Dydak generalized the above theorem on the case of metrizable noncompact spaces [**31**, **33**]. The condition $\dim X < \infty$ cannot be removed due to the existence of infinite-dimensional compacta of finite integral cohomological dimension [**24**, **38**]. Nevertheless, Dydak proved in [**31**] that condition (3) implies condition (1) if X is a metrizable space which is an absolute neighborhood extensor for metrizable spaces. There is a hope that Theorem 2.1 remains valid in the class of C-spaces (see next section for the definition of C-spaces).

1505? **Problem 2.1.** *Does Theorem 2.1 hold if the compactum X is a C-space?*

Another generalization of Theorem 2.1 belongs to Cencelj and Dranishnikov [**8**–**10**] and weakens the requirement of simply connectedness. Namely, Theorem 2.1 remains true for nilpotent complexes. The condition on the fundamental group of L cannot be dropped completely. Indeed, if X is a two-dimensional disk and L is a complex such that $\pi_1(L)$ is nontrivial and $\widetilde{H}_*(L) = 0$ then the implication (2) $\Rightarrow$ (1) of the above theorem does not hold [**35**]. An example of such L can be found, for instance, in [**47**, Example 2.38, p. 142].

Finally, consider the equivalence (1) $\Leftrightarrow$ (3) and take $\mathbb{R}P^2$ as a simplest example of a complex which is not nilpotent. Note that $H_1(\mathbb{R}P^2) \approx \pi_1(\mathbb{R}P^2) \approx \mathbb{Z}_2$ and hence, for any metrizable compactum X, $\text{e-dim}\, X \leq \mathbb{R}P^2$ implies $\dim_{\mathbb{Z}_2} X \leq 1$

due to Theorem 2.1. What can be said about the converse implication? The results of Levin [**51**] imply the existence of a metrizable compactum X such that $\dim_{\mathbb{Z}_2} X = 1$ and $\text{e-dim}\, X \not\leq \mathbb{R}P^2$. Nevertheless, this compactum is infinite-dimensional and the following well-known problem is still open.

Problem 2.2. *Does* $\dim_{\mathbb{Z}_2} X \leq 1$ *imply* $\text{e-dim}\, X \leq \mathbb{R}P^2$ *for a finite dimensional compactum* X*?* 1506?

Since the infinite projective space $\mathbb{R}P^\infty$ is the space of type $K(\mathbb{Z}_2, 1)$, the above problem can be restated as follows: does $\mathbb{R}P^\infty \in AE(X)$ imply $\mathbb{R}P^2 \in AE(X)$ for a finite dimensional compactum X? Dydak and Levin proved in [**35**] that the answer is positive if $\dim X \leq 3$.

The study of universal spaces of a given dimension occupies one of the central places in dimension theory. Recall that a space U is called *a universal space* for a class of topological spaces $\mathcal{C}$ if $U \in \mathcal{C}$ and any space X from $\mathcal{C}$ admits an embedding in U. Here are several classical papers devoted to the topic of universal spaces of a given dimension: [**5**, **55**, **56**, **61**, **63**, **66**, **73**]. In connection with the universal spaces theme one should mention the following unsolved problem.

Problem 2.3 (West [**72**])**.** *Does there exist a universal metric compactum of a given integral cohomological dimension?* 1507?

Employing the concept of extension dimension, the above problem can be generalized as follows.

Problem 2.4 (Chigogidze [**12**])**.** *Let* $\mathcal{C}_L$ *denote the class of all metrizable compacta* X *such that* $\text{e-dim}\, X \leq [L]$*, where* L *is a countable and locally finite complex. Characterize all such complexes* L *for which* $\mathcal{C}_L$ *contains a universal space.* 1508?

Problem 2.3 is obtained from Problem 2.4 by letting $L = K(\mathbb{Z}, n)$. Everywhere below, by $[L]$-universal compactum we mean a universal object for the class $\mathcal{C}_L$.

Problem 2.4 has partial solutions. Results of Chigogidze [**12**, Theorem 2.5] and Dydak [**33**] imply that a universal compactum exists in the case when L is finite or, more generally, finitely dominated. A standard way of obtaining such a universal compactum consists in construction of $[L]$-invertible map $f\colon X_L \to \mathbb{I}^\omega$, where X_L is a metric compactum with $\text{e-dim}\, X_L = [L]$.

Definition ([**12**])**.** A map $f\colon X \to Y$ is called $[L]$-*invertible* if for each space Z with $\text{e-dim}\, Z \leq [L]$ and for any map $g\colon Z \to Y$ there exists a map $h\colon Z \to X$ satisfying the conditions $f \circ h = g$.

Note that $[L]$-invertibility of L and universality of $\mathbb{I}^\omega$ for all metrizable compacta guarantees $[L]$-universality of X_L.

A universal object exists for any (countable and locally finite) complex L if we enlarge the class of spaces. Dydak and Mogilski [**36**] proved that for a given n there exists a Polish space X of integral cohomological dimension n which contains a topological copy of any separable metric space Y with $\dim_{\mathbb{Z}} Y \leq n$. This result was generalized by Olszewski [**62**], who proved the existence of a universal separable metric space of given extension dimension $[L]$ (where L is a countable and locally finite CW complex). As a corollary, this implies the existence of universal separable metrizable space of a given cohomological dimension with respect to any countable Abelian coefficient group.

There is an important connection between the existence of an $[L]$-universal compactum, $[L]$-invertible mappings, and the following question [**11**]: for a given complex L, does $\text{e-dim}\, X \leq L$ imply $\text{e-dim}\, \beta X \leq L$ for any space X? The study of this connection led to the introduction of a new class of CW complexes.

Definition ([**48**, **49**])**.** We say that a complex L is *quasi-finite* if for every finite subcomplex P of L there exists a finite subcomplex eP of L containing P such that the pair $P \subset eP$ is $[L]$-connected for Polish spaces. The latter means that any map $f\colon A \to P$, defined on a closed subset A of a Polish space X with $\text{e-dim}\, X \leq L$ can be extended to a map of X into eP.

The following theorem [**11**, **48**] reveals the relation between $[L]$-universality and quasi-finite complexes. Equivalences (ii)–(vi) of this theorem are due to Chigogidze [**11**].

Theorem 2.2. *Let L be a countable and locally finite CW complex. Then the following conditions are equivalent:*

(i) *L is quasi-finite;*
(ii) $\text{e-dim}\, \beta X \leq [L]$ *whenever X is a (Tychonoff) space with* $\text{e-dim}\, X \leq [L]$*;*
(iii) $\text{e-dim}\, \beta X \leq [L]$ *whenever X is normal and* $\text{e-dim}\, X \leq [L]$*;*
(iv) $\text{e-dim}\, \beta(\bigoplus\{X_t : t \in T\}) \leq [L]$ *whenever T is arbitrary and each X_t, $t \in T$, is a separable metrizable space with* $\text{e-dim}\, X_t \leq [L]$*;*
(v) $\text{e-dim}\, \beta(\bigoplus\{X_t : t \in T\}) \leq [L]$ *whenever T is arbitrary and each X_t, $t \in T$, is a Polish space with* $\text{e-dim}\, X_t \leq [L]$*;*
(vi) *There exists an $[L]$-invertible map $f\colon X \to \mathbb{I}^\omega$, where X is a metrizable compactum with* $\text{e-dim}\, X \leq [L]$.

From the results of Dranishnikov [**23**], Dydak [**32**], Dydak and Walsh [**37**], and Levin [**52**], we know that for each $n \geq 2$ there exists a (metrizable separable) space X with integral cohomological dimension $\dim_{\mathbb{Z}} X \leq n$ and $\dim_{\mathbb{Z}} \beta X > n$. Therefore the Eilenberg–Mac Lane complex $K(\mathbb{Z}, n)$ is not quasi-finite for all $n \geq 2$. In fact, the results of Levin [**52**] imply that $K(G, 2)$ is not quasi-finite for any nontrivial Abelian group G.

Quasi-finite complexes provide a negative answer to the following question by Chigogidze [**13**] and Dydak [**33**]: suppose that $[L]$-universal compactum exists; is it true that the extension type of L contains a finitely dominated complex? There exists a quasi-finite complex which is not equivalent to a finitely dominated complex [**48**].

In the light of the preceding discussion, a natural question to ask is the following.

1509? **Problem 2.5.** *Let L be a complex such that $[L]$-universal compactum exists. Is it true that L is quasi-finite?*

Further, the example of a quasi-finite complex in [**48**] is a bouquet of finite complexes. What are other possible examples?

Problem 2.6. *Is there a quasi-finite complex which is not equivalent to a bouquet of finite complexes?* 1510?

Recalling main results about universal spaces in classical dimension n, one may vary Problem 2.4 as follows.

Problem 2.7. *Characterize all complexes L for which $\mathcal{C}_L$ contains a universal object which is an absolute extensor in dimension $[L]$ for Polish spaces (or metrizable compacta).* 1511?

As usual, a space Y is called an absolute (neighborhood) extensor in dimension $[L]$, shortly $Y \in A(N)E([L])$, for a given class of spaces $\mathcal{C}$ if $Y \in A(N)E(X)$ for all X from $\mathcal{C}$ such that $\operatorname{e-dim} X \leq [L]$.

If a complex L is finite, then $[L]$-universal compact absolute extensors in dimension L exist [**12**]. On the other hand, Zarichnyi [**74**] proved that there is no universal compactum of a given integral cohomological dimension which is an absolute extensor with respect to metrizable compacta of given cohomological dimension. Thus, in the Problem 2.7 the complex L cannot be the Eilenberg–Mac Lane complex $K(\mathbb{Z}, n)$, $n \geq 2$.

There is some hope that quasi-finite complexes may be candidates to provide a solution to Problem 2.7. Namely, it is shown in [**50**] that if there exists an $[L]$-universal compactum which is an absolute extensor in dimension $[L]$ for Polish spaces then L must be quasi-finite.

3. Selections and C-property

All spaces in this section are supposed to be paracompact and all maps continuous. By a perfect space we mean a space without isolated points.

Recall that a X is a *C-space* if for any sequence $\{\nu_n\}_{n=1}^{\infty}$ of open covers of X there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of disjoint open families in X such that each γ_n refines ν_n and $\bigcup_{n=1}^{\infty} \nu_n$ is a cover of X. Every countable-dimensional (a countable union of 0-dimensional subsets) metric space is a C-space [**40**]. R. Pol constructed a metrizable compact C-space which is not countable-dimensional [**65**].

Problem 3.1 (V. Gutev)**.** *Let $f\colon X \to Y$ be an open surjective map between the metrizable spaces X and Y such that Y is a C-space and each fiber $f^{-1}(y)$, $y \in Y$ is (zero-dimensional) compact and perfect. Does there exist a surjective map $g\colon X \to Y \times \mathbb{I}$ such that $f = \pi_Y \circ g$, where $\pi_Y : Y \times \mathbb{I} \to Y$ is the projection?* 1512?

According to a result of Bula [**7**, Theorem 1], Problem 3.1 has an affirmative answer in case X and Y are metrizable and Y is finite-dimensional. Gutev [**44**, Theorem 1.1] extended the Bula theorem for arbitrary metrizable X and countable-dimensional Y.

Observe that Problem 3.1 is equivalent to the following one:

1513? **Problem 3.2** (V. Gutev)**.** *Is it true that under the hypotheses of Question 2.1 there exists a map* $h\colon X \to \mathbb{I}$ *with* $h(f^{-1}(y)) = \mathbb{I}$ *for every* $y \in Y$*, equivalently,* $f(h^{-1}(t)) = Y$ *for all* $t \in \mathbb{I}$?

Levin and Rogers [**53**, Theorem 1.3] proved that Problem 3.2 has a positive solution in the class of compact space. Therefore, the answer to Problem 3.1 is also "yes" for compact X and Y. In fact, one can try to solve a simplified version of Problem 3.1 and Problem 3.2. It is easily seen that a positive answer to one of this questions implies a positive answer to next question.

1514? **Problem 3.3** (V. Gutev). *Let* X*,* Y *and* f *be as in Problem 3.1. Are there closed sets* $F, H \subset X$ *such that* $F \cap H = \emptyset$ *and* $f(F) = f(H) = Y$*?*

Note that Dranishnikov [**25**] constructed an open surjection $f\colon X \to Y$ of metrizable compacta having all fibers homeomorphic to the Cantor set and such that there are no disjoint closed sets $F, H \subset X$ with $f(F) = f(H) = Y$. Hence, Problems 3.1 and 3.2 have a negative answer if there is no dimensional restrictions on Y.

There exists an equivalent version of Problem 3.3 in terms of semi-continuous selections. Recall that a set-valued map $\varphi\colon Y \to \mathcal{S}(X)$, where $\mathcal{S}(X)$ denotes the family of all nonempty subsets of X, is called lower (resp., upper) semi-continuous if for every open set $U \subset X$ the set $\{y \in Y : \varphi(y) \cap U \neq \emptyset\}$ (respectively, $\{y \in Y : \varphi(y) \subset U\}$) is open in Y. By $\mathcal{C}(X)$ we denote the family of compact nonempty subsets of X.

1515? **Problem 3.4** (V. Gutev)**.** *Let* Y *be a metrizable* C*-space,* X *be metrizable and* $\varphi\colon Y \to \mathcal{C}(X)$ *be an l.s.c. map such that each* $\varphi(y)$*,* $y \in Y$*, is perfect. Does there exist a u.s.c. map* $\theta\colon Y \to \mathcal{C}(X)$ *with* $\theta(y) \subset \varphi(y)$ *and* $\varphi(y) \setminus \theta(y) \neq \emptyset$ *for every* $y \in Y$*?*

The existence of a u.s.c. map $\theta\colon Y \to \mathcal{C}(X)$ satisfying the conditions from Problem 3.4 is equivalent to the existence of two u.s.c. maps $\theta_i\colon Y \to \mathcal{C}(X)$, $i = 1, 2$, such that $\theta_1(y) \cap \theta_2(y) = \emptyset$ and $\theta_i(y) \subset \varphi(y)$ for all $y \in Y$ and $i = 1, 2$ (in such a case we say that φ admits disjoint u.s.c. selections). Actually, Dranishnikov's example mentioned above is based on this observation, he constructed an open surjection $f\colon X \to Y$ such that the map $\varphi(y) = f^{-1}(y)$ does not admit any disjoint u.s.c. selections.

On the other hand, there exist a few characterizations of paracompact C-spaces in terms of selections for set-valued maps. One of them was established by Uspenskij [**71**, Theorem 1.3] and another one by Gutev–Valov [**45**]. So, it is interesting whether the selection condition from Problem 3.4 also characterizes C-spaces.

1516? **Problem 3.5.** *Is it true that a metrizable space* Y *is a* C*-space if and only if any l.s.c. map* $\varphi\colon Y \to \mathcal{C}(X)$ *with perfect point-images* $\varphi(y)$*,* $y \in Y$*, and metrizable* X *admits disjoint u.s.c. selections*?

4. Parametrization of the disjoint n-disks property

In this section, unless stated otherwise, all spaces are Tychonoff.

The following property was introduced in [**4**] as a parametrization of the well know *disjoint n-disks property*. We say that a space X has the m-$\overline{\mathrm{DD}}^{\{n,k\}}$-*property*, where m,n,k are positive integers or infinity, if for if for any open cover $\mathcal{U}$ of X and two maps $f\colon \mathbb{I}^m\times\mathbb{I}^n\to X$, $g\colon \mathbb{I}^m\times\mathbb{I}^k\to X$ there exist maps $f'\colon \mathbb{I}^m\times\mathbb{I}^n\to X$, $g'\colon \mathbb{I}^m\times\mathbb{I}^k\to X$ such that $f'\sim_{\mathcal{U}} f$, $g'\sim_{\mathcal{U}} g$, and $f'(\{z\}\times\mathbb{I}^n)\times g'(\{z\}\times\mathbb{I}^k\} = \emptyset$ for all $z\in\mathbb{I}^m$. Here $f'\sim_{\mathcal{U}} f$ means that f' is $\mathcal{U}$-homotopic to f.

The importance of the m-$\overline{\mathrm{DD}}^{\{n,k\}}$-property is justified by the following results established in [**4**]:

(1) Let $X\in m$-$\overline{\mathrm{DD}}^{\{n,n\}}$ be a locally contractible and completely metrizable space and $p\colon K\to M$ be a perfect map between metrizable C-spaces with $\dim M\le m$ and $\dim f\le n$. Then the function space $C(K,X)$ equipped with the source limitation topology contains a dense G_δ-subset consisting of maps that are injective on each fiber of p.

(2) Let m,n,k,d,l be non-negative integers, L be a metrizable space with the 0-$\overline{\mathrm{DD}}^{\{0,0\}}$-property and D be a metrizable space with the 0-$\overline{\mathrm{DD}}^{\{0,d+l\}}$-property. If $m+n+k<2d+l$, then the product $D^d\times L^l$ has the m-$\overline{\mathrm{DD}}^{\{n,k\}}$-property.

It follows from the above two results that $\mathbb{D}^d\times\mathbb{R}^l\in m$-$\overline{\mathrm{DD}}^{\{n,k\}}$ for any m,n,k,d,l with $m+n+k<2d+l$, where $\mathbb{D}$ is a dendrite with a dense set of end-points and $\mathbb{R}$ is the real line. The last statement with $m=d=0$ and $l=2n+1$ is actually the Lefschetz–Menger–Pontrjagin embedding theorem; the case $m=l=0$ and $d=n+1$ is the embedding theorem of Bowers [**6**]; the case $m=0$, $d=n$ and $l=1$ is the embedding theorem of Sternfeld [**68**]; the case $m=0$ and $d=0$ is close to the embedding theorem from Banakh–Trushchak [**3**], while for $l=0$ and $m=0$ it is close to that one from Banakh–Cauty–Trushchak–Zdomskyy [**1**]; finally, letting $d=0$ we obtain the Pasynkov theorem [**64**] asserting that for a map $p\colon X\to Y$ between compacta the function space $C(X,\mathbb{R}^{\dim Y+2\dim(p)+1})$ contains a dense G_δ-set of maps that are injective on each fiber of the map p.

However, another generalization of Pasynkov's result due to H. Toruńczyk [**69**] is not covered by the statements (1) and (2):

> If $p\colon X\to Y$ is a map between compacta, then $C(X,\mathbb{R}^{\dim X+\dim(p)+1})$ contains a dense G_δ-set of maps that are injective on each fiber of the map p.

Since the Euclidean space $\mathbb{R}^d$ has the m-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties for all m,n,k with $m+n+k<d$, we may ask whether the mentioned theorem of H. Toruńczyk [**69**] is true in the following more general form.

Problem 4.1 ([**4**]). *Does any map $p\colon K\to M$ between finite-dimensional compact* 1517?
metric spaces embed into the projection $\mathrm{pr}\colon M\times X\to M$ *along a Polish AR-space X possessing the m-$\overline{\mathrm{DD}}^{\{n,k\}}$-property for all m,n,k with $m+n+k\le\dim(K)+\dim(p)$?*

Let us also note that the above result of H. Toruńczyk would follow from [**4**, Theorem 1] if the following problem had an affirmative answer.

Problem 4.2 ([**4**]). *Let $f\colon X \to Y$ be a k-dimensional map between finite-dimensional metrizable compacta. Is it true that there is a map $g\colon Y \to Z$ to a compact space Z with $\dim Z \le \dim X - k$ such that the map $g \circ f$ is still k-dimensional?*

Next two questions concern the minimal dimension of spaces with m-$\overline{\mathrm{DD}}^{\{n,n\}}$. It is known [**4**] that the smallest possible dimension of compact metrizable AR with $X \in m$-$\overline{\mathrm{DD}}^{\{n,n\}}$ is either $n+\lfloor \frac{m+1}{2} \rfloor$ or $n+\lceil \frac{m+1}{2} \rceil$, where $\lfloor r \rfloor = \max\{k \in \mathbb{Z} : k \le r\}$ and $\lceil r \rceil = \min\{k \in \mathbb{Z} : k \ge r\}$.

1518? **Problem 4.3.** *What is the smallest possible dimension of Polish spaces with m-$\overline{\mathrm{DD}}^{\{n,n\}}$? Is it $n + \lceil \frac{m+1}{2} \rceil$?*

1519? **Problem 4.4.** *What is the smallest possible dimension of metrizable compacta X such that $X \times \mathbb{I}^n$ contains a copy of the n-dimensional Menger cube? Is it $\lceil \frac{m}{2} \rceil$?*

The last question in this section is a reformulation of the well known problem of finding a characterization of codimension one manifold factors, see [**22**, **46**].

1520? **Problem 4.5.** *Let $X \times \mathbb{R}$ is an n-manifold with $n \ge 5$. Does X have the 1-$\overline{\mathrm{DD}}^{\{1,1\}}$-property?*

5. Locally connected continua

By a *continuum*, we mean a compact connected Hausdorff space. A *compact ordered space* is a compact space with topology induced by a linear order. An *arc* is a compact ordered space which is connected. Equivalently, an arc is a continuum with exactly two non cut-points. Let P be a topological property. A topological space X is said to be *rim-P* if it has a basis of open sets with boundaries having the property P. Some of the spaces with natural rim-properties are *rim-finite* spaces, *rim-countable* spaces, *rim-metrizable* spaces, *rim-scattered* spaces and rim-compact spaces.

The Hahn–Mazurkiewicz Theorem (1914) characterizes the continuous images of the closed unit interval as locally connected metric continua. A theorem of Alexandroff characterizes the continuous images of the Cantor set as the class of compact metric spaces. In the non-metric case, continuous images of arcs and more generally, of compact ordered spaces are quite restricted and interesting. Mardešić (1960) gave an example of a locally connected continuum which is not a continuous image of an arc. Treybig (1964) showed that continuous images of compact ordered spaces do not contain a non-metric product of (infinite) compact spaces. In 1967, Mardešić [**54**] proved an important result: every continuous image of a compact ordered space is rim-metrizable. Heath, Lutzer and Zenor (1973) proved that continuous images of ordered compacta are monotonically normal. Nikiel (1988) characterized the continuous images of arcs in the non-metric case. He also proved that each hereditarily locally connected continuum is a continuous image of an arc and rim-countable, see [**57**]. Nikiel, Tymchatyn and Tuncali (1991) gave an example of a rim-countable locally connected continuum which is not a continuous image of an arc.

Following these results, the study of images of arcs/compact ordered spaces developed in several directions. One study focused on the behavior of images of arcs under inverse limits. Nikiel, Tymchatyn and Tuncali [**60**](1993) proved that the inverse limit of an inverse sequence of images of arcs with monotone bonding maps is a continuous image of an arc. They also proved that each one-dimensional continuous image of an arc can be obtained as an inverse limit of inverse sequence of rim-finite continua with monotone bonding maps. This result extends the similar theorem of Nikiel (1989) in the metric case, and indicates that in the 1-dimensional case, images of arcs behave like metric locally connected continua.

Tuncali [**70**] proved that continuous images of rim-metrizable continua do not contain a non-metric product of nondegenerate continua. These results suggest that some properties of images of compact ordered space/arcs depend on the boundary structure of basic open sets. Nikiel, Treybig and Tuncali [**58**] (1995) showed that continuous images of rim-metrizable locally connected continua are not necessarily rim-metrizable, hence Mardešić's 1967 result cannot be generalized. This result shows that the class of rim-metrizable continua is large. In 1989, Nikiel asked if every monotonically normal compactum is a continuous image of a compact ordered space. M.E. Rudin [**67**] (2001) answered that question affirmatively. Note that Ostaszewski (1978) proved that a separable, monotonically normal space is hereditarily Lindelöf. Therefore, each separable, monotonically normal, compact space is perfectly normal. On the other hand, under the continuum hypothesis, Filippov [**41**] (1969) constructed a perfectly normal and locally connected continuum which is nonmetrizable and has a basis of open sets with 0-dimensional metrizable boundaries. Gruenhage [**42**] (1990) also constructed an example of a perfectly normal locally connected continuum which is nonmetrizable, rim-metrizable, and not arcwise connected. Note that a product of $[0, 1]$ with a Souslin line is a perfectly normal, localy connected continuum which is not rim-metrizable. Following these results, an interesting problem to consider is the following:

Problem 5.1. *Characterize locally connected, rim-metrizable, perfectly normal* 1521?
continua.

Recently, Daniel, Nikiel, Treybig, Tymchatyn and Tuncali in a sequence of papers have been investigating various properties of continuous images of arcs, Suslinian continua, rim-metrizable continua, and perfectly normal compact spaces, [**14–21**]. They proved that each Suslinian continuum is perfectly normal and rim-metrizable, [**16**]. A continuum is said to be *Suslinian* if it does not contain an uncountable collection of mutually disjoint continua. Lelek introduced Suslinian continua in 1971. Using inverse limit techniques, Daniel, Nikiel, Treybig, Tymchatyn and Tuncali showed that locally connected Suslinian continua must have weight ω_1 and under the Souslin Hypothesis such continua are metrizable. In [**2**], these results are improved. It is proved that all Suslinian continua must have weight ω_1, and under the Souslin Hypothesis, all Suslinian continua are metrizable. In [**18**], it is also proved that each homogenous Suslinian continuum X must be locally connected, and moreover, if X is seperable, then it must be metrizable. Another interesting result is that under Souslin Hypothesis each perfectly normal compact space of weight ω_1 contains an uncountable, upper semi-continuous, almost null family of nondegenerate, pairwise disjoint, closed subsets, [**17**]. Moreover, if X is locally connected continuum, the members of this family can be chosen to be continua.

Recently, Todd Eisworth [**39**] announced that each separable monotonically normal compact space admits two-to-one map onto a metric space. These results are related to the following well-known questions. First one is due to M.E. Rudin and the second one is due to D.H. Fremlin.

1522? **Problem 5.2.** *Is it consistent that each perfectly normal, locally connected continua is metrizable?*

1523? **Problem 5.3.** *Is it consistent that every pefectly normal compact space admits a two-to-one continuous map onto a metric space?*

It is not difficult to see that a locally connected, perfectly normal continuum X with small inductive dimension $\mathrm{ind}(X) = 1$ is rim-metrizable. Filippov's 1969 example is such a continuum. Moreover, the product $X \times [0, 1]$ of a locally connected perfectly normal continuum X and $[0, 1]$ is locally connected and perfectly normal again. However, $X \times [0, 1]$ is rim-metrizable if and only if X is metrizable. This follows from the fact that rim-metrizable continua do not contain a nonmetric product of nondegenerate continua, [**70**]. These show why Problem 5.1 is natural to consider. Concerning Problem 5.2 and 5.3, readers are also referred to the article by Gary Gruenhage and Justin Moore titled "Perfect compacta and basis problems in topology" in this volume [**43**].

Daniel and Treybig, [**21**] showed that if there is an example of a locally connected Suslinian continuum which is not a continuous image of an arc, there is such a continuum X which is separable. Therefore, it will be interesting to know the answer to the following question.

1524? **Problem 5.4.** *Is a locally connected, rim-metrizable continuum X with no nondegenerate metric continuum rim-finite?*

In [**20**], it was shown that such a continuum X is rim-finite with the additional assumption that X contains no separable subcontinuum. It is known that each rim-finite continuum is a continous image of an arc. Also, Suslinian nonseparable continua are rim-finite on some open set, [**16**].

In addition, in [**16**, **21**] various interesting properties of Suslinian continua were investigated. There are some interesting problems concerning Suslinian continua remain to be answered.

1525? **Problem 5.5.** *Is a separable Suslinian continuum hereditarily separable?*

1526? **Problem 5.6.** *If X is a locally connected Suslinian continuum, is X connected by arcs (ordered continua)?*

Recently, in [**2**] a new cardinal invariant is introduced. Namely,

$$\mathrm{Sln}(X) = \sup\{|\mathcal{C}| : \mathcal{C} \text{ is a disjoint family of nondegenerate subcontinua of } X\}$$

defined for any continuum X and is called the *Suslinian number* of X. Thus a continuum X is Suslinian if and only $\mathrm{Sln}(X) \leq \aleph_0$. It is clear that $\mathrm{Sln}(X) \leq \mathrm{Sln}(Y)$ for any pair $X \subset Y$ of continua. It is convenient to extend the definition of $\mathrm{Sln}(X)$ to all Tychonov spaces by letting

$$\mathrm{Sln}(X) = \min\{\mathrm{Sln}(Y) : Y \text{ is a continuum containing } X\}$$

for a Tychonov space X. Like many other cardinal invariants the Suslinian number is monotone.

For any Tychonov space X, the hereditary Lindelöf number of any space X is bounded from above by the Suslinian number of X, *see also* ip-ktv-bft This generalizes the result of Daniel, Nikiel, Treybig, Tuncali and Tymchatyn that each Suslinian continua is perfectly normal, [**16**]. Since each Suslinian continua is rim-metrizable, it is natural to ask the following question.

Problem 5.7. *Is* $\text{rim-w}(X) \leq \mathrm{Sln}(X)$ *for any compact Hausdorff space?* 1527?

Note that for a given a topological space X $\text{rim-w}(X) = \min\{\sup_{U\in\mathcal{B}} w(\partial U) : \mathcal{B}$ is a base of the topology of $X\}$ is the *rim-weight* of X.

In addition to problems stated above, there are number of questions concerning rim-properties of locally connected continua. We list them below. For further reading, we refer readers to [**15**] and [**59**]. Note that some of these problems were listed before in various papers cited in this section.

Problem 5.8. *Is each rim-scattered locally connected continuum rim-metrizable?* 1528?

Note that Drozdovskiĭ and Filippov [**30**], gave an example of a rim-scattered, rim-metrizable locally connected continuum which is not rim-countable.

Problem 5.9. *Let X be a rim-metrizable locally connected continuum. Does X admit a basis of open F_σ-sets with metrizable boundaries?* 1529?

Problem 5.10. *Is a continous image of a rim-countable continuum rim-metrizable?* 1530?

Recall that, the continous images of rim-metrizable compact spaces are not necessarily rim-metrizable, [**58**].

In [**59**], it was proved that if X is a continous image of an arc, then the three classical dimension numbers $\mathrm{ind}(X)$, $\mathrm{Ind}(X)$ and $\dim(X)$ are equal. Moreover,

$$\mathrm{ind}(X) = \max\{1, \sup\{\mathrm{ind}(M) : M \subset X \text{ and } M \text{ is closed and metrizable}\}\}.$$

Problem 5.11. *If X is a locally connected rim-metrizable continuum, What is the relation among* $\mathrm{ind}(X)$, $\mathrm{Ind}(X)$, $\dim(X)$ *and* $\max\{1, \sup\{\mathrm{ind}(M) : M \subset X$ *and M is closed and metrizable*$\}\}$*?* 1531?

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