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DIMENSION OF MAPS, UNIVERSAL SPACES, AND HOMOTOPY

A. Karasev and V. Valov

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Abstract. This paper is a survey of some recent results in dimension theory. The main topics under consideration are: dimension of maps in the classical and extension dimension theories, universal spaces (in particular, universal compacta) in extension dimension theory, and $[L]$-homotopy. A number of theorems included in the survey are accompanied by proofs.

CONTENTS

1. Preliminaries

1.1. Introduction. The study of universal spaces and dimensional properties of maps is a classical direction in dimension theory. The goal of this paper is to provide an overview of some recent related results. We did not aim at full coverage of all achievements in this very broad field. Instead, we included proofs of some theorems, thus hoping to convey the key ideas to the reader.

The paper is organized as follows. We begin by introducing our notation and some preliminary facts in Sec. 1.2. In Sec. 1.3, we recall the definition of extension dimension and related concepts. Further, we state Dranishnikov's theorem, which describes extension dimension in terms of cohomological dimension, and discuss some related questions. Section 2, which opens the main part of this paper, is devoted to dimension of maps whose ranges are metrizable C-spaces. In particular, we present generalized versions of the parametric decomposition theorem and of the parametric Hurewicz theorem on regularly branched maps. A number of results on embeddings in $\mathbb{R}^n$ avoiding given collections of planes or having "small" intersections with them constitute Sec. 3. Various generalizations of theorems on dimension-lowering and dimension-raising maps to extension dimension are discussed in Sec. 4. Section 5 is devoted to universal spaces in extension theory. We are mainly interested in the existence of universal compacta of given extension dimension. In relation to this problem, we introduce a special class of CW-complexes, called quasi-finite complexes, and study their properties. "Classical" universal spaces have many useful properties (in addition to universality). In particular, they are absolute extensors in the corresponding dimension. We analyze the possibility of obtaining extension-dimension analogs of universal spaces with a similar property in Sec. 6. In Sec. 7, we describe a general approach to quasi-finite complexes and show that these complexes behave well in many respects, thus

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resembling finite complexes. The last section of this paper is devoted to $[L]$-homotopy theory, which was initiated by Chigogidze and applies to spaces of given extension dimension.

1.2. Notation and basic definitions. We assume that all maps under consideration are continuous and all spaces are at least Tychonoff, although we largely deal with metrizable or even separable metrizable spaces. Recall that a Polish space is a separable completely metrizable space.

The weight of a space X is denoted by $w(X)$; $\mathrm{cov}(X)$ is the set of all open covers of X. As usual, $C(X, Y)$ denotes the set of all maps from X to Y. By $X \vee Y$ we denote the bouquet of (pointed) spaces X and Y. The cone $\mathrm{Cone}(X)$ over X is defined as $\mathrm{Cone}(X) = X \times [0,1]/X \times \{1\}$. The Stone–Čech compactification of X is denoted by βX. As usual, id_X denotes the identity map of X.

For a family of sets $\mathcal{A}$ and a set B, we define the star of B in $\mathcal{A}$ as $\mathrm{St}(B, \mathcal{A}) = \bigcup\{A \in \mathcal{A} \mid A \cap B \neq \varnothing\}$. We say that a cover $\nu \in \mathrm{cov}(X)$ is a strong star refinement of a cover $\omega \in \mathrm{cov}(X)$ if, for each $V \in \nu$, there exists a $W \in \omega$ such that $\mathrm{St}(V, \nu) \subset W$.

By D^n we denote the unit n-ball in Euclidean n-space $\mathbb{R}^n$, and by S^n, the n-sphere treated as the boundary of D^{n+1}. The notation $\mathbb{R}P^n$ is used for the projective n-space. We denote the unit interval by $\mathbb{I}$.

The Lebesgue (covering) dimension of a space X is denoted by $\dim X$. For a map $f : X \to Y$, we define the dimension of f by

$$\dim f = \sup\{\dim f^{-1}(y) \mid y \in Y\}.$$

We also write $\text{e-dim}\, f \leq L$ if $\text{e-dim}\, f^{-1}(y) \leq L$ for all y in Y. Here L is a CW-complex and e-dim stands for extension dimension, which is defined in Sec. 1.3. A map $f : X \to Y$ between metrizable spaces is said to be uniformly 0-dimensional (see [84]) if there exists a metric on X such that it generates the topology of X and, for every $\varepsilon > 0$, each point of $f(X)$ has a neighborhood U in Y for which $f^{-1}(U)$ is a union of disjoint open subsets of X with diameters $<\varepsilon$.

We recall that a space X is called a C-space if, for any sequence $\{\nu_n\}_{n=1}^{\infty}$ of open covers of X, there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of disjoint open families in X such that every γ_n refines ν_n. The property C was introduced by Haver [73] for compact metric spaces. Addis and Gresham [1] extended Haver's definition to more general spaces.

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_m$, and $\mathbb{Q}$ we denote the positive integers, the integers, the residues modulo m, and the rational numbers, respectively.

Let G be an Abelian group. By $H_n(X; G)$ we denote the nth singular homology group of X with coefficients in G. We set $H_n(X) = H_n(X; \mathbb{Z})$. The notation $\widetilde{H}_n(X; G)$ is used for reduced homology groups. By $M(G, n)$ we denote a Moore space of type (G, n), i.e., a CW-complex such that $H_n(M(G, n)) \approx G$ and $\widetilde{H}_i(M(G, n)) \approx 0$ for all $i \neq n$. As usual, $K(G, n)$ is the Eilenberg–MacLane complex, i.e., a CW-complex such that $\pi_n(K(G, n)) \approx G$ and $\pi_i(K(G, n)) \approx 0$ for all $i \neq n$.

By $H^k(X; G)$ we denote the kth Čech cohomology group of the space X with coefficients in G. According to a theorem of Huber [75], for any paracompact Hausdorff space X and any countable coefficient group G, there exists a natural isomorphism between the group of all homotopy classes of maps from X to $K(G, n)$ and the group $H^n(X; G)$. By $H^n(X, A; G)$, where A is a subset of X, we denote the relative Čech cohomology group of the pair (X, A).

We define the cohomological dimension $\dim_G X$ of a paracompact space X with respect to the coefficient group G in the standard way as

$$\dim_G X = \sup\{n : \text{there exists a closed set } A \subset X \text{ with } H^n(X, A; G) \neq 0\}.$$

Throughout the paper, except in Secs. 4 and 7, by a complex we mean a countable locally finite connected CW-complex; the letter L is reserved to denote complexes.

1.3. Extension dimension. In this section, we assume that all spaces are separable and metrizable. The general case was considered in [25] (and in Sec. 7 in this paper). We say that a space Y is an absolute (neighborhood) extensor for X and write $Y \in A(N)E(X)$ if any map from an arbitrary closed subspace A of X to Y can be extended to a map from the entire space X to Y (respectively, to a map from some open neighborhood of A to Y). In this notation, the classical theorems characterizing the Lebesgue and cohomological dimension in terms of extension of maps are stated as follows:

$$\dim X \leq n \Leftrightarrow S^n \in AE(X) \quad \text{and} \quad \dim_G X \leq n \Leftrightarrow K(G,n) \in AE(X).$$

The main idea behind the definition of extension dimension, which is due to Dranishnikov [39], is to consider an arbitrary complex L instead of S^n or $K(G,n)$. Namely, we say that the extension dimension of a space X does not exceed a complex L and write $\text{e-dim}\, X \leq L$ if $L \in AE(X)$. It was observed in [40] that the natural order on the positive integers can be generalized to a (partial) order on complexes. Following Dranishnikov [40], we say that $L \leq K$ if $L \in AE(X)$ implies $K \in AE(X)$ for any space X. Clearly, $S^n \leq S^m$ if and only if $n \leq m$. The relation $L \leq K$ determines a preorder on complexes and generates an equivalence relation. Obviously, this equivalence relation can be described in terms of extension of maps; namely, a complex L is equivalent to a complex K if $L \in AE(X) \Leftrightarrow K \in AE(X)$ for any space X. The equivalence class of a complex L is called the extension type of L and denoted by $[L]$. The homotopy extension property of ANR-spaces implies that if L is homotopy equivalent to K, then $[L] = [K]$. However, the converse is not true. For example, $[S^n] = [S^n \vee S^m]$ if $n \leq m$.

We say that the extension dimension of a space X is equal to an extension type $[L]$ if $\text{e-dim}\, X \leq L$ and $L \leq K$ for any complex K with $\text{e-dim}\, X \leq K$. Sometimes, we use the notation $\text{e-dim}\, X \leq [L]$ instead of $\text{e-dim}\, X \leq L$.

The preorder relation on complexes determines a partial order relation on extension types; we denote it by the same symbol $\leq$. We emphasize that there are many incomparable complexes. For example, results of [45] imply that $\mathbb{R}P^2$ is not comparable with any sphere S^n, where $n \geq 2$. It can be shown that the minimum of the extension types of complexes L and K is $[L \vee K]$. We refer the reader to [27, 43, 56] for more information about extension dimension and extension types.

1.4. Extension dimension in terms of cohomological dimension. The following theorem, which is due to Dranishnikov, is one of the central results in extension dimension theory. This theorem establishes a relation between the extension and cohomological dimensions of a metrizable compactum X with finite covering dimension for a simply connected complex L.

Theorem 1.1 (see [39]). *Let X be a compact metrizable space, and let L be a complex. If* $\text{e-dim}\, X \leq L$, *then* $\dim_{H_n(L)} \leq n$ *for all positive integers n. If* $\dim X < \infty$ *and L is a simply connected complex, then the following conditions are equivalent*:

(1) $\text{e-dim}\, X \leq L$;
(2) $\dim_{H_n(L)} \leq n$ *for all* $n > 0$;
(3) $\dim_{\pi_n(L)} \leq n$ *for all* $n > 0$.

This theorem suggests several directions of research. The first of them is generalization to noncompact metrizable spaces; this was accomplished by Dydak [52, 54]. It is also natural to try to replace the finite-dimensionality of X and the simple connectedness of L by weaker assumptions. The condition $\dim X < \infty$ cannot be removed, because there exist infinite-dimensional compacta of finite integral cohomological dimension [37]. However, it can be dispensed with under an additional condition on X. Namely, Dydak proved in [52] that (3) implies (1) if X is a metrizable space which is an absolute neighborhood extensor for metrizable spaces. To the best of our knowledge, the following question is still open.

Problem 1.1. *Does Theorem* 1.1 *hold if the compactum X is a C-space?*

As for the simple connectedness requirement, Cencelj and Dranishnikov proved in the series of papers [19–21] that Theorem 1.1 remains true for nilpotent complexes. Nevertheless, the condition on the fundamental group of L cannot be removed completely. Indeed, if X is the 2-disk and L is a complex such that $\pi_1(L)$ is nontrivial and $\widetilde{H}_*(L) = 0$, then the implication (2) $\Rightarrow$ (1) in the above theorem does not hold [58]. An example of such an L can be found, for instance, in [72, Example 2.38, p. 142].

Finally, consider the equivalence (1) $\Leftrightarrow$ (3). The simplest example of a complex which is not nilpotent is $\mathbb{R}P^2$. Note that $H_1(\mathbb{R}P^2) \approx \pi_1(\mathbb{R}P^2) \approx \mathbb{Z}_2$; hence, for any compact metrizable space X, $\text{e-dim}\, X \leq \mathbb{R}P^2$ implies $\dim_{\mathbb{Z}_2} X \leq 1$ (by Theorem 1.1). What can be said about the converse implication? Results of Levin [90] imply the existence of a compact metrizable space X such that $\dim_{\mathbb{Z}_2} X = 1$ and $\text{e-dim}\, X \not\leq \mathbb{R}P^2$. Nevertheless, this compact space is infinite-dimensional, and the following well-known problem is still open.

Problem 1.2. *Does* $\dim_{\mathbb{Z}_2} X \leq 1$ *imply* $\text{e-dim}\, X \leq \mathbb{R}P^2$ *for any finite-dimensional compact space* X*?*

Since the infinite projective space $\mathbb{R}P^\infty$ is of type $K(\mathbb{Z}_2, 1)$, this problem can be reformulated as follows: Does $\mathbb{R}P^\infty \in AE(X)$ imply $\mathbb{R}P^2 \in AE(X)$ for any finite-dimensional compact space X? Dydak and Levin proved in [58] that the answer is positive if $\dim X \leq 3$.

2. Finite-Dimensional Maps between Metrizable Spaces

In this section, we describe a general method for proving some theorems concerning dimensionally restricted maps, which was applied earlier in [129–132]. The "algorithmic" form of this method used in this paper is borrowed from [133] (see also [32]). It is based on selection theorems proved by V. Gutev and the second author in [70, 71].

2.1. Preliminaries. First, we provide some information about the source limitation topology. This topology can be described as follows: If (M, d) is a metric space, then a set $U \subset C(X, M)$ is open if for every $g \in U$ there exists a continuous function $\alpha : X \to (0, \infty)$ such that $\overline{B}(g, \alpha) \subset U$. Here $\overline{B}(g, \alpha)$ denotes the set $\{h \in C(X, M) : d(g(x), h(x)) \leq \alpha(x) \text{ for each } x \in X\}$. The source limitation topology does not depend on the metric d if X is paracompact [86]. Moreover, $C(X, M)$ with this topology has the Baire property provided that (M, d) is a complete metric space. Obviously, the source limitation topology coincides with the topology of uniform convergence generated by d for compact X. In this section, unless otherwise specified, all function spaces are assumed to be endowed with the source limitation topology.

We will need the following simplified versions of selection theorems from [70, 71].

Theorem 2.1 (see [70])**.** *For a paracompact space* X*, the following conditions are equivalent*:

(a) X *is a* C*-space*;

(b) *suppose that* Y *is a Banach space and* $\Phi : X \to Y$ *is a closed-and-convex-valued lower semicontinuous map. Then, for every sequence of upper semicontinuous maps* $\psi_n : X \to Y$ *such that every* ψ_n *has a closed graph and* $\psi_n(x) \cap \Phi(x)$ *is a* Z*-set in* $\Phi(x)$ *for any* $x \in X$ *and* $n \in \mathbb{N}$*, there exists a single-valued continuous map* $f : X \to Y$ *with*

$$f(x) \in \Phi(x) \setminus \bigcup_{n=1}^{\infty} \psi_n(x)$$

for each $x \in X$.

Theorem 2.2 (see [71])**.** *Suppose that* X *is a paracompact space with* $\dim(X) \leq n$*,* Y *is a completely metrizable space,* $\varphi : X \to Y$ *is a closed-valued lower semicontinuous map such that* $\{\varphi(x) : x \in X\}$ *is equi-*LC^{n-1} *in* Y *and each* $\varphi(x)$*, where* $x \in X$*, is* C^{n-1}*, and* $\psi : X \to Y$ *is a set-valued map having a closed graph such that* $\psi(x) \cap \varphi(x)$ *is a* Z_n*-set in* $\varphi(x)$ *for every* $x \in X$*. Then the set of all continuous* (*single-valued*) *selections of* φ *with the source limitation topology contains a dense* G_δ*-subset of maps* g *such that* $g(x) \in \varphi(x) \setminus \psi(x)$ *for every* $x \in X$.

Recall that a set-valued map $\varphi : X \to Y$ is lower semicontinuous (lsc) if $\varphi^{-1}(U) = \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$, and a map $f : X \to Y$ is a *selection* for $\varphi : X \to Y$ if $f(x) \in \varphi(x)$ for every $x \in X$.

Note that all countable-dimensional metrizable spaces (that is, spaces that are countable unions of finite-dimensional subsets), in particular, all finite-dimensional ones, form a proper subclass in the class of C-spaces (there exists a metric C-compactum which is not countable-dimensional [112]).

Let $m \geq 0$. A family $\mathcal{S}$ of subsets of a space Y is *equi-LC^m in* Y [97] if, for any $y \in Y$ and any neighborhood U of y, there exists a neighborhood V of y such that, for every $S \in \mathcal{S}$, any continuous image of S^k $(k \leq m)$ in $V \cap S$ is contractible in $U \cap S$. A space S is called a C^m space if every continuous image of S^k $(k \leq m)$ in S is contractible in S. A closed subset F of a metrizable space M is said to be a Z_m-set in M if the set $C(\mathbb{I}^m, M\backslash F)$ is dense in $C(\mathbb{I}^m, M)$ with respect to the uniform convergence topology, where $\mathbb{I}^m$ is the m-cube. If F is a Z_m-set in M for every $m \in \mathbb{N}$, then we say that F is a Z-set in M.

Throughout this section, K is a closed convex subset of $\mathbb{R}^p$ and $f : X \to Y$ is a perfect surjective map between paracompact spaces. It can be shown that the set $C^*(X, K)$ of bounded maps from X to K is open in $C(X, K)$ with respect to the source limitation topology, provided that $\mathbb{R}^p$ is endowed with the Euclidean metric ϱ. Therefore, $C^*(X, K)$ with this topology has the Baire property. Suppose that, for every $y \in Y$, $\mathcal{P}(y) \subset C^*(X, K)$ is a set such that if $h \in C^*(X, K)$ and $h|f^{-1}(y) = g|f^{-1}(y)$ for some $g \in \mathcal{P}(y)$, then $h \in \mathcal{P}(y)$. This means that the set $\mathcal{P}(y)$ is determined by the restrictions $g|f^{-1}(y)$. For this reason, we sometimes consider $\mathcal{P}(y)$ as a class of functions on $f^{-1}(y)$. Let $\mathcal{P}(H) = \bigcap_{y \in H} \mathcal{P}(y)$, where $H \subset Y$. In what follows, we use the set-valued map $\psi : Y \to C^*(X, K)$ defined by $\psi(y) = C^*(X, K)\backslash\mathcal{P}(y)$.

Lemma 2.3 (see [133])**.** *Suppose that, for any $y \in Y$ and $g \in \mathcal{P}(y)$, there exists a neighborhood V_y of y in Y and a $\delta_y > 0$ such that $h \in \mathcal{P}(V_y)$ provided that $h|f^{-1}(V_y)$ is δ_y-close to $g|f^{-1}(V_y)$. Then $\mathcal{P}(Y)$ is open in $C^*(X, K)$ with respect to the source limitation topology. Moreover, ψ has a closed graph with respect to the topology of uniform convergence on $C^*(X, K)$.*

Proof. We use some ideas from [10]. Let $(y_0, g_0) \in Y \times C^*(X, K)\backslash G_\psi$, where $C^*(X, K)$ is endowed with the topology of uniform convergence and G_ψ is the graph of ψ. Then $g_0 \notin \psi(y_0)$, i.e., $g_0 \in \mathcal{P}(y_0)$. Take V_{y_0} and $\delta_{y_0} > 0$ satisfying the hypotheses of the lemma, and let W denote the δ_{y_0}-neighborhood of g_0 in $C^*(X, K)$. Then $V_{y_0} \times W$ is a neighborhood of (y_0, g_0) disjoint from G_ψ. Thus, G_ψ is closed.

To show that $\mathcal{P}(Y)$ is open in $C^*(X, K)$ with respect to the source limitation topology, take $g_0 \in \mathcal{P}(Y)$. For every $y \in Y$, we have $g_0 \in \mathcal{P}(y)$. Choose neighborhoods V_y and positive numbers $\delta_y \leq 1$ satisfying the conditions of the lemma. We can assume that $\{V_y : y \in Y\}$ is a locally finite cover of Y and consider the lsc set-valued map $\varphi : Y \to (0, 1]$ defined by $\varphi(y) = \cup\{(0, \delta_z] : y \in V_z\}$. According to [115, Theorem 6.2, p. 116], φ has a continuous selection $\beta : Y \to (0, 1]$; let $\alpha = \beta \circ f$. It remains to show that if $g \in C^*(X, K)$ and $\varrho\big(g_0(x), g(x)\big) < \alpha(x)$ for all $x \in X$, then $g \in \mathcal{P}(Y)$. Take such a g and let $y \in Y$. There exists a $z \in Y$ such that $y \in V_z$ and $\alpha(x) \leq \delta_z$ for all $x \in f^{-1}(y)$. Consider a map $h \in C^*(X, K)$ coinciding with g on $f^{-1}(y)$ and satisfying the inequality $\varrho\big(h(x), g_0(x)\big) \leq \delta_z$ for each $x \in X$. According to the choice of V_z, we have $h \in \mathcal{P}(y)$. Hence $g \in \mathcal{P}(y)$, because $g|f^{-1}(y) = h|f^{-1}(y)$. Therefore, $\mathcal{P}(Y)$ is open in $C^*(X, K)$. □

Lemma 2.4 (see [133])**.** *Suppose that a point $y \in Y$ and the set $\mathcal{P}(y)$ treated as a subset of $C(f^{-1}(y), K)$ satisfy the following condition*:

- *For every $k \in \mathbb{N}$ (for $k = m$), the set of all maps $h \in C(\mathbb{I}^k \times f^{-1}(y), K)$ such that $h|(\{z\} \times f^{-1}(y)) \in \mathcal{P}(y)$ for any $z \in \mathbb{I}^k$ is dense in $C(\mathbb{I}^k \times f^{-1}(y), K)$ with respect to the topology of uniform convergence.*

Then, for any $\alpha : X \to (0, \infty)$ and $g_0 \in C^*(X, K)$, $\psi(y) \cap \overline{B}(g_0, \alpha)$ is a Z-set (respectively, Z_m-set) in $\overline{B}(g_0, \alpha)$ considered as a subset of $C^*(X, K)$ with the topology of uniform convergence, provided that $\psi(y) \subset C^*(X, K)$ is closed.

Proof. We follow the scheme of the proof of [129, Lemma 2.8]. All function spaces in this proof are endowed with the topology of uniform convergence. We consider only the case where the set of maps

$h \in C(\mathbb{I}^k \times f^{-1}(y), K)$ such that $h|(\{z\} \times f^{-1}(y)) \in \mathcal{P}(y)$ for $z \in \mathbb{I}^k$ is dense in $C(\mathbb{I}^k \times f^{-1}(y), K)$ for each k. The case $k = m$ is similar. We must show that, given k, $\delta > 0$, and a map $u : \mathbb{I}^k \to \overline{B}(g_0, \alpha)$, there exists a map $v : \mathbb{I}^k \to \overline{B}(g_0, \alpha) \backslash \psi(y)$ which is δ-close to u in $C(\mathbb{I}^k \times f^{-1}(y), K)$. To this end, note that u generates the function $h \in C^*(\mathbb{I}^k \times X, K)$ defined by $h(z, x) = u(z)(x)$, and $\varrho(h(z, x), g_0(x)) \leq \alpha(x)$ for any $(z, x) \in \mathbb{I}^k \times X$. Since $f^{-1}(y)$ is compact, we can find $\lambda \in (0, 1)$ for which $\lambda \sup\{\alpha(x) : x \in f^{-1}(y)\} < \delta/2$. Let us define $h_1 \in C(\mathbb{I}^k \times f^{-1}(y), K)$ by $h_1(z, x) = (1 - \lambda)h(z, x) + \lambda g_0(x)$. Then, for every $(z, x) \in \mathbb{I}^k \times f^{-1}(y)$, we have

(1) $\varrho(h_1(z, x), g_0(x)) \leq (1 - \lambda)\alpha(x) < \alpha(x)$,

(2) $\varrho(h_1(z, x), h(z, x)) \leq \lambda\alpha(x) < \delta/2$.

Let $q < \min\{r, \delta/2\}$, where

$$r = \inf\{\alpha(x) - d(h_1(z, x), g_0(x)) : (z, x) \in \mathbb{I}^k \times f^{-1}(y)\} > 0.$$

By assumption, there exists a map $h_2 \in C(\mathbb{I}^k \times f^{-1}(y), K)$ such that $\varrho(h_2(z, x), h_1(z, x)) < q$ and $h_2|(\{z\} \times f^{-1}(y)) \in \mathcal{P}(y)$ for any $(z, x) \in \mathbb{I}^k \times f^{-1}(y)$. By (1) and (2), for all $(z, x) \in \mathbb{I}^k \times f^{-1}(y)$, we have

(3) $\varrho(h_2(z, x), h(z, x)) < \delta$ and $\varrho(h_2(z, x), g_0(x)) < \alpha(x)$.

Since both $\mathbb{I}^k$ and $f^{-1}(y)$ are compact, it follows that $u_2(z)(x) = h_2(z, x)$ defines a map $u_2 : \mathbb{I}^k \to C(f^{-1}(y), K)$. The required map v is the lifting of u_2. Obviously, the restriction map $\pi : \overline{B}(g_0, \alpha) \to C(f^{-1}(y), K)$ defined by $\pi(g) = g|f^{-1}(y)$ is continuous.

Claim. *The map $\pi : \overline{B}(g_0, \alpha) \to \pi(\overline{B}(g_0, \alpha))$ is open.*

It is sufficient to show that

$$\pi(\overline{B}(g_0, \alpha) \cap B_\varepsilon(g)) = \pi(\overline{B}(g_0, \alpha)) \cap B_\varepsilon(\pi(g))$$

for any $g \in \overline{B}(g_0, \alpha)$ and $\varepsilon > 0$, where $B_\varepsilon(g)$ and $B_\varepsilon(\pi(g))$ are open balls in $C^*(X, K)$ and $C(f^{-1}(y), K)$, respectively, both with the uniform metric generated by ϱ. Let $p \in \pi(\overline{B}(g_0, \alpha)) \cap B_\varepsilon(\pi(g))$. Then $\varrho(p(x), g_0(x)) \leq \alpha(x)$ and $\varrho(p(x), g(x)) < \eta < \varepsilon$ for any $x \in f^{-1}(y)$ and some positive number η. Consider the closed-and-convex-valued map $\Phi : X \to K$ defined by $\Phi(x) = p(x)$ if $x \in f^{-1}(y)$ and

$$\Phi(x) = \overline{B_{\alpha(x)}(g_0(x)) \cap B_\eta(g(x))}$$

if $x \notin f^{-1}(y)$ (recall that $B_{\alpha(x)}(g_0(x))$ and $B_{\eta(x)}(g(x))$ are open balls in K). Since $g \in \overline{B}(g_0, \alpha)$, it follows that $\Phi(x) \neq \emptyset$ for every $x \in X$. Moreover, the continuity of α, g, and g_0 implies the lower semicontinuity of Φ. Therefore, by Michael's convex-valued selection theorem, Φ admits a selection $g_1 \in C(X, K)$. Thus, $\pi(g_1) = p$ and $g_1 \in \overline{B}(g_0, \alpha) \cap B_\varepsilon(g)$. Hence

$$\pi(\overline{B}(g_0, \alpha)) \cap B_\varepsilon(\pi(g)) \subset \pi(\overline{B}(g_0, \alpha) \cap B_\varepsilon(g));$$

the reverse inclusion is trivial.

To complete the proof, take $z \in \mathbb{I}^k$ and consider the set-valued map $\phi_z : X \to K$ defined by $\phi_z(x) = u_2(z)(x)$ for $x \in f^{-1}(y)$ and

$$\phi_z(x) = \overline{B_\alpha(x)(g_0(x)) \cap B_\mu(u(z)(x))}$$

for $x \notin f^{-1}(y)$, where $\mu < \delta$ is a positive number such that $\varrho(h_2(z, x), h(z, x)) < \mu$ for any $(z, x) \in \mathbb{I}^k \times f^{-1}(y)$ (this is possible because of (3)). Note that the set $A_z(x) = \overline{B_\alpha(x)(g_0(x)) \cap B_\mu(u(z)(x))}$ contains $u(z)(x)$ for every $x \in X$. By (3), $A_z(x)$ also contains $u_2(z)(x)$ when $x \in f^{-1}(y)$. Therefore, $\phi_z(x) \neq \emptyset$ for all $x \in X$, and the map ϕ_z is lower semicontinuous and convex-valued. Applying the Michael convex-valued selection theorem, we obtain a map $g_z \in C^*(X, K)$, which is a selection for ϕ_z. Now let us lift the map u_2 to a map $v : \mathbb{I}^k \to \overline{B}(g_0, \alpha)$ such that v is δ-close to u. To this end, we define $\theta : \mathbb{I}^k \to C^*(X, K)$ by $\theta(z) = \overline{\pi^{-1}(u_2(z)) \cap B_\delta(u(z))}$. We have $g_z \in \theta(z) \neq \emptyset$, $z \in \mathbb{I}^k$. On the other hand, since π is open θ is lower semicontinuous [96, Example 1.1* and Proposition 2.5]. Obviously, every $\theta(z)$ is convex and closed in $C^*(X,K)$, which is, in turn, closed and convex in the Banach space of all bounded continuous functions from X to $\mathbb{R}^p$. Applying again the Michael selection theorem [96, Theorem 3.2″], we obtain a continuous selection $v : \mathbb{I}^k \to C^*(X,K)$ for θ. The map v takes $\mathbb{I}^k$ to $\overline{B}(g_0,\alpha)$, and v is δ-close to u. Moreover, we have $\pi(v(z)) = u_2(z)$ for any $z \in \mathbb{I}^k$, and the map $u_2(z)$, which coincides with the restriction $h_2|(\{z\}\times f^{-1}(y))$, belongs to $\mathcal{P}(y)$. Hence $v(z) \in \mathcal{P}(y)$ for $z \in \mathbb{I}^k$, i.e., $v(\mathbb{I}^k) \subset \overline{B}(g_0,\alpha)\backslash\psi(y)$. □

Proposition 2.5 (see [133]). *Suppose that Y is a C-space (with $\dim Y \le m$) and the family $\{\mathcal{P}(y)\}_{y\in Y}$ satisfies the following conditions:*

(a) *the set-valued map $\psi : Y \to C^*(X,K)$ defined by $\psi(y) = C^*(X,K)\backslash\mathcal{P}(y)$ has a closed graph;*
(b) *for any $y \in Y$, $g \in C^*(X,K)$, and a continuous function $\alpha : X \to (0,\infty)$, the set $\psi(y) \cap \overline{B}(g,\alpha)$ is a Z-set (respectively, Z_m-set) in $\overline{B}(g,\alpha)$, where $\overline{B}(g,\alpha)$ is considered as a subspace of $C^*(X,K)$ with the topology of uniform convergence.*

Then $\mathcal{P}(Y)$ is dense in $C^*(X,K)$ with respect to the source limitation topology.

Proof. It suffices to show that, for any $g_0 \in C^*(X,K)$ and a continuous function $\alpha : X \to (0,\infty)$, there exists a $g \in \overline{B}(g_0,\alpha)\cap\mathcal{P}(Y)$. We endow $C^*(X,K)$ with the topology of uniform convergence and consider the constant convex-valued map $\phi : Y \to C^*(X,K)$ defined by $\phi(y) = \overline{B}(g_0,\alpha_1)$, where $\alpha_1(x) = \min\{\alpha(x),1\}$. Conditions (a) and (b) allow us to apply Theorem 2.1 (respectively, Theorem 2.2) to obtain a continuous map $h : Y \to C^*(X,K)$ such that $h(y) \in \phi(y)\backslash\psi(y)$ for every $y \in Y$. Note that h is a map from Y to $\overline{B}(g_0,\alpha_1)$ and $h(y) \in \mathcal{P}(y)$ for every $y \in Y$. Thus, setting $g(x) = h(f(x))(x)$ for $x \in X$, we obtain a bounded map $g \in \overline{B}(g_0,\alpha)$ such that $g|f^{-1}(y) = h(y)|f^{-1}(y)$, $y \in Y$. Therefore, $g \in \mathcal{P}(y)$ for all $y \in Y$, i.e., $g \in \overline{B}(g_0,\alpha)\cap\mathcal{P}(Y)$. □

2.2. Dimensionally restricted maps. Our first application of the results of Sec. 2.1 is a generalization of the following theorem due to Pasynkov [109] (announced in 1975) and Torunczyk [127].

Theorem 2.6 (Pasynkov–Torunczyk). *Let $f : X \to Y$ be a map between finite-dimensional compact metric spaces. Then the following conditions are equivalent:*

(a) $\dim f \le n$;
(b) *the map $f \times g : X \to Y \times \mathbb{I}^k$ is $(n-k)$-dimensional for almost every map $g : X \to \mathbb{I}^k$, where $k \le n$;*
(c) *for every $0 \le k \le n-1$, there exists a σ-compact subset $A \subset X$ such that $\dim A \le k$ and $\dim f|(X\backslash A) \le n-k-1$.*

Condition (b) in Theorem 2.6 is a parametric version of Hurewicz' result [77] that every n-dimensional compact metrizable space admits a 0-dimensional map to $\mathbb{I}^n$, and (c) is a parametric version of the decomposition theorem for covering dimension. Pasynkov [109] asked whether Theorem 2.6 remains true for infinite-dimensional spaces. Theorem 2.10 below shows that this is so for C-spaces (recall that all finite-dimensional paracompact spaces are C-spaces, but the class of C-spaces contains also infinite-dimensional spaces). Moreover, an even stronger form of Theorem 2.10 is valid. Namely, the implication (a)⇒(b) holds if f is σ-perfect, and (a)⇒(c) holds if f is σ-closed (see [129, Theorems 1.3 and 1.4]). For compact X and Y, Theorem 2.10 was proved by Turygin [134].

However, the question whether Theorem 2.6 remains valid without any dimensional restrictions on Y is open. Significant progress was made by Sternfeld and Levin. In [123], Sternfeld proved that if $f : X \to Y$ is an n-dimensional map between compact metric spaces, then $\dim(f\times g) \le 1$ for almost all $g \in C(X,\mathbb{I}^n)$; equivalently, there exists a σ-compact $(n-1)$-dimensional subset A of X such that $\dim(f|(X\backslash A)) \le 1$. Levin [89] improved Sternfeld's result as follows.

Theorem 2.7 (see [89]). *If $f : X \to Y$ is an n-dimensional map between compact metric spaces, then $\dim(f\times g) \le 0$ for almost all maps $g \in C(X,\mathbb{I}^{n+1})$, or, equivalently, there exists a σ-compact n-dimensional set $A \subset X$ such that $\dim f|(X\backslash A) \le 0$.*

Turygin [134] found a relationship between simplicial approximation of maps and the validity of Theorem 2.6 without dimensional restrictions. Following Uspenskij [135], we say that a map $f : X \to Y$ between compact spaces can be approximated by a k-dimensional simplicial map if, for any open covers ω_X and ω_Y of X and Y, respectively, there exists a commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{p_X} & \mathcal{K}_X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{p_Y} & \mathcal{K}_Y \end{array},$$

where p_X is an ω_X-map, p_Y is an ω_Y-map, and p is a k-dimensional simplicial map between polyhedra $\mathcal{K}_X$ and $\mathcal{K}_Y$. Recall that a map $g : X \to Y$ is called an ω-map with respect to an open cover ω of X if there exists an open cover of Y whose preimage under g refines ω. Uspenskij [135] asked whether every n-dimensional map between metrizable compacta can be approximated by n-dimensional simplicial maps.

Theorem 2.8 (see [134])**.** *Let $f : X \to Y$ be an n-dimensional map between compact metrizable spaces. Then almost all maps $g \in C(X, \mathbb{I}^n)$ satisfy* $\dim(f \times g) \leq 0$ *if and only if f can be approximated by n-dimensional simplicial maps.*

The following assertion is a corollary of Theorems 2.7 and 2.8.

Corollary 2.9 (see [134])**.** *Every n-dimensional map between compact metrizable spaces can be approximated by $(n+1)$-dimensional simplicial maps.*

The Pasynkov–Torunczyk theorem has the following generalization.

Theorem 2.10 (see [129])**.** *If $f : X \to Y$ is a perfect map between metrizable spaces and Y is a C-space, then the following conditions are equivalent*:

(a) $\dim f \leq n$;

(b) *for any $k \leq n$, the space $C(X, \mathbb{I}^k)$ with the source limitation topology contains a dense G_δ-subset of maps g for which $f \times g : X \to Y \times \mathbb{I}^k$ are $(n-k)$-dimensional*;

(c) *for any $0 \leq k \leq n-1$, there exists an F_σ-set $A_k \subset X$ such that* $\dim A_k \leq k$ *and* $\dim f|(X \backslash A_k) \leq n - k - 1$.

Proof. (a)⇒(b) Take $A \subset X$ and $\varepsilon > 0$. We say that A is $(n-k, \varepsilon)$-discrete if A can be covered by a finite family γ of open subsets of X such that each of these subsets is of diameter $\leq\varepsilon$ and any point of X is contained in at most $n-k+1$ elements of γ. Following the general scheme from Sec. 2, for any $\varepsilon > 0$ and $y \in Y$, we define the set $\mathcal{P}_\varepsilon(y)$ set of all maps $g \in C(X, \mathbb{I}^k)$ such that $f^{-1}(y) \cap g^{-1}(z)$ is $(n-k, \varepsilon)$-discrete for every $z \in g(f^{-1}(y))$. It suffices to show that every $\mathcal{P}_\varepsilon(Y)$ is open and dense in $C(X, \mathbb{I}^k)$ with the source limitation topology; indeed, then $\mathcal{P} = \bigcap\limits_{i=1}^{\infty} \mathcal{P}_{1/i}$ is a dense set of type G_δ in $C(X, \mathbb{I}^k)$. Moreover, every fiber of $f \times g$, where $g \in \mathcal{P}$, should be at most $(n-k)$-dimensional.

Lemma 2.11. *If $\varepsilon > 0$ and $y_0 \in Y$, then, for every $g \in \mathcal{P}_\varepsilon(y_0)$, there exists a neighborhood V of y_0 in Y and a $\delta > 0$ such that $h \in \mathcal{P}_\varepsilon(V)$ provided that $h|f^{-1}(V)$ is δ-close to $g|f^{-1}(V)$.*

Proof. Suppose that this is not so for some $g_0 \in \mathcal{P}_\varepsilon(y_0)$ and let $\{V_i\}_{i\geq 1}$ be a local base at y_0 in Y. Then, for every i, there exists a map $g_i \in C(X, \mathbb{I}^k)$ and points $y_i \in V_i$ and $z_i \in g_i(f^{-1}(y_i))$ such that $g_i|f^{-1}(V_i)$ is $1/i$-close to $g_0|f^{-1}(V_i)$ but none of the $g_i^{-1}(z_i) \cap f^{-1}(y_i)$ is $(n-k, \varepsilon)$-discrete. Since f is perfect, the set $F = \bigcup\limits_{i=0}^{\infty} f^{-1}(y_i)$ is compact. Hence $g_0(F)$ is compact as well, and we can assume that $\lim z_i = z_0$ for some $z_0 \in g_0(f^{-1}(y_0))$. Since $g_0 \in \mathcal{P}_\varepsilon(y_0)$, it follows that $g_0^{-1}(z_0) \cap f^{-1}(y_0)$ is $(n-k, \varepsilon)$-discrete. Thus, there exists an $(n-k, \varepsilon)$-discrete open set $U \subset X$ containing $g_0^{-1}(z_0) \cap f^{-1}(y_0)$. The closedness of the map $f \times g$ allows us to find neighborhoods V_0 and W_0 of y_0 and z_0, respectively, for which $f^{-1}(V_0) \cap g_0^{-1}(W_0) \subset U$. Now take a neighborhood W of z_0 in $\mathbb{I}^k$ and $\eta > 0$ such that the distance between W and $\mathbb{I}^k \backslash W_0$ is $\geq \eta$. Finally, choose j so that $V_j \subset V_0$, $z_j \in W$, and $1/j < \eta$. We have

$$f^{-1}(y_j) \cap g_j^{-1}(z_j) \subset f^{-1}(V_j);$$

hence $g_0(x)$ is $1/j$-close to z_j for every $x \in f^{-1}(y_j) \cap g_j^{-1}(z_j)$. Thus,

$$f^{-1}(y_j) \cap g_j^{-1}(z_j) \subset g_0^{-1}(W_0).$$

This implies

$$f^{-1}(y_j) \cap g_j^{-1}(z_j) \subset f^{-1}(V_0) \cap g_0^{-1}(W_0) \subset U.$$

Therefore, $f^{-1}(y_j) \cap g_j^{-1}(z_j)$ is $(n-k, \varepsilon)$-discrete, which contradicts the choice of g_j, y_j, and z_j. $\square$

It follows from Lemmas 2.11 and 2.3 that, for every $\varepsilon > 0$, the set $\mathcal{P}_\varepsilon(Y)$ is open in $C(X, \mathbb{I}^k)$ with the source limitation topology and the set-valued map $\psi_\varepsilon : Y \to C(X, \mathbb{I}^k)$ defined by $\psi_\varepsilon(y) = C(X, \mathbb{I}^k) \backslash \mathcal{P}_\varepsilon(y)$ has a closed graph when $C(X, \mathbb{I}^k)$ is endowed with the topology of uniform convergence.

Lemma 2.12. *For any $\varepsilon > 0$, $m \geq 1$, and $y \in Y$, the set of all maps $g \in C\big(\mathbb{I}^m \times f^{-1}(y), \mathbb{I}^k\big)$ such that $g|\big(\{z\} \times f^{-1}(y)\big)$ is $(n-k)$-dimensional for each $z \in \mathbb{I}^m$ is dense in $C(\mathbb{I}^m \times f^{-1}(y), \mathbb{I}^k)$.*

Proof. Lemma 2.12 is a simple version of a more general result of Pasynkov [109] (see Theorem 2.6(b)). However, Lemma 2.12 has a relatively easy proof, and there is no need to use Pasynkov's result, the proof of which is quite involved.

First, note that it suffices to prove the lemma for $k = n$. Indeed, if $h \in C(\mathbb{I}^m \times f^{-1}(y), \mathbb{I}^k)$ and $\eta > 0$, then we can lift h to a map $h_1 : \mathbb{I}^m \times f^{-1}(y) \to \mathbb{I}^n$ for which $h = p \circ h_1$, where $p : \mathbb{I}^n \to \mathbb{I}^k$ is the canonical projection. Take $g_1 \in C(\mathbb{I}^m \times f^{-1}(y), \mathbb{I}^n)$ such that it is η-close to h_1 and $\pi \times g_1$ is 0-dimensional. The map $g = p \circ g_1$ is η-close to h, and $\pi \times g$ is $(n-k)$-dimensional. Thus, we assume that $k = n$.

According to Lemma 2.13 below, there exists a σ-compact set $A \subset \mathbb{I}^m \times f^{-1}(y)$ such that $\dim A \leq n-1$ and the restriction of the projection $\pi : \mathbb{I}^m \times f^{-1}(y) \to \mathbb{I}^m$ to $\mathbb{I}^m \times f^{-1}(y) \backslash A$ is 0-dimensional. Now we use an idea of Levin [89, $(i) \Leftrightarrow (ii)$]. Consider $A = \bigcup_{i=1}^{\infty} A_i$, where each A_i is compact and $A_i \subset A_{i+1}$. By the Hurewicz theorem [77], almost all maps $g \in C\big(\mathbb{I}^m \times f^{-1}(y), \mathbb{I}^n\big)$ are n-to-1 on every A_i. Let g be such a map. Since $A_i \subset A_{i+1}$ for every i, it follows that g is n-to-1 on A. Take $z \in \mathbb{I}^m$ and $a \in \mathbb{I}^n$. Then $\big(\{z\} \times f^{-1}(y)\big) \cap g^{-1}(a)$ is the union of the sets

$$\big(\{z\} \times f^{-1}(y)\big) \cap \big(g^{-1}(a) \backslash A\big), \quad g^{-1}(a) \cap A.$$

Since $g^{-1}(a) \cap A$ is finite, we have

$$\dim \big(\{z\} \times f^{-1}(y)\big) \cap g^{-1}(a) = \dim \big(\{z\} \times f^{-1}(y)\big) \backslash A \leq 0.$$

The last inequality follows from the fact that the restriction of π to $\mathbb{I}^m \times f^{-1}(y) \backslash A$ is 0-dimensional. $\square$

Lemmas 2.12 and 2.4 imply that, for any $\varepsilon > 0$, $\alpha : X \to (0, \infty)$, and $g \in C(X, \mathbb{I}^k)$, the set $\psi_\varepsilon(y) \cap \overline{B}(g, \alpha)$ is a Z-set in $\overline{B}(g, \alpha)$, where $\overline{B}(g, \alpha)$ is considered as a subset of $C(X, \mathbb{I}^k)$ with the topology of uniform convergence. Applying Proposition 2.5, we conclude that, for every $\varepsilon > 0$, $\mathcal{P}_\varepsilon(Y)$ is dense in $C(X, \mathbb{I}^k)$ with the source limitation topology.

(b)⇒(c) Suppose that A_{n-1} is already defined. For $k < n-1$, we can find an F_σ-subset $A_k \subset A_{n-1}$ for which $\dim A_k \leq k$ and $\dim(A_{n-1} \backslash A_k) \leq n-k-2$ (we can do this by induction; the first step consists of representing A_{n-1} as the union of 0-dimensional G_δ-subsets B_j, where $j = 1, 2, \ldots, n$, and setting $A_{n-2} = \bigcup_{j=1}^{j=n-1} B_j$). Thus, we need only construct A_{n-1}. First, we prove the following analog of Sternfeld's lemma [123, Lemma 1], which was proved for compact metrizable spaces.

Lemma 2.13 (see [129]). *Suppose that M is metrizable and K is a compact metric space with $\dim K \leq n$. Then there exists an F_σ-set $B \subset M \times K$ such that $\dim B \leq n-1$ and $\pi_M|(M \times K)\backslash B$ is 0-dimensional, where $\pi_M : M \times K \to M$ is the projection.*

Proof. As in [123], the proof reduces to the case of $n = 1$ and $K = [0,1]$. Thus, let us prove the existence of a 0-dimensional F_σ-set B in $M \times \mathbb{I}$ such that any set $(\{y\} \times \mathbb{I})\backslash B$, where $y \in M$, is 0-dimensional. Suppose that Z is a 0-dimensional metrizable space and $h : Z \to M$ is a perfect surjection. According to [110, Proposition 9.1], there exists a map $g : Z \to Q$ such that $h \times g : Z \to M \times Q$ is a closed embedding, where Q denotes the Hilbert cube. Next, let D be the Cantor set; take a surjection $p : D \to Q$ admitting an averaging operator between the function spaces $C(D)$ and $C(Q)$ [111] (such maps are called Milyutin maps). According to [33], there exists a lower semicontinuous compact-valued map $\phi : Q \to D$ with $\phi(y) \subset p^{-1}(y)$ for every $y \in Q$. Applying Michael's 0-dimensional selection theorem [98], we obtain a continuous selection q for the map $\phi \circ g$. Obviously, $h \times q : Z \to M \times D$ is a closed embedding; thus, $Z_0 = (h \times q)(Z)$ is a 0-dimensional closed subset of $M \times D$. Finally, considering D as a subset of $\mathbb{I}$, we set $Z_r = \{(h(z), q(z) + r) : z \in Z\} \subset M \times \mathbb{I}$ for every rational $r \in \mathbb{I}$, where the sum $q(z) + r$ is taken in $\mathbb{R}$ modulo 1. Each Z_r is a closed subset of $M \times \mathbb{I}$ homeomorphic to Z; therefore, $B = \bigcup\{Z_r : r \text{ is rational}\}$ is 0-dimensional, and F_σ in $M \times \mathbb{I}$. Moreover, $(\{y\} \times \mathbb{I})\backslash B$ is 0-dimensional for every $y \in M$. □

Now we are ready to complete the proof of (b)⇒(c). Take $g : X \to \mathbb{I}^n$ such that $f \times g : X \to Y \times \mathbb{I}^n$ is 0-dimensional. By Lemma 2.13, there exists an F_σ-set $B \subset Y \times \mathbb{I}^n$ for which $\dim B \leq n-1$ and every $(\{y\} \times \mathbb{I}^n)\backslash B$ with $y \in Y$ is 0-dimensional. The set $A_{n-1} = (f \times g)^{-1}(B)$ has type F_σ in X. Since $f \times g$ is perfect, it follows from the generalized Hurewicz theorem on closed maps lowering dimension [119] that $\dim A_{n-1} \leq n-1$ and $\dim(f^{-1}(y)\backslash A_{n-1}) \leq 0$ for every $y \in Y$.

(c)⇒(a) The existence of an F_σ-set $A_{n-1} \subset X$ such that $\dim A_{n-1} \leq n-1$ and $\dim f|(X\backslash A_{n-1}) \leq 0$ implies that every $f^{-1}(y)$ with $y \in Y$ is the union of $f^{-1}(y) \cap A_{n-1}$ and $f^{-1}(y)\backslash A_{n-1}$. Since $\dim f^{-1}(y) \cap A_{n-1} \leq n-1$ and $\dim f^{-1}(y)\backslash A_{n-1} \leq 0$, it follows that $\dim f^{-1}(y) \leq n$. □

As an application of Theorem 2.10, we obtain a parametric version of the following Bogatyi's decomposition theorem for metrizable spaces [6]: For every metrizable n-dimensional space M, there exist countably many 0-dimensional G_δ-subsets $M_k \subset M$ such that $M = \bigcup\limits_{i=1}^{i=n+1} M_{k(i)}$ for all pairwise distinct $k(1), \ldots, k(n+1)$ in $\mathbb{N}$.

Proposition 2.14 (see [129]). *Let $f : X \to Y$ be a closed n-dimensional surjection from a metrizable space X onto a metrizable C-space Y. Then there exists a sequence $\{A_k\}$ of G_δ-subsets of X such that every restriction $f|A_k$ is 0-dimensional and $X = \bigcup\limits_{k \in P} A_k$ for any $P \subset \mathbb{N}$ of cardinality $n+1$.*

Proof. Following the proof of [129, Theorem 1.3], we take closed sets $X_i \subset X$ with $i \geq 0$ and a map $g : X \to \mathbb{I}^n$ such that $f|X_0$ is perfect, $X\backslash X_0 = \bigcup\limits_{i \geq 1} X_i$, $\dim(f \times g) = 0$, and each $g|X_i$ with $i \geq 1$ is uniformly 0-dimensional. According to Bogatyi's theorem, there exists a sequence of 0-dimensional G_δ-subsets $B_k \subset \mathbb{I}^n$ such that $\mathbb{I}^n$ is the union of any $n+1$ elements of this sequence. Let $A_k = (f \times g)^{-1}(Y \times B_k)$ for $k \in \mathbb{N}$. The only nontrivial condition we need to check is that each restriction $f|A_k$ is 0-dimensional, i.e., $\dim f^{-1}(y) \cap A_k \leq 0$ for all $y \in Y$ and $k \geq 1$. Given y and k, we have $f^{-1}(y) \cap A_k = \bigcup\limits_{i \geq 0} g_i^{-1}(B_k)$, where g_i denotes the restriction $g|(f^{-1}(y) \cap X_i)$. Since every $g_i^{-1}(B_k)$ is closed in $f^{-1}(y) \cap A_k$, it suffices to show that the sets $g_i^{-1}(B_k)$ with $i \geq 0$ are 0-dimensional. For $i = 0$, this follows from the Hurewicz dimension-lowering theorem [76], because g_0 is a perfect 0-dimensional map. For $i \geq 1$, it suffices to note that $g|X_i$ is uniformly 0-dimensional and the preimage of any 0-dimensional set under a uniformly 0-dimensional map is again 0-dimensional. □

2.3. Mappings lowering the dimension of fibers of finite-dimensional maps. In this section, we consider yet another application of Theorem 2.10 and the results of Sec. 2.1. Answering a question of R. Pol, Uspenskij [136] proved the following theorem: Let $f : X \to Y$ be a light map (i.e., every fiber $f^{-1}(y)$ is 0-dimensional) between compact spaces, and let $\mathcal{A}$ be the set of all functions $g : X \to \mathbb{I} = [0,1]$ such that $g(f^{-1}(y))$ is 0-dimensional for all $y \in Y$. Then $\mathcal{A}$ is a dense G_δ-subset of the function space $C(X,\mathbb{I})$, provided that Y is a C-space (the case of countable-dimensional Y was proved earlier by Torunczyk). We extend this result as follows.

Theorem 2.15 (see [130])**.** *Let $f : X \to Y$ be a σ-perfect surjection such that $\dim f \le n$ and Y is a paracompact C-space. Suppose that*

$$\mathcal{P} = \{g \in C(X,\mathbb{I}^{n+1}) : \dim g(f^{-1}(y)) \le n \text{ for each } y \in Y\}.$$

Then $\mathcal{P}$ is a dense G_δ-set in $C(X,\mathbb{I}^{n+1})$ with respect to the source limitation topology.

Proof. All function spaces in this proof, unless otherwise specified, are endowed with the source limitation topology. Recall that f is said to be σ-perfect if there exists a sequence $\{X_i\}$ of closed subsets of X such that every restriction $f|X_i$ is perfect and the sets $f(X_i)$ are closed in Y.

First, let us reduce the proof of Theorem 2.15 to the case where f is perfect. Take a sequence $\{X_i\}$ of closed subsets of X such that every map $f_i = f|X_i : X_i \to Y_i = f(X_i)$ is perfect and every $Y_i \subset Y$ is closed. Consider the maps $\pi_i : C(X,\mathbb{I}^{n+1}) \to C(X_i,\mathbb{I}^{n+1})$ defined by $\pi(g) = g|X_i$ and the sets

$$\mathcal{P}_i = \{g \in C(X_i,\mathbb{I}^{n+1}) : \dim g(f_i^{-1}(y)) \le n \text{ for each } y \in Y_i\}.$$

If Theorem 2.15 holds for perfect maps, then every $\mathcal{P}_i$ is a dense G_δ-set in $C(X_i,\mathbb{I}^{n+1})$; hence so are the sets $\pi_i^{-1}(\mathcal{P}_i)$ in $C(X,\mathbb{I}^{n+1})$, because the maps π_i are open and surjective. Finally, note that $\mathcal{P}$ is the intersection of all $\pi_i^{-1}(\mathcal{P}_i)$, which completes the proof, because $C(X,\mathbb{I}^{n+1})$ has the Baire property. Thus, we can assume that f is perfect.

We can also assume that f is 0-dimensional. Indeed, if $\dim f \le n$, then, by Theorem 2.10, we have $\dim(f \times g) \le 0$ for almost all maps $g \in C(X,\mathbb{I}^n)$. For any such $g \in C(X,\mathbb{I}^n)$, almost all functions $h \in C(X,\mathbb{I})$ have the property $\dim h(f^{-1}(y) \cap g^{-1}(z)) \le 0$ for any $(y,z) \in Y \times \mathbb{I}^n$. The last inequality implies that, for every $y \in Y$, the projection map $p_y : (g \times h)(f^{-1}(y)) \to g(f^{-1}(y))$ defined by $p_y\big(g(z), h(z)\big) = g(z)$ for $z \in f^{-1}(y)$ is 0-dimensional. Obviously, p_y is perfect, and $\dim g(f^{-1}(y)) \le n$ implies $\dim(g \times h)(f^{-1}(y)) \le n$ by the Hurewicz dimension-lowering theorem. Therefore, we can assume that f is a perfect 0-dimensional map.

For every open set V in $\mathbb{I}$ and every $y \in Y$, let $\mathcal{P}_V(y)$ be the set of all $g \in C(X,\mathbb{I})$ such that V is not contained in $g(f^{-1}(y))$. For $H \subset Y$, we also let $\mathcal{P}_V(H)$ be the intersection of all $\mathcal{P}_V(y)$ for $y \in H$. Uspenskij's argument from [136] implies that it suffices to show that every set $\mathcal{P}_V(Y)$ is dense and open in $C(X,\mathbb{I})$. Indeed, choose a countable base $\mathcal{B}$ in $\mathbb{I}$. Since a subset of $\mathbb{I}$ is at most 0-dimensional if and only if it does not contain any $V \in \mathcal{B}$, it follows that $\mathcal{P}$ is the intersection of all $\mathcal{P}_V(Y)$ over $V \in \mathcal{B}$. Since $C(X,\mathbb{I})$ has the Baire property, $\mathcal{P}$ is a dense G_δ-set in $C(X,\mathbb{I})$.

Lemma 2.16. *Suppose that $y_0 \in Y$ and $V \in \mathcal{B}$. Then, for every $g \in \mathcal{P}_V(y_0)$, there exists a neighborhood W of y_0 in Y and a $\delta > 0$ such that $h \in \mathcal{P}_V(W)$ provided that $h|f^{-1}(W)$ is δ-close to $g|f^{-1}(W)$.*

Proof. If $g \in \mathcal{P}_V(y_0)$, then there exists a $z_0 \in V \backslash g(f^{-1}(y_0))$. Let δ be a positive number such that 2δ is smaller than the distance between z_0 and $g(f^{-1}(y_0))$. Since f is perfect, y_0 has a neighborhood W in Y such that $|g(x) - z_0| > \delta$ for all $x \in f^{-1}(W)$. Therefore, $z_0 \notin h(f^{-1}(W))$ for every $h \in C(X,\mathbb{I})$ such that $h|f^{-1}(W)$ is δ-close to $g|f^{-1}(W)$, and any such h belongs to $\mathcal{P}_V(W)$. □

Lemma 2.17 (see [130])**.** *For any $y \in Y$ and $m \ge 1$, the set of all maps $g \in C(\mathbb{I}^m \times f^{-1}(y),\mathbb{I})$ such that $\dim g(\{z\} \times f^{-1}(y)) \le 0$ for every $z \in \mathbb{I}^m$ is dense in $C(\mathbb{I}^m \times f^{-1}(y),\mathbb{I})$.*

Proof. Since f^{-1} is 0-dimensional, the set $C(y)$ of all $h \in C(f^{-1}(y), \mathbb{R})$ such that $h(f^{-1}(y))$ is finite is dense in $C(f^{-1}(y), \mathbb{R})$. Hence, by the Stone–Weierstrass theorem, all polynomials in elements of the family $\gamma = \{t \cdot h : t \in C(\mathbb{I}^m, \mathbb{R}), h \in C(y)\}$ form a dense subset $\mathcal{K}$ of $C(\mathbb{I}^m \times f^{-1}(y), \mathbb{R})$. Fix a retraction $r : \mathbb{R} \to \mathbb{I}$ and consider the map $u_r : C(\mathbb{I}^m \times f^{-1}(y), \mathbb{R}) \to C(\mathbb{I}^m \times f^{-1}(y), \mathbb{I})$ defined by $u_r(h) = r \circ h$. The set $u_r(\mathcal{K})$ is dense in $C(\mathbb{I}^m \times f^{-1}(y), \mathbb{I})$. The observation that $g(\{z\} \times f^{-1}(y))$ is finite for every $g \in u_r(\mathcal{K})$ and $z \in \mathbb{I}^m$ completes the proof. □

Lemmas 2.16 and 2.17, together with the results of Sec. 2.1, imply that $\mathcal{P}_V(Y)$ is dense and open in $C(X, \mathbb{I})$ for every $V \in \mathcal{B}$. This completes the proof of Theorem 2.15. □

Corollary 2.18 (see [130]). *If X, Y, and f satisfy the conditions of Theorem 2.15 and $n+1 \leq m \leq \omega$, then there exists a dense G_δ-set $\mathcal{P}_m$ in $C(X, \mathbb{I}^m)$ with the source limitation topology such that $\dim g(f^{-1}(y)) \leq n$ for every $g \in \mathcal{P}_m$ and $y \in Y$.*

Proof. As in the proof of Theorem 2.15, we can assume that f is perfect. First, consider the case where m is an integer $\geq n+1$. Let $\exp_{n+1}$ be the family of all subsets of $A = \{1, 2, \ldots, m\}$ having cardinality precisely $n+1$, and let $\pi_B : \mathbb{I}^m \to \mathbb{I}^B$ denote the corresponding projections for $B \in \exp_{n+1}$. It can be shown that

$$C(X, \mathbb{I}^m) = C(X, \mathbb{I}^B) \times C(X, \mathbb{I}^{A \setminus B});$$

thus, each projection $p_B : C(X, \mathbb{I}^m) \to C(X, \mathbb{I}^B)$ is open. By Theorem 2.15, every set

$$\mathcal{P}_B = \{g \in C(X, \mathbb{I}^B) : \dim g(f^{-1}(y)) \leq n \text{ for all } y \in Y\}$$

is dense and has type G_δ in $C(X, \mathbb{I}^B)$; hence so is the set $p_B^{-1}(\mathcal{P}_B)$ in $C(X, \mathbb{I}^m)$. Consequently, the intersection $\mathcal{P}_m$ of all $\mathcal{P}_B$ with $B \in \exp_{n+1}$ is dense as well, and G_δ in $C(X, \mathbb{I}^m)$. Moreover, if $g \in \mathcal{P}_m$ and $y \in Y$, then $\dim \pi_B(g(f^{-1}(y))) \leq n$ for any $B \in \exp_{n+1}$. According to a result of Nöbeling [63, Problem 1.8.C], these inequalities imply $\dim g(f^{-1}(y))) \leq n$.

Now suppose that $m = \omega$. Let $\exp_{\geq n+1}$ denote the family of all finite sets $B \subset \omega$ of cardinality $|B| \geq n+1$. We use the above notation; for any $B \in \exp_{\geq n+1}$, $\pi_B : Q = \mathbb{I}^\omega \to \mathbb{I}^B$ and $p_B : C(X, Q) \to C(X, \mathbb{I}^B)$ denote the corresponding projections. The intersection $\mathcal{P}_\omega$ of all $p_B^{-1}(\mathcal{P}_B)$ is a dense G_δ-set in $C(X, Q)$. We must only show that $\dim g(f^{-1}(y)) \leq n$ for any $g \in \mathcal{P}_\omega$ and $y \in Y$. Take an increasing sequence $\{B(k)\}$ in $\exp_{\geq n+1}$ which covers ω and consider the inverse sequence $\mathcal{S} = \{\pi_{B(k)}(g(f^{-1}(y))), \pi_k^{k+1}\}$, where $\pi_k^{k+1} : \pi_{B(k+1)}(g(f^{-1}(y))) \to \pi_{B(k)}(g(f^{-1}(y)))$ are the natural projections. Obviously, $g(f^{-1}(y))$ is the limit space of $\mathcal{S}$. Moreover, $g \in \mathcal{P}_\omega$ implies $\pi_{B(k)} \circ g \in \mathcal{P}_{B(k)}$ for any k; thus, all $\pi_{B(k)}(g(f^{-1}(y)))$ are at most n-dimensional. Hence $\dim g(f^{-1}(y)) \leq n$. □

2.4. Regularly branched maps. In this section, we present a parametric version of Hurewicz's theorem [77] on regularly branched maps. Recall that a map $g : X \to Z$ is said to be regularly branched (this term was introduced by Dranishnikov, Repovš, and Ščepin in [47]) if $\dim B_n(g) \leq n \cdot \dim X - (n-1) \cdot \dim Z$ for any $n \geq 1$, where $B_n(g)$ denotes the set $\{z \in Z : |g^{-1}(z)| \geq n\}$. We extend the notion of regularly branched maps as follows: A map $g : X \to Z$ is said to be regularly branched with respect to a fixed map $f : X \to Y$ (briefly, f-regularly branched) if

$$\dim B_n(f \times g) \leq n \cdot \big(\dim f + \dim Y\big) - (n-1) \cdot \big(\dim Z + \dim Y\big)$$

for every $n \geq 1$.

Obviously, if f is a constant map (and Y is a point), the notions of f-regularly branched and regularly branched maps coincide.

Hurewicz Theorem. *Suppose that X is a finite-dimensional compact metrizable space and $p \geq 1$. Then the set of all regularly branched maps $g : X \to \mathbb{R}^p$ contains a dense G_δ-subset of the space $C(X, \mathbb{R}^p)$.*

First, let us prove a noncompact version of a result by Levin and Lewis [93, Proposition 4.4]. Note that, for separable metrizable spaces, Proposition 2.19 follows from [127, Lemma 2].

Proposition 2.19 (see [133]). *Suppose that $f : X \to Y$ is a perfect 0-dimensional map between metrizable spaces and $\dim Y \le m$. Then $C(X, \mathbb{I})$ with the source limitation topology contains a dense G_δ-subset of maps g such that each fiber of $f \times g$ contains at most $m+1$ points.*

Proof. Take a map $\theta : X \to Q$ for which $f \times \theta : X \to Y \times Q$ is an embedding (such a θ exists according to [110, 131]), where Q is the Hilbert cube. Let $\{W_i\}_{i \in \mathbb{N}}$ be a countable base of open sets in Q, and let $\mathcal{A}$ be the collection of the closures of $\theta^{-1}(W_i)$ in X for $i \ge 1$. There are countably many families $\Gamma = \{A_1, A_2, \dots, A_{m+2}\}$ consisting of $m+2$ disjoint elements of $\mathcal{A}$. For any such Γ and $y \in Y$, let $\mathcal{P}_\Gamma(y)$ denote the set of all $g \in C(X, \mathbb{I})$ such that every $g^{-1}(z) \cap (f^{-1}(y)$, where $z \in \mathbb{I}$, meets at most $m+1$ elements of Γ. As in Sec. 2, for $H \subset Y$, we set $\mathcal{P}_\Gamma(H) = \cap\{\mathcal{P}_\Gamma(y) : y \in H\}$. Since the intersection of all $\mathcal{P}_\Gamma(Y)$ consists of maps g such that every fiber of $f \times g$ contains at most $m+1$ points, it suffices to show that any $\mathcal{P}_\Gamma(Y)$ is open and dense in $C(X, \mathbb{I})$ with respect to the source limitation topology.

Lemma 2.20 (see [133]). *If $\Gamma = \{G_1, \dots, G_{m+2}\}$ and $y \in Y$, then, for every $g \in \mathcal{P}_\Gamma(y)$ there exists a neighborhood V of y in Y and a $\delta > 0$ such that $h \in \mathcal{P}_\Gamma(V)$ provided that $h|f^{-1}(V)$ is δ-close to $g|f^{-1}(V)$.*

Proof. Suppose this is not true for some $g_0 \in \mathcal{P}_\Gamma(y)$ and let $\{V_i\}_{i \ge 1}$ be a local base at y in Y. Then, for every i, there exist functions $g_i \in C(X, \mathbb{I})$ and points $y_i \in V_i$ and $z_i \in \mathbb{I}$ such that $g_i|f^{-1}(V_i)$ is $1/i$-close to $g_0|f^{-1}(V_i)$ but $g_i^{-1}(z_i) \cap f^{-1}(y_i)$ meets all of the $\le m+2$ elements of Γ. Since f is closed, we can assume that $U_i = f^{-1}(V_i) \subset g_0^{-1}(W_i)$, where the U_i and W_i are the $1/i$ neighborhoods of $f^{-1}(y)$ and $g_0(f^{-1}(y))$ in X and $\mathbb{I}$, respectively, and $z_i \in W_i$. Passing to subsequences, we can also assume that $\lim z_i = z_0 \in g_0(f^{-1}(y))$. Then $g_0^{-1}(z_0) \cap f^{-1}(y)$ intersects at most $m+1$ elements of Γ, say, the first $m+1$ elements. Take points $a_i \in g_i^{-1}(z_i) \cap f^{-1}(y_i)$ and $b_i \in f^{-1}(y)$ such that $a_i \in G_{m+2}$ and $dist(a_i, b_i) \le 1/i$ for all i. Again, we can assume that $\lim b_i = b_0$ for some $b_0 \in f^{-1}(y)$. Then $\lim a_i = b_0 \in g_0^{-1}(z_0) \cap f^{-1}(y)$, and $b_0 \notin G_{m+2}$. This implies $a_i \notin G_{m+2}$ for almost all i, which contradicts the choice of the points a_i. □

Lemma 2.21. *For any Γ and $y \in Y$, the set of all functions $g \in C\big(\mathbb{I}^m \times f^{-1}(y)\big)$ such that $g|\big(\{z\} \times f^{-1}(y)\big) \in \mathcal{P}_\Gamma(y)$ for every $z \in \mathbb{I}^m$ is dense in $C\big(\mathbb{I}^m \times f^{-1}(y)\big)$.*

Proof. According to [93, Proposition 4.4], every $h \in C\big(\mathbb{I}^m \times f^{-1}(y)\big)$ can be approximated by functions $g \in C\big(\mathbb{I}^m \times f^{-1}(y)\big)$ such that every $g^{-1}(t) \cap \big(\{z\} \times f^{-1}(y)\big)$, where $z \in \mathbb{I}^m$ and $t \in \mathbb{I}$, contains at most $m+1$ points. This implies $g|\big(\{z\} \times f^{-1}(y)\big) \in \mathcal{P}_\Gamma(y)$ for each $z \in \mathbb{I}^m$. □

Combining the last two lemmas and the results of Sec. 2.1, we see that every $\mathcal{P}_\Gamma(Y)$ is dense and open in $C(X, \mathbb{I})$ with respect to the source limitation topology. This completes the proof of Proposition 2.19. □

The following theorem is a parametrization of Hurewicz's result.

Theorem 2.22 (see [133]). *Suppose that $f : X \to Y$ is a σ-perfect map between finite-dimensional metrizable spaces and $p \ge 1$. Then the space $C^*(X, \mathbb{R}^p)$ with the source limitation topology contains a dense G_δ-set $\mathcal{H}$ consisting of f-regularly branched maps.*

Proof. First, let show that the proof of Theorem 2.22 can be reduced to the case of perfect f. Suppose that X is the union of an increasing sequence of its closed subsets X_i such that every restriction $f_i = f|X_i$ is perfect and every $Y_i = f(X_i) \subset Y$ is closed. Then, applying Theorem 2.22 to any map $f_i : X_i \to Y_i$ and using the observation that the maps $\pi_i : C^*(X, \mathbb{R}^p) \to C^*(X_i, \mathbb{R}^p)$ defined by $\pi_i(g) = g|X_i$ are surjective and open, we conclude that there exists a dense G_δ-set $G \subset C^*(X, \mathbb{R}^p)$ consisting of maps g such that $g_i = g|X_i$ is f_i-regularly branched for every i. Take $g \in G$ and $n \ge 1$. For any i, the set $B_n(f_i \times g_i)$ is of type F_σ in $(f_i \times g_i)(X_i)$ (see [63]) and $(f_i \times g_i)(X_i) \subset Y \times \mathbb{R}^p$ is closed (recall that each $Y_i \subset Y$ is closed and the map $f_i \times g_i : X_i \to Y_i \times \mathbb{R}^p$ is perfect). Thus, all of the sets $B_n(f_i \times g_i)$ are of type F_σ in $Y \times \mathbb{R}^p$. Moreover,

$$\dim B_n(f_i \times g_i) \le n \cdot \big(\dim f_i + \dim Y_i\big) - (n-1) \cdot \big(p + \dim Y_i\big) \le n \cdot \big(\dim f + \dim Y\big) - (n-1) \cdot \big(p + \dim Y\big).$$

Therefore,

$$\dim \bigcup_{i=1}^{\infty} B_n(f_i \times g_i) \le n \cdot \big(\dim f + \dim Y\big) - (n-1) \cdot \big(p + \dim Y\big).$$

On the other hand,

$$B_n(f \times g) \subset \bigcup_{i=1}^{\infty} B_n(f_i \times g_i).$$

Consequently,

$$\dim B_n(f \times g) \le n \cdot \big(\dim f + \dim Y\big) - (n-1) \cdot \big(p + \dim Y\big)$$

for any $g \in G$ and $n \ge 1$. Hence G consists of f-regularly branched maps. Thus, we can assume that f is perfect. Moreover, we can also assume that $p > \dim f$, because, by definition, every $g \in C(X, \mathbb{R}^p)$ is f-regularly branched if $p \le \dim f$.

Finally, by Theorem 2.10(b), the map f can be assumed to be 0-dimensional. Thus, below, we assume that f is a perfect 0-dimensional map, $p \ge 1$, and $\dim Y = m$. Let $l = l(m,p) = [m/p] + 1$, where $[m/p]$ denotes the integer part of m/p.

Let us show that $f \times g$ is at most l-to-1 for almost all maps $g \in C^*(X, \mathbb{R}^p)$ by induction on p. For $p = 1$, this follows from Proposition 2.19. Suppose that $p > 1$ and let $m = (l-1)p + t$ for $0 \le t < p$. We have $Y = Y_1 \cup Y_2$, where Y_1 is an F_σ-subset of Y with $\dim Y_1 \le m - l = (l-1)(p-1) + t - 1$ and $\dim Y_2 \le l - 1$. Suppose also that $g = g_1 \times g_2 : X \to \mathbb{R}^{p-1} \times \mathbb{R}$. Since $[(m-l)/(p-1)] + 1 = l$, the induction hypothesis implies that g_1 can be approximated by a map $g_1^* : X \to \mathbb{R}^{p-1}$ such that $f \times g_1^*$ is at most l-to-1 on $f^{-1}(Y_1)$. Let B denote the union of all fibers of $f \times g_1^*$ containing more than l points. Then B has type F_σ in X and is disjoint from $f^{-1}(Y_1)$; hence $f(B) \subset Y_2$. By the induction hypothesis, g_2 can be approximated by a map $g_2^* : X \to \mathbb{R}$ such that $f \times g_2^*$ is at most l-to-1 on $f^{-1}(f(B))$. Thus, g is approximated by the map $g^* = g_1^* \times g_2^*$, and $f \times g^*$ is at most l-to-1. This implies that the maps $g \in C^*(X, \mathbb{R}^p)$ such that $f \times g$ is at most l-to-1 form a dense subset of $C^*(X, \mathbb{R}^p)$. To complete the induction construction, we must show that this set is of type G_δ in $C^*(X, \mathbb{R}^p)$. To this end, following the proof of Proposition 2.19, we take a map $\theta : X \to Q$ such that $f \times \theta : X \to Y \times Q$ is an embedding and a countable base $\{W_i\}_{i \in \mathbb{N}}$ of open sets in Q. Consider the collection $\mathcal{A}$ of all closures of $\theta^{-1}(W_i)$ in X for $i \ge 1$. There are countably many families $\Gamma = \{A_1, A_2, \dots, A_{l+1}\}$ consisting of $l+1$ disjoint elements of $\mathcal{A}$; for any such Γ and $y \in Y$, let $\mathcal{P}_\Gamma(y)$ denote the set of all $g \in C^*(X, \mathbb{R}^p)$ such that every $g^{-1}(z) \cap f^{-1}(y)$ with $z \in \mathbb{R}^p$ meets at most l elements of Γ. As in the proof of Proposition 2.19, it can be shown that any set $\mathcal{P}_\Gamma(Y) = \cap\{\mathcal{P}_\Gamma(y) : y \in Y\}$ is open in $C^*(X, \mathbb{R}^p)$. Therefore, the maps $g \in C^*(X, \mathbb{R}^p)$ for which $f \times g$ is at most l-to-1 form a G_δ-set in $C^*(X, \mathbb{R}^p)$, being the intersection of all $\mathcal{P}_\Gamma(Y)$.

Now we can complete the proof of Theorem 2.22. Let $Y_i \subset Y$, where $0 \le i \le m$, be F_σ-subsets of Y such that $Y_0 \subset Y_1 \subset \cdots \subset Y_m$, $\dim Y_i \le i$, and $\dim Y \backslash Y_i \le m - i - 1$. The above considerations imply that $C^*(X, \mathbb{R}^p)$ contains a dense G_δ-subset G of maps g such that $f \times g$ is at most $l(i,p)$-to-1 on $f^{-1}(Y_i)$ for every $0 \le i \le m$. Moreover, by Theorem 2.15, we can additionally assume that $g(f^{-1}(y))$ is 0-dimensional for all $y \in Y$ and all $g \in G$. It remains to show that every $g \in G$ is f-regularly branched. Take $g \in G$ and $n \ge 1$ and let $\pi_Y : Y \times \mathbb{R}^p \to Y$ be the projection onto Y. Since $B_n(f \times g)$ is an F_σ-set in $(f \times g)(X)$ and $\pi_Y|(f \times g)(X)$ is a perfect map, it follows that $\pi_Y\big(B_n(f \times g)\big)$ is an F_σ-set in Y. Moreover, since each $g(f^{-1}(y))$ is 0-dimensional, it follows that $\dim B_n(f \times g)$ is at most the dimension of $\pi_Y\big(B_n(f \times g)\big)$. On the other hand, if $(f \times g)^{-1}(y, z)$ contains $\ge n$ points, then $y \notin Y_{p(n-1)-1}$. Hence $\pi_Y\big(B_n(f \times g)\big)$ is contained in $Y \backslash Y_{p(n-1)-1}$. Consequently, $\dim \pi_Y\big(B_n(f \times g)\big) \le m - (n-1)p$, whence $\dim B_n(f \times g) \le m - (n-1)p$. Since $n(\dim f + \dim Y) - (n-1)(p + \dim Y) = m - (n-1)p$, the last inequality shows that g is regularly f-branched. ☐

Corollary 2.23 (see [133]). *Suppose that k, p, m, and n are integers such that $k+m+1 \le (p-k)n$. Then, for any σ-perfect map $f : X \to Y$ with $\dim f \le k$ and $\dim Y \le m$, the space $C^*(X, \mathbb{R}^p)$ contains a dense G_δ-set consisting of maps g such that $|(f \times g)^{-1}(z)| \le n$ for every $z \in Y \times \mathbb{R}^p$.*

Corollary 2.23 follows directly from Theorem 2.22. Indeed, if the conditions of this corollary hold and $g \in C^*(X, \mathbb{R}^p)$ is f-regularly branched, then

$$\dim B_{n+1}(f \times g) \le (n+1)(k+m) - n(p+m) \le -1.$$

Thus, $f \times g$ is $\le n$-to-one for all f-regularly branched maps. Theorem 2.22 has also the following corollary (actually, it follows from Corollary 2.23), which was proved in [132]; it provides positive solutions to two conjectures of Bogatyi, Fedorchuk, and van Mill [9].

Corollary 2.24 (see [132]). *Suppose that $f : X \to Y$ is a σ-perfect map with $\dim f \le k$ and $\dim Y \le m$. Then, for any $p \ge 1$, $C^*(X, \mathbb{R}^{p+k})$ contains a dense G_δ-set consisting of maps g such that $|(f \times g)^{-1}(z)| \le \max\{k+m-p+2, 1\}$ for all $z \in Y \times \mathbb{R}^p$.*

For $p \ge 2k+m+1$, Corollary 2.23 (as well as Corollary 2.24) implies the existence of a dense G_δ-set G in $C^*(X, \mathbb{R}^p)$ with the source limitation topology such that $f \times g$ is one-to-one for every $g \in G$. Therefore, all maps $f \times g$, where $g \in G$, are embeddings provided that f is perfect. We obtain the parametric version of the Nöbeling–Pontryagin embedding theorem which was proved in [109, 110, 131].

The first version of [133] contained the question whether the set $\mathcal{H}$ in Theorem 2.22 can consist of maps g such that

$$\dim B_n(f \times g) \le n \cdot \dim X - (n-1) \cdot \big(p + \dim Y\big)$$

for every $n \ge 1$. The referee of [133] and S. A. Bogatyi independently gave a negative answer. The example suggested by Bogatyi is as follows. Let T be a compact metrizable space such that it cannot be embedded in $\mathbb{R}^{2m}$ with $m \ge 2$ and $\dim T \le m$. Consider the disjoint sum $X = \mathbb{I}^m \oplus T$ and the map $f : X \to \mathbb{I}^m$ defined by $f(x) = x$ for $x \in \mathbb{I}^m$ and $f(x) = x_0 \in \mathbb{I}^m$ for $x \in T$. The existence of a map $g : X \to \mathbb{R}^{m+2}$ with the above property would imply that g is an embedding of T into $\mathbb{R}^{m+2}$, which cannot exist, because $m+2 \le 2m$.

In this connection, we mention the following result of Torunczyk, which is not covered by Theorem 2.22.

Theorem 2.25 (see [128]). *Suppose that $f : X \to Y$ is a map of finite-dimensional compact metrizable spaces X and Y such that $\dim f = k$ and $\dim X = n$. Then there exists a map $g \in C(X, \mathbb{R}^{n+k+1})$ for which $f \times g$ is an embedding.*

Note that, according to [5, Corollary 11], for any m, there exists a polyhedron X with $\dim X = m$ such that every map $g \in C(X, \mathbb{R}^{m+1})$ has a fiber containing at least $m+1$ points. Therefore, the inequality in the definition of a regularly branched map

$$\dim B_n(f \times g) \le n \cdot \big(\dim f + \dim Y\big) - (n-1) \cdot \big(\dim Z + \dim Y\big)$$

cannot be improved.

3. Roberts-Type Embeddings and Conversion of the Tverberg Transversal Theorem

This section contains some generalizations of theorems due to Roberts [116], Boltyanskii [12], and Berkowitz and Roy [3]. The results are taken from the joint paper [10] of Bogatyi and the second author.

Throughout this section, $\Pi^k \subset \mathbb{R}^m$ denotes a k-plane in $\mathbb{R}^m$ (not necessarily coordinate). We write $\Pi^t \subset \Pi^T$ if Π^t and Π^T are two coordinate planes, $t \le T$, and Π^t is a linear subspace of Π^T.

3.1. Conversion of the Tverberg transversal theorem. Our first goal in this section is to prove Theorem 3.1 below, which is a converse of the transversal Tverberg theorem and implies the Berkowitz–Roy theorem [3, 68]. The first part of Theorem 3.1 for $t=0$, $T=m$, and $d-t+1\le q$ was stated as a conjecture in [7, Conjecture 2] and [11, Conjecture 4.2].

Recall that a real number v is said to algebraically depend on real numbers $u_1,\dots,u_k$ if v satisfies the equation $p_0(u)+p_1(u)v+\dots+p_n(u)v^n=0$, where $p_0(u),\dots,p_n(u)$ are polynomials in $u_1,\dots,u_k$ with rational coefficients not all equal to 0. A finite set of real numbers is algebraically independent if none of these numbers algebraically depends on the others.

Theorem 3.1 (see [10])**.** *Suppose that $A_{i,j}$, where $i=1,2,\dots,q$ and $j=1,2,\dots,n_i+1$, are points in $\mathbb{R}^m$ such that the set of their coordinates is algebraically independent, $0\le t\le d\le T\le m$, and Π^d is a d-plane in $\mathbb{R}^m$ parallel to some coordinate planes $\Pi^t\subset\Pi^T\subset\mathbb{R}^m$. If either $d-t+1\le q$ and $n_1+n_2+\dots+n_q+1\le(m-d)(q-1)-(T-d)(d-t)$ or $q\le d-t+1$ and $n_1+n_2+\dots+n_q+1\le(m-T)(q-1)$, then there exists an $i\in\{1,2,\dots,q\}$ such that Π^d does not intersect the linear hull $\Pi(M_i)$ of the set $M_i=\{A_{i,1},\dots,A_{i,n_i+1}\}$.*

Proof. Suppose that Π^d intersects the linear hull $\Pi(M_i)$ of each set M_i and let $Y_i\in\Pi^d\cap\Pi(M_i)$ for $i=1,\dots,q$. It is sufficient to show that, under this assumption, we have either $n_1+n_2+\dots+n_q+1>(m-d)(q-1)-(T-d)(d-t)$ (if $q\ge d-t+1$) or $n_1+n_2+\dots+n_q+1>(m-T)(q-1)$ (if $1\le q\le d-t+1$). For this purpose, we need the following proposition (see [3, 69]), which follows from the properties of algebraically independent sets.

Proposition 3.2 (see [3, 69])**.** *Suppose that $A\subset\mathbb{R}$ is an algebraically independent set and $B\subset\mathbb{R}$ is a set such that every element of A algebraically depends on B. Then the cardinality of A does not exceed that of B.*

We can assume that Π^t and Π^T are determined by the first t and T coordinates, respectively. Let π be the projection of $\mathbb{R}^m$ onto the space $\mathbb{R}^{m-t}$ determined by the last $m-t$ coordinates. Then $\Pi^{d-t}=\pi(\Pi^d)$ is a $(d-t)$-plane in $\mathbb{R}^{m-t}$ parallel to the coordinate plane $\Pi^{T-t}=\pi(\Pi^T)$. Moreover, the set of coordinates of all points $B_{i,j}=\pi(A_{i,j})$ are algebraically independent (because it consists of some coordinates of $A_{i,j}$). Therefore, we can reduce the proof to the case of $t=0$ by considering the space $\mathbb{R}^{m-t}$ determined by the last $m-t$ coordinates and the projections $B_{i,j}$, $\pi(\Pi^d)$, and $\pi(\Pi^T)$ onto this space.

Since $Y_i\in\Pi(M_i)$, there exist numbers $\{\lambda_{i,j}\}_{j=1}^{n_i+1}$ such that $\lambda_{i,1}+\dots+\lambda_{i,n_i+1}=1$ and

(4)
$$Y_i=\sum_{j=1}^{n_i+1}\lambda_{i,j}A_{i,j}=\sum_{j=1}^{n_i}\lambda_{i,j}A_{i,j}+\big(1-\sum_{j=1}^{n_i}\lambda_{i,j}\big)A_{i,n_i+1}$$

for all $i=1,\dots,q$.

For each i, at least one of the numbers $\{\lambda_{i,j}\}_{j=1}^{n_i+1}$, say $\lambda_{i,1}$, is different from zero (we can even suppose that all $\lambda_{i,j}$ are nonzero; otherwise, we reduce the sets M_i by removing the corresponding point $A_{i,j}$). Therefore, by (4), $A_{i,1}$ can be represented as a linear combination of the points $A_{i,2},\dots,A_{i,n_i+1},Y_i$. This gives n_i additional numbers $\lambda_{i,1},\dots,\lambda_{i,n_i}$.

As a result, we express all coordinates of the points $A_{i,j}$ (where $i=1,\dots,q$ and $j=1,\dots,n_i+1$) in terms of the coordinates of the points $A_{i,j}$ (where $i=1,\dots,q$ and $j=2,\dots,n_i+1$) and $Y_1,\dots,Y_q$ and the numbers $\lambda_{i,j}$ (where $i=1,\dots,q$ and $j=1,\dots,n_i$).

I. Suppose that $q\le d+1$. Since Π^d is parallel to Π^T, the plane $Y_1+\Pi^T$ contains all points $Y_2,\dots,Y_q$. Thus, we have

$(5)_{\mathrm{I}}$
$$Y_i=Y_1+\sum_{j=1}^{T}\alpha_{i,j}\mathbf{e}_j,\quad i=2,\dots,q,$$

where $\mathbf{e}_j$ denotes the jth unit coordinate vector.

Therefore, each coordinate of any point $A_{i,j}$ (where $i = 1, \dots, q$ and $j = 1, \dots, n_i + 1$) algebraically depends on the set of all coordinates of the points Y_1 and $A_{i,j}$ (where $i = 1, \dots, q$ and $j = 2, \dots, n_i + 1$) and the numbers $\lambda_{i,j}$ (where $i = 1, \dots, q$ and $j = 1, \dots, n_i$) and $\alpha_{i,j}$ (where $i = 2, \dots, q$ and $j = 1, \dots, T$).

Hence, by Proposition 3.2, we have $qm \le m + n_1 + \cdots + n_q + T(q-1)$. It follows that

$(6)_{\mathrm{I}}$ $n_1 + \cdots + n_q \ge (m-T)(q-1)$.

Note that $(6)_I$ does not change when m, d, and T are replaced by $m-t$, $d-t$, and $T-t$, respectively. Therefore, $(6)_{\mathrm{I}}$ remains true for any t with $0 \le t \le d$.

II. Suppose that $q \ge d+1$. All points $Y_1, \dots, Y_q$ belong to Π^d; hence there are $d+1$ of them, say $Y_1, \dots, Y_{d+1}$, such that each Y_j with $j = d+2, \dots, q$ is their linear combination. Note that such $d+1$ points exist even if the linear hull of $Y_1, \dots, Y_q$ is of dimension $<d$. Therefore,

$$Y_j = \sum_{i=1}^{d} \beta_{j,i} Y_i + \bigl(1 - \sum_{i=1}^{d} \beta_{j,i}\bigr) Y_{d+1},$$

where $j = d+2, \dots, q$, for some numbers $\{\beta_{j,i}\}$ with $i = 1, \dots, d$ and $j = d+2, \dots, q$.

It follows that equalities $(5)_{\mathrm{I}}$ hold for $i = 2, \dots, d+1$. Thus, each coordinate of any point $A_{i,j}$ with $i = 1, \dots, q$ and $j = 1, \dots, n_i + 1$ algebraically depends on the set of all coordinates of the points Y_1 and $A_{i,j}$ with $i = 1, \dots, q$ and $j = 2, \dots, n_i+1$ and the numbers $\lambda_{i,j}$ with $i = 1, \dots, q$ and $j = 1, \dots, n_i$, $\beta_{j,i}$ with $j = d+2, \dots, q$ and $i = 1, \dots, d$, and $\alpha_{i,j}$ with $i = 2, \dots, d+1$ and $j = 1, \dots, T$.

According to Proposition 3.2, we have $qm \le m + n_1 + \cdots + n_q + (q-d-1)d + Td$, or, equivalently, $n_1 + \cdots + n_q \ge (m-d)(q-1) - (T-d)d$. Replacing m, d, and T in the last inequality by $m-t$, $d-t$, and $T-t$, respectively, we obtain

$(6)_{\mathrm{II}}$ $n_1 + \cdots + n_q \ge (m-d)(q-1) - (T-d)(d-t)$.

The inequalities $(6)_{\mathrm{I}}$ and $(6)_{\mathrm{II}}$ imply the required assertion. □

Corollary 3.3 (see [10])**.** *Suppose that K is a finite simplicial complex, $\theta : K \to \mathbb{R}^m$ is a semilinear map, and $\varepsilon > 0$. Then there exists a semilinear map $g : K \to \mathbb{R}^m$ such that $d(g(v), \theta(v)) < \varepsilon$ for each vertex v of K and, for any integers n, d, t, and T such that $0 \le t \le d \le m-n-1$ and $d \le T \le m$ and any d-plane $\Pi^d \subset \mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$, the number q of pairwise disjoint simplexes of K of dimension $\le n$ whose images under g intersect Π^d satisfies the inequalities $q \le d+1-t+\frac{n+(n+T-m)(d-t)}{m-n-d}$ (if $n \ge (m-n-T)(d-t)$) and $q \le 1+\frac{n}{m-n-T}$ (if $n \le (m-n-T)(d-t)$).*

Proof. The following argument is due to Roberts [116]. Let $\{r_i\}$ be an algebraically independent infinite set (an infinite set is said to be algebraically independent if every finite subset of this set is algebraically independent). We set $R_i = \{q + r_i : q \in Q\}$, where Q is the set of rational numbers. Then each R_i is dense in $\mathbb{R}$, and all of the R_i are pairwise disjoint. Moreover, any finite set M containing at most one point from each R_i, where $i = 1, 2, \dots$, is algebraically independent.

Consider the vertices v_i of K and the algebraically independent sets R_j mentioned above. For each i, choose a point $A_i = (A_i(1), \dots, A_i(m)) \in \mathbb{R}^m$ such that $\operatorname{dist}\bigl(\theta(v_i), A_i\bigr) < \varepsilon$ and $A_i(s) \in R_{(i-1)m+s}$. The set $\{A_i(k)\}$ of all coordinates of the points A_i is algebraically independent. We define a map $g : K \to \mathbb{R}^m$ by setting $g(v_i) = A_i$ and extend it by linearity over every simplex of K. Obviously, $g(v_{i_1}) \ne g(v_{i_2})$ for $i_1 \ne i_2$. Moreover, the restriction of g to any n-simplex of K is one-to-one. Let Π^d be a d-plane in R^m parallel to some coordinate planes $\Pi^t \subset \Pi^T$, and let q be the number of disjoint at most n-dimensional simplexes $\sigma_i = \langle v_{i,j} : j = 1, \dots, n_i + 1\rangle$ in K whose images $g(\sigma_i) = \langle A_{i,j} : j = 1, \dots, n_i + 1\rangle$ under g meet Π^d. Let show that

$$q \le N_1 = d + 1 - t + \frac{n + (n+T-m)(d-t)}{m-n-d}$$

if $n \ge (m-n-T)(d-t)$.

Suppose that $q \geq d+1-t$. Since Π^d intersects all images $g(\sigma_i) \subset \Pi\big(\{A_{i,j} : j = 1, \ldots, n_i+1\}\big)$ for $i = 1, \ldots, q$, it follows from Theorem 3.1 that $n_1 + \cdots + n_q \geq (m-d)(q-1) - (T-d)(d-t)$. Consequently,

(7) $nq \geq (m-d)(q-1) - (T-d)(d-t)$,

because $n_i \leq n$ for each i. Inequality (7) is equivalent to the required inequality $q \leq N_1$.

If $q \leq d+1-t$, then $q \leq N_1$, because the assumption $n \geq (m-n-T)(d-t)$ implies $d+1-t \leq N_1$.

Now let us show that

$$q \leq N_2 = 1 + \frac{n}{m-n-T}$$

for $n \leq (m-n-T)(d-t)$.

Suppose that $q \leq d-t+1$. Then, by Theorem 3.1, we have $n_1 + \cdots + n_q \geq (m-T)(q-1)$. Therefore,

(8) $nq \geq (m-T)(q-1)$,

which is equivalent to $q \leq N_2$.

Finally, suppose that $q \geq d-t+1$. We have already proved that $q \leq N_1$. On the other hand, the assumption $n \leq (m-n-T)(d-t)$ implies $d-t+1 \geq N_1$. Therefore, $q = d-t+1 = N_1$, whence $n = (m-n-T)(d-t)$. Thus, $q = d-t+1 = N_1 = N_2$. ☐

The idea of employing algebraically independent sets to prove general position theorems, such as Corollary 3.3, was first suggested by Roberts [116]. This idea was also applied by Berkowitz and Roy in [3], where they stated a version of Corollary 3.3 for $t = 0$ and $T = m$. A proof of the Berkowitz–Roy theorem was given by Goodsell in [69, Theorem A.1] (see also [68] for yet another application of the Berkowitz–Roy theorem).

3.2. A Roberts-type embedding theorem. Roberts proved in [116, Theorem 1.2] that if X is a compact metrizable space of dimension $\leq n$ and $n+1 \leq d \leq 2n+1$, then $C(X, \mathbb{R}^{2n+1})$ with the topology of uniform convergence contains a dense G_δ-subset consisting of maps g such that $\dim g(X) \cap \Pi^d \leq d-n-1$ for every d-plane $\Pi^d \subset \mathbb{R}^{2n+1}$. Using this result and the Hurewicz theorem about metrizable compactifications preserving dimension, Roberts derived the existence of such embeddings for separable metrizable spaces of dimension $\leq n$. In this section, we obtain a generalization of Roberts' result; the latter follows from Corollary 3.8 below and the Nöbeling–Pontryagin embedding theorem.

Theorem 3.4 (see [10])**.** *Let $f : X \to Y$ be a perfect map between paracompact spaces such that $\dim f \leq n$ and $\dim Y = 0$. Then, for every $m \geq n+1$, $C^*(X, \mathbb{R}^m)$ contains a dense G_δ-set $\mathcal{H}$ consisting of maps g such that $\dim g(f^{-1}(y)) \cap \Pi^d \leq n+d-m$ for any $y \in Y$ and any d-plane $\Pi^d \subset \mathbb{R}^m$ with $m-n \leq d \leq m$.*

Proof. First, we prove Theorem 3.4 in the special case of $d = m-n$. Take spaces X and Y and a map f satisfying the conditions of the theorem. In this section, unless otherwise specified, ϱ denotes the Euclidean metric on $\mathbb{R}^m$; an ε-disjoint set in $\mathbb{R}^m$ is a set which can be covered by a disjoint family of open sets of diameter $< \varepsilon$ in $\mathbb{R}^m$. We say that a set $A \subset \mathbb{R}^m$ is of type (d, ε) if $\Pi^d \cap A$ is ε-disjoint for every d-plane Π^d in $\mathbb{R}^m$.

Let $\mathcal{H}_\varepsilon(y)$, where $y \in Y$ and $\varepsilon > 0$, denote the set of maps $g \in C^*(X, \mathbb{R}^m)$ such that $g(f^{-1}(y))$ is of type $(m-n, \varepsilon)$. Following the general scheme from Sec. 2, we set

$$\mathcal{H}_\varepsilon(B) = \bigcap\{\mathcal{H}_\varepsilon(y) : y \in B\}$$

for every $B \subset Y$. Since $C^*(X, \mathbb{R}^m)$ with the source limitation topology has the Baire property, it suffices to show that each of the sets $\mathcal{H}_\varepsilon(Y)$ is open and dense in $C^*(X, \mathbb{R}^m)$. Indeed, if this is so, then

$$\mathcal{H} = \bigcap_{k=1}^{\infty} \mathcal{H}_{1/k}(Y)$$

is a dense G_δ-set in $C^*(X,\mathbb{R}^m)$. Moreover, if $g \in \mathcal{H}$ and $y \in Y$, then $g(f^{-1}(y)) \cap \Pi$ is at most 0-dimensional for every $(n-m)$-plane $\Pi \subset \mathbb{R}^m$.

Lemma 3.5. *Suppose that $A \subset X$ is a compact set, $\varepsilon > 0$, and the set $g_0(A)$ is of type (d,ε) for some $g_0 \in C^*(X,\mathbb{R}^m)$ and d. Then there exists a neighborhood U of A in X and a number $\delta > 0$ such that $\overline{g(U)}$ is of type (d,ε) provided that $g \in C^*(X,\mathbb{R}^m)$ and $g|U$ is δ-close to $g_0|U$.*

Proof. Assume the opposite. For every $i \geq 1$, take a neighborhood U_i of A such that $U_i \subset g_0^{-1}(W_i)$, where W_i is the $1/i$-neighborhood of $g_0(A)$. There exists a $g_i \in C^*(X,\mathbb{R}^m)$ and a d-plane Π_i^d such that $g_i|U_i$ is $1/i$-close to $g_0|U_i$ but $\overline{g_i(U_i)} \cap \Pi_i^d$ is not ε-disjoint. Choose points $z_i \in \overline{g_i(U_i)} \cap \Pi_i^d$ and $x_i \in U_i$ such that $\rho\big(g_i(x_i), z_i\big) \leq 1/i$ for $i \geq 1$. Obviously, $K = \{z_i\}_{i=1}^\infty \cup g_0(A)$ is a compact set intersecting each Π_i^d. Hence, there exists a subsequence of $\{\Pi_i^d\}_{i=1}^\infty$ converging to a d-plane Π_0^d. We assume that $\{\Pi_i^d\}_{i=1}^\infty$ itself converges to Π_0^d.

Let V be an open subset of $\mathbb{R}^m$ containing $g_0(A) \cap \Pi_0^d$ and being the union of a finite disjoint open family in $\mathbb{R}^m$ whose elements are of diameter $< \varepsilon$. Since each $\overline{g_i(U_i)} \cap \Pi_i^d$ is not ε-disjoint, there exist points $a_i \in U_i$ and $b_i \in \overline{g_i(U_i)} \cap \Pi_i^d$ such that V does not contain the set $\{g_i(a_i), b_i\}_{i=1}^\infty$. We can also require that $\varrho\big(b_i, g_i(a_i)\big) \leq 1/i$ for all i. This implies the existence of a point $b \in g_0(A)$ and a subsequence of $\{b_i\}$ converging to b. Without loss of generality, we can assume that $\lim b_i = b$. Then $b \in \Pi_0^d$, because $\{\Pi_i^d\}$ converges to Π_0^d. Hence $b \in g_0(A) \cap \Pi_0^d \subset V$. Consequently, $b_i \in V$ for some i, which contradicts the choice of b_i. ☐

Corollary 3.6. *Suppose that $g_0 \in \mathcal{H}_\varepsilon(y_0)$ for some $y_0 \in Y$ and $\varepsilon > 0$. Then there exists a neighborhood V of y_0 in Y and a number $\delta > 0$ such that $g \in \mathcal{H}_\varepsilon(V)$ for every $g \in C^*(X,\mathbb{R}^m)$ with $g|f^{-1}(V)$ δ-close to $g_0|f^{-1}(V)$.*

Proof. Applying Lemma 3.5 to $f^{-1}(y_0)$, we obtain $\delta > 0$ and a neighborhood U of $f^{-1}(y_0)$ such that $\overline{g(U)}$ is of type $(m-n,\varepsilon)$ provided that $g \in C^*(X,\mathbb{R}^m)$ and $g|U$ is δ-close to $g_0|U$. Since f is closed, we can find a neighborhood V of y_0 in Y for which $f^{-1}(V) \subset U$. Obviously, V is as required. ☐

Corollary 3.6 combined with Lemma 2.3 implies that $\mathcal{H}_\varepsilon(Y)$ is open in $C^*(X,\mathbb{R}^m)$ for each $\varepsilon > 0$, and the set-valued map $\psi_\varepsilon : Y \to C^*(X,\mathbb{R}^m)$ defined by $\psi_\varepsilon(y) = \mathcal{H}_\varepsilon(y)$ has a closed graph when $C^*(X,\mathbb{R}^m)$ is endowed with the topology of uniform convergence.

Let us show that $\mathcal{H}_\varepsilon(Y)$ is dense in $C^*(X,\mathbb{R}^m)$. For any space M and any $\varepsilon > 0$, let $C_{(n,\varepsilon)}(M,\mathbb{R}^m)$ denote the set of all $g \in C^*(M,\mathbb{R}^m)$ such that $g(M)$ is of type $(m-n,\varepsilon)$ for $n \leq m-1$.

Lemma 3.7. *Suppose that M is an n-dimensional compact space and $m \geq n+1$. Then $C_{(n,\varepsilon)}(M,\mathbb{R}^m)$ is dense in $C(M,\mathbb{R}^m)$ for any $\varepsilon > 0$.*

Proof. Take $g_0 \in C(M,\mathbb{R}^m)$ and $\delta > 0$. Representing g_0 as the composition of two maps $q_1 : M \to Z$ and $q_2 : Z \to \mathbb{R}^m$, where Z is a compact metrizable space of dimension $\leq n$, and considering Z and q_2 instead of M and g_0, we reduce the proof to the case where M is a compact metrizable space. Take a positive number η satisfying the conditions

(9) $5\eta < \delta/2$ and $9\eta(r+1) < \varepsilon$, where $r = n(m+1-n)$.

Since $\dim M \leq n$, a standard procedure yields a finite n-dimensional complex K and maps $h : M \to K$ and $\theta : K \to \mathbb{R}^m$ such that $\theta \circ h$ is $\delta/2$-close to g_0. Moreover, we can assume that

(10) $\operatorname{diam}\big(\theta(\sigma)\big) < \eta$ for every simplex $\sigma \in K$.

It is sufficient to find a map $g : K \to \mathbb{R}^m$ $\delta/2$-close to θ for which $g \circ h \in C_{(n,\varepsilon)}(M,\mathbb{R}^m)$. To this end, we apply Corollary 3.3 (with $d = m-n$, $t = 0$, $T = m$, $\varepsilon = \eta$, and n replaced by $n-1$) to obtain a semilinear map $g : K \to \mathbb{R}^m$ such that $\varrho\big(g(v), \theta(v)\big) \leq \eta$ for all vertexes v of K and, for each $(m-n)$-plane $\Pi \subset \mathbb{R}^m$, the number q of disjoint at most $(n-1)$-dimensional simplexes of K whose images under g intersect Π is at most $r = n(m+1-n)$. We can choose g so that, in addition, $g(v_i) \neq g(v_j)$ for any different vertices v_i and v_j of K.

Let v_i and v_j be two vertices of the same simplex $\triangle \in K$. Then, according to (10) and the choice of g, we have

$$\varrho\big(g(v_i), g(v_j)\big) \le \varrho\big(g(v_i), \theta(v_i)\big) + \varrho\big(\theta(v_i), \quad \theta(v_j)\big) + \varrho\big(\theta(v_j), g(v_j)\big) < 3\eta.$$

Consequently,

(11) $g(\triangle)$ is of diameter $<3\eta$ for any simplex $\triangle \in K$.

Condition (11) implies $\varrho\big(g(y), \theta(y)\big) < 5\eta$ for all $y \in K$. Hence, by (9), g and θ are $\delta/2$-close to each other.

It remains to show that $g \circ h \in C_{(n,\varepsilon)}(M, \mathbb{R}^m)$, or, equivalently, $g(K)$ is of type $(m-n, \varepsilon)$. For this purpose, we use an idea of [116, proof of 2.3, p. 568]. Take an $(m-n)$-plane $\Pi \subset \mathbb{R}^m$. It suffices to prove that each component of $g(K) \cap \Pi$ is of diameter $\le 9(r+1)\eta$, because, by (6), $9(r+1)\eta < \varepsilon$. Suppose that $\varrho(a,b) > 9(r+1)\eta$ for some component P of $g(K) \cap \Pi$ and some points $a, b \in P$. Consider an arc ab in P. According to (11), every subarc of ab of diameter $\ge 3\eta$ must contain at least one point of the boundary of a simplex from $g(K)$; hence it must contain a point of a simplex from $g(K)$ of dimension $\le n-1$. Take points $a_i \in ab$, where $i = 1, \ldots, r+1$, such that

$$3(3i-2)\eta < \varrho(a, a_i) \le 3(3i-1)\eta$$

for $i = 1, \ldots, r+1$ and each a_i belongs to a simplex $g(\sigma_i) \in g(K)$ with σ_i of dimension $\le n-1$. We have $\varrho(a_i, a_j) > 6\eta$ for $i \ne j$; according to (11), this implies $g(\sigma_i) \cap g(\sigma_j) = \emptyset$. Thus, we have obtained $r+1$ disjoint simplexes $\sigma_i \in K$, where $i = 1, \ldots, r+1$, of dimension at most $n-1$ whose images under g meet Π. This is a contradiction, because, according to the choice of g, K contains at most $n(m-n+1) = r$ pairwise disjoint at most $(n-1)$-dimensional simplexes whose images under g intersect Π. □

It can be shown that $\mathcal{H}_\varepsilon(Y)$ is dense in $C^*(X, \mathbb{R}^m)$. We have $\dim Y \le 0$; thus, by Lemma 2.4, it suffices to prove that each $C(\mathbb{I}^0 \times f^{-1}(y), \mathbb{R}^m)$, where $y \in Y$, contains a dense set consisting of maps g such that $g\big(\{z\} \times f^{-1}(y)\big)$ is of type $(m-n, \varepsilon)$ for each $z \in \mathbb{I}^0$ (here $\mathbb{I}^0$ is the 0-cube). This follows from Lemma 3.7. The application of Proposition 2.5 completes the proof of Theorem 3.4 in the case of $d = m-n$.

Consider the general case. Let us show that every $g \in \mathcal{H}$ is as required, i.e., any set $A = g(f^{-1}(y)) \cap \Pi^d$, where $y \in Y$ and $\Pi^d \subset \mathbb{R}^m$ is a d-plane with $m-n+1 \le d \le m$, is at most $(n+d-m)$-dimensional. Take some $(m-n)$-plane $\Pi^{m-n} \subset \Pi^d$ and consider the orthogonal projection p of Π^d onto the $(n+d-m)$-plane $\Pi^{n+d-m} \subset \Pi^d$ being the orthogonal complement of Π^{m-n} in Π^d. The compact set $B = p\big(g(f^{-1}(y)) \cap \Pi^d\big) \subset \Pi^{n+d-m}$ is of dimension $\le n+d-m$. Obviously, any fiber of the map $p|_A : A \to B$ is the intersection of $g(f^{-1}(y))$ and some $(m-n)$-plane. Thus, by the choice of g, $p|_A$ has zero-dimensional fibers. Therefore, by the Hurewicz dimension-lowering theorem, $\dim A \le \dim B \le n+d-m$. □

Corollary 3.8 (see [10])**.** *Suppose that X is a normal space with $\dim X = n$ and $m \ge n+1$. Then $C^*(X, \mathbb{R}^m)$ with the topology of uniform convergence contains a dense G_δ-set $\mathcal{H}$ consisting of maps g such that $\overline{g(X)} \cap \Pi^d$ is at most $(n+d-m)$-dimensional for every d-plane $\Pi^d \subset \mathbb{R}^m$ with $m-n \le d \le m$.*

Proof. Considering the Čech–Stone compactification βX of X and the function space $C(\beta X, \mathbb{R}^m)$ instead of X and $C^*(X, \mathbb{R}^m)$, respectively, we can assume that X is compact. Then, we take a constant map on X for f and apply Theorem 3.4. □

3.3. Generalized Boltyanskii-type theorems. In this section, we generalize some results of Boltyanskii [12]. For example, a special case of Theorem 3.9 (with compact X, one-point Y, $t = 0$, and $T = m$) implies the Boltyanskii theorem about k-regular maps [12, Theorem 1], and Theorem 3.13 with compact X, one-point Y, and $r = 0$ gives another result of Boltyanskii [12, Theorem 4].

Theorem 3.4 does not apply to $d \le m-n-1$. However, the following theorem shows that, in this case, an even stronger assertion is valid; namely, we can find a residual subset of $C^*(X,\mathbb{R}^m)$ consisting of maps g such that $g(f^{-1}(y)) \cap \Pi^d$ is finite for any $y \in Y$ and any d-plane in $\mathbb{R}^m$.

Theorem 3.9 (see [10])**.** *Suppose that $f : X \to Y$ is a perfect map between metrizable spaces, $\dim f \le n$, and $\dim Y \le 0$. Then $C^*(X,\mathbb{R}^m)$ contains a dense G_δ-set $\mathcal{K}$ consisting of maps g such that, for any integers d, t, and T satisfying the inequalities $0 \le t \le d \le T \le m$ and $d \le m-n-1$ and any d-plane $\Pi^d \subset \mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$, any set $f^{-1}(y) \cap g^{-1}(\Pi^d)$ with $y \in Y$ contains at most q points, where $q = d+1-t+\dfrac{n+(n+T-m)(d-t)}{m-n-d}$ if $n \ge (m-n-T)(d-t)$ and $q = 1+\dfrac{n}{m-n-T}$ otherwise.*

Proof. It is sufficient to show that, for any integers d, t, and T satisfying the inequalities $0 \le t \le d \le T \le m$ and $d \le m-n-1$ and any coordinate planes $\Pi^t \subset \Pi^T$ in $\mathbb{R}^m$, $C^*(X,\mathbb{R}^m)$ contains a dense G_δ-set consisting of maps g such that $f^{-1}(y) \cap g^{-1}(\Pi^d)$ contains at most q points for every $y \in Y$ and every d-plane $\Pi^d \subset \mathbb{R}^m$ parallel to $\Pi^t \subset \Pi^T$, where q is the integer part of $N_1 = d+1-t+\dfrac{n+(n+T-m)(d-t)}{m-n-d}$ if $n \ge (m-n-T)(d-t)$ and of $N_2 = 1+\dfrac{n}{m-n-T}$ otherwise. Thus, take integers n, d, t, and T satisfying the above inequalities and coordinate planes $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$. Note that q cannot be smaller than 1 and let $\mathcal{P}$ be the set of all d-planes in $\mathbb{R}^m$ parallel to $\Pi^t \subset \Pi^T$. We say that a subset A of an arbitrary metric space M is of *cotype* (q,ε,g), where $\varepsilon > 0$ and $g \in C^*(M,\mathbb{R}^m)$, if, for any $\Pi^d \in \mathcal{P}$, the set $A \cap g^{-1}(\Pi^d)$ can be covered by at most q disjoint open subsets of M of diameter $\le\varepsilon$. Let $\mathcal{K}_\varepsilon(y)$, where $y \in Y$ and $\varepsilon > 0$, denote the set of all maps $g \in C^*(X,\mathbb{R}^m)$ such that $f^{-1}(y)$ is of cotype (q,ε,g). By $\mathcal{K}_\varepsilon(H)$, where $H \subset Y$, we denote the intersection of all $\mathcal{K}_\varepsilon(y)$ over $y \in H$. We have reduced the proof to showing that any $\mathcal{K}_\varepsilon(Y)$, where $\varepsilon > 0$, is open and dense in $C^*(X,\mathbb{R}^m)$.

Lemma 3.10. *Let A be a compact subset of X of cotype (q,ε,g_0) for some $g_0 \in C^*(X,\mathbb{R}^m)$ and $\varepsilon > 0$. Then there exists a neighborhood U of A in X and a $\delta > 0$ such that U is of cotype (q,ε,g) for every $g \in C^*(X,\mathbb{R}^m)$ with $g|U$ δ-close to $g_0|U$.*

Proof. Suppose that the lemma is false. For every $i \ge 1$, take a $1/i$-neighborhood U_i of A such that $g_0(U_i)$ is contained in the $1/i$-neighborhood of $g_0(A)$ in $\mathbb{R}^m$. There exists a $g_i \in C^*(X,\mathbb{R}^m)$ and a d-plane $\Pi_i^d \in \mathcal{P}$ such that $g_i|U_i$ is $1/i$-close to $g_0|U_i$ but $g_i^{-1}(\Pi_i^d) \cap U_i$ is not covered by any family of $\le q$ open disjoint sets in X with diameters $\le\varepsilon$. As in the proof of Lemma 3.5, passing to subsequences, we can assume that $\{\Pi_i^d\}_{i=1}^\infty$ converges to a d-plane Π_0^d. All Π_i^d are from $\mathcal{P}$ (i.e., parallel to $\Pi^t \subset \Pi^T$); hence so is Π_0^d. Therefore, $A \cap g_0^{-1}(\Pi_0^d)$ is covered by a disjoint family $\{V_j\}$ of $\le q$ open sets in X with diameters $\le\varepsilon$. Take points $x_i \in g_i^{-1}(\Pi_i^d) \cap U_i \setminus V$ and $y_i \in A$ such that $\operatorname{dist}(x_i,y_i) \le 1/i$ for $i = 1,2,\dots$, where $V = \bigcup V_j$. We can assume that the sequence x_i converges to some $x_0 \in A$ (recall that A is compact). Then $\{g_i(x_i)\}$ converges to $g_0(x_0)$ and $g_0(x_0) \in g_0(A) \cap \Pi_0^d$. Therefore, $x_0 \in A \cap g_0^{-1}(\Pi_0^d) \subset V$. Thus, $x_i \in V$ for almost all i, which contradicts the assumption. ◻

Lemma 3.10 implies that, if $g_0 \in \mathcal{K}_\varepsilon(y_0)$ for some $y_0 \in Y$ and $\varepsilon > 0$, then there exists a neighborhood V of y_0 in Y and a $\delta > 0$ such that $g \in \mathcal{K}_\varepsilon(V)$ for every $g \in C^*(X,\mathbb{R}^m)$ with $g|f^{-1}(V)$ δ-close to $g_0|f^{-1}(V)$. Hence, by Lemma 2.3, $\mathcal{K}_\varepsilon(Y)$ is open in $C^*(X,\mathbb{R}^m)$ for each $\varepsilon > 0$, and the corresponding set-valued map $\psi_\varepsilon : Y \to C^*(X,\mathbb{R}^m)$ has a closed graph when $C^*(X,\mathbb{R}^m)$ is endowed with the topology of uniform convergence.

Lemma 3.11. *Let M be a metrizable at most n-dimensional compact space. Then the set $\mathcal{K}_0(M,\mathbb{R}^m)$ of all $g \in C(M,\mathbb{R}^m)$ such that $g^{-1}(\Pi^d)$ contains at most q points for every $\Pi^d \in \mathcal{P}$ is dense in $C(M,\mathbb{R}^m)$.*

Proof. Let Ω be the collection of all disjoint families $\{\overline{V}_1, \overline{V}_2, \dots, \overline{V}_{q+1}\}$ of $q+1$ elements such that each V_j belongs to a fixed base for M. We set

$$C_\Gamma = \{g \in C(M, \mathbb{R}^m) : g^{-1}(\Pi^d) \text{meets at most } q \text{ elements of } \Gamma \text{ for every } \Pi^d \in \mathcal{P}\}$$

for $\Gamma \in \Omega$.

Obviously, Ω is countable and $\mathcal{K}_0(M, \mathbb{R}^m)$ is the intersection of all sets C_Γ over $\Gamma \in \Omega$. Thus, we must show that each C_Γ is dense and open in $C(M, \mathbb{R}^m)$.

Claim 1. *Every C_Γ is open in $C(M, \mathbb{R}^m)$.*

Take $\Gamma \in \Omega$ and $g_0 \in C_\Gamma$. Suppose that, for every i, there exists a $g_i \notin C_\Gamma$ such that g_i is $1/i$-close to g_0. Then we can find $\Pi_i^d \in \mathcal{P}$ such that $g_i^{-1}(\Pi_i^d)$ meets every element of Γ. As in Lemma 3.5, we can assume that the sequence $\{\Pi_i^d\}$ converges to some $\Pi_0^d \in \mathcal{P}$. Then $g_0^{-1}(\Pi_0^d)$ intersects at most q elements of Γ, say, the first q elements. For every i, choose a point $x_i \in g_i^{-1}(\Pi_i^d) \cap \overline{V}_{q+1}$. Since M is compact, we can assume that the sequence $\{x_i\}$ converges to some $x_0 \in \overline{V}_{q+1}$. Thus, $\{g_i(x_i)\}$ converges to $g_0(x_0) \in \Pi_0^d$, and $x_0 \in g_0^{-1}(\Pi_0^d) \cap \overline{V}_{q+1}$, which contradicts the assumption.

Claim 2. *Every C_Γ is dense in $C(M, \mathbb{R}^m)$.*

Suppose that $\Gamma = \{\overline{V}_1, \overline{V}_2, \dots, \overline{V}_{q+1}\}$, $g_0 \in C_\Gamma$, and $\delta > 0$. There exists an open cover ω of M with $\operatorname{mesh}(\omega) \le r/3$, where $r = \min\{\operatorname{dist}(\overline{V}_i, \overline{V}_j) : i \ne j\}$, and a semilinear map $h : L \to \mathbb{R}^m$ such that $g = h \circ \pi$ is δ-close to g_0. Here L is the polyhedron underlying the nerve of ω and $\pi : M \to L$ is the canonical map. According to Corollary 3.3, we can assume that, for any $\Pi^d \in \mathcal{P}$, the number of pairwise-disjoint at most n-dimensional simplexes $\sigma \in L$ with $h(\sigma) \cap \Pi^d \ne \varnothing$ does not exceed q. We can also assume that ω is of order at most $n+1$, so that L is at most n-dimensional. If there exists a plane $\Pi^* \in \mathcal{P}$ for which $g^{-1}(\Pi^*)$ intersects every $\overline{V}_i$, choose $x_i \in g^{-1}(\Pi^*) \cap \overline{V}_i$ and let ω_i be the family of all elements of ω containing the point x_i. Then each family ω_i, where $i = 1, \dots, q+1$, generates a simplex $\sigma_i \in L$ of dimension $\le n$ such that $h(x_i) \in h(\sigma_i) \cap \Pi^*$ and $\sigma_i \cap \sigma_j = \emptyset$ for $i \ne j$. This contradicts the choice of h. □

Lemma 3.11 implies that each $C(\mathbb{I}^0 \times f^{-1}(y), \mathbb{R}^m)$, where $y \in Y$, contains a dense set of maps g such that $g|\big(\{z\} \times f^{-1}(y)\big)$ belongs to $\mathcal{K}_\varepsilon(y)$ for every $z \in \mathbb{I}^0$. Hence, by the results of Sec. 2.1, $\mathcal{K}_\varepsilon(Y)$ is dense in $C^*(X, \mathbb{R}^m)$. □

Theorem 3.9 with one-point Y, $m = n+2$, $d = 1$, $t = 0$, and $T = r$ implies the following assertion.

Corollary 3.12. *Let X be a compact metrizable space with $\dim X \le n$. Then $C(X, \mathbb{R}^{n+2})$ contains a dense G_δ-set consisting of maps g such that, for any positive integer $r \le n+2$ and any line $\Pi^1 \subset \mathbb{R}^{n+2}$ parallel to some coordinate plane Π^r in $\mathbb{R}^{n+2}$, the preimage $g^{-1}(\Pi^1)$ contains at most $n+r$ points.*

Theorem 3.9, together with [142, Theorem 5.1], implies the following interesting fact: If X is an n-dimensional compact metrizable space, $n \ge 3$, and $m \ge 2n+1$, then almost every embedding of X in $\mathbb{R}^m$ is tame.

Theorem 3.13 below is an infinite-dimensional version of Theorem 3.9. As mentioned above, for compact X and one-point Y, Theorem 3.13 was proved by Boltyanskii [12, Theorem 4] under the additional assumption that $r = 0$.

Theorem 3.13 (see [10]). *Let $f : X \to Y$ be a perfect map from a metrizable space X to a metrizable C-space Y. For any integers d and r, let $\mathcal{P}(d, r)$ denote the family of all d-planes $\Pi^d \subset l_2$ parallel to some coordinate plane $\Pi^r \subset l_2$. Then $C^*(X, l_2)$ contains a dense G_δ-subset consisting of maps g such that, for any $y \in Y$ and $\Pi^d \in \mathcal{P}(d, r)$, the intersection $f^{-1}(y) \cap g^{-1}(\Pi^d)$ contains at most $d+1-r$ points if $r \le d$ and at most one point if $r \ge d$.*

Proof. Given integers d and r, let $\mathcal{Q}$ be the set of all d-planes parallel to the r-plane of first r coordinates in l_2. As in Theorem 3.9, we introduce the notion of a (q, ε, g)-cotype subset of a metric space M by considering the planes $\Pi^d \in \mathcal{Q}$ and taking $q = d + 1 - r$ if $d \geq r$ and $q = 1$ otherwise. For $y \in Y$ and $\varepsilon > 0$, let $\mathcal{F}_\varepsilon(y)$ denote the set of all maps $g \in C^*(X, l_2)$ such that $f^{-1}(y)$ is of cotype (q, ε, g). It suffices to show that each $\mathcal{F}_\varepsilon(Y) = \bigcap\{\mathcal{F}_\varepsilon(y) : y \in Y\}$ is open and dense in $C^*(X, l_2)$. Following the scheme of the proof of Theorem 3.9, we can show that all $\mathcal{F}_\varepsilon(Y)$ are open in $C^*(X, l_2)$ and that Lemma 3.14 below implies the density of $\mathcal{F}_\varepsilon(Y)$.

Lemma 3.14. *Let M be a compact metrizable space, and let $\varepsilon > 0$. Then the set $\mathcal{F}_\varepsilon(M, l_2)$ of all $g \in C(M, l_2)$ such that M is of cotype (q, ε, g) is dense in $C(M, l_2)$.*

Proof. Take $g_0 \in C(M, l_2)$ and $\lambda > 0$. There exist maps $\varphi : M \to K$ and $h : K \to l_2$ such that K is a finite complex and $h \circ \varphi$ is $\lambda/2$-close to g_0. We can assume that each fiber of φ has diameter $<\varepsilon$. For every $A \subset \mathbb{N}$, we identify $\mathbb{R}^A$ with the subspace

$$\{y \in l_2 : y_i = 0 \text{ for all } i \notin A\}$$

of l_2 and denote the canonical projection $\pi_A : l_2 \to \mathbb{R}^A$ by π_A. Let $\dim K = n$. Take any finite set $A \subset \mathbb{N}$ satisfying the following conditions:

(i) $\{1, \dots, r\} \subset A$;
(ii) $|A| \geq r + d + 2n + n \cdot |d - r| + 1$, where $|A|$ is the cardinality of A.

Since every projection π_A is open, it follows that the map $\Lambda_A : C(K, l_2) \to C(K, \mathbb{R}^A)$ defined by $\Lambda_A(g) = \pi_A \circ g$ is open as well.

First, suppose that $r \leq d$. If $T = m$, then $(m - n - T)(d - r) = -n(d - r) \leq 0$, whence $n \geq (m - n - T)(d - r)$. Therefore, by Theorem 3.9 with $m = T = |A|$ and $t = r$, there exists a dense G_δ-set $\mathcal{F}_A$ in $C(K, \mathbb{R}^A)$ consisting of maps g such that $g^{-1}(\Pi^d)$ contains at most $1 + d - r + \dfrac{n + n(d - r)}{|A| - n - d}$ points for every d-plane $\Pi^d \subset \mathbb{R}^A$ parallel to the r-plane of first r coordinates in $\mathbb{R}^A$. Since $|A| \geq r + d + 2n + n \cdot (d - r) + 1$, it follows that any preimage $g^{-1}(\Pi^d)$, where $g \in \mathcal{F}_A$, contains at most $1 + d - r$ points. The set $\Lambda_A^{-1}(\mathcal{F}_A)$ is dense and has type G_δ in $C(K, l_2)$. Hence there exists a $g \in \Lambda_A^{-1}(\mathcal{F}_A)$ which is $\lambda/2$-close to h. The map $\overline{g} = g \circ \varphi$ is λ-close to g_0. It remains to show that $\overline{g} \in \mathcal{F}_\varepsilon(M, l_2)$. To this end, take $\Pi^d \in \mathcal{Q}$. Since $\Pi^d_A = \pi_A(\Pi^d)$ is a d-plane in $\mathbb{R}^A$ parallel to the r-plane of first r coordinates in $\mathbb{R}^A$, it follows that the preimage $g^{-1}(\pi_A^{-1}(\Pi^d_A))$ contains $\leq 1 + d - r$ points. Formally, the plane Π^d_A may have dimension $<d$, but in this case, the estimation on the cardinality of preimage is even better. The inclusion $\Pi^d \subset \pi_A^{-1}(\Pi^d_A)$ implies that $g^{-1}(\Pi^d)$ contains $\leq 1 + d - r$ points. Hence $(\overline{g})^{-1}(\Pi^d) = \varphi^{-1}(g^{-1}(\Pi^d))$ consists of $\leq 1 + d - r$ fibers of φ. Since any fiber of φ has diameter $<\varepsilon$, it follows that $\overline{g} \in \mathcal{F}_\varepsilon(M, l_2)$.

Suppose that $d \leq r$. We again apply Theorem 3.9 to A but with $|A| = m$, $t = 0$, and $T = r$. Obviously, in this case, we have $(m - n - T)(d - t) > n$. Hence there exists a dense G_δ-set $\mathcal{F}_A$ in $C(K, \mathbb{R}^A)$ consisting of maps g such that $g^{-1}(\Pi^d)$ contains at most one point for every d-plane in $\mathbb{R}^A$ parallel to the r-plane of first r coordinates in $\mathbb{R}^A$. As above, we take $g \in \Lambda_A^{-1}(\mathcal{F}_A)$ which is $\lambda/2$-close to h and show that $\overline{g} = g \circ \varphi \in \mathcal{F}_\varepsilon(M, l_2)$. □

This completes the proof of Theorem 3.13. □

3.4. Problems and possible improvements. In this section, we discuss some possible improvements of the results obtained in Secs. 3.1–3.3.

Conjecture 1 (see [10]). *Let $f : X \to Y$ be a map of finite-dimensional compact metrizable spaces. Then, $C(X, \mathbb{R}^m)$ contains a dense G_δ-set consisting of maps φ such that, for any integers d, t, and T satisfying the inequalities* $0 \leq t \leq d \leq T \leq m$ *and* $\dim f + d + 1 \leq m$ *and any d-plane* $\Pi^d \subset \mathbb{R}^m$

parallel to some coordinate planes $\Pi^t \subset \Pi^T$ *in* $\mathbb{R}^m$*, each set* $f^{-1}(y) \cap \varphi^{-1}(\Pi^d)$ *with* $y \in Y$ *contains at most*

$$1 + \frac{\dim Y + \dim f + (T-d)(d-t)}{m - \dim f - d}$$

points.

Obviously, Theorem 3.9 implies that Conjecture 1 is true for 0-dimensional Y.

Conjecture 2 (see [10]). *Let* $f : X \to Y$ *be a map of finite-dimensional compact spaces. Then* $C(X, \mathbb{R}^m)$ *contains a dense* G_δ*-set consisting of maps* φ *such that*

$$\dim\bigl(\varphi\bigl(f^{-1}(y)\bigr) \cap \Pi^d\bigr) \le \dim f + d - m$$

for any d*-plane* $\Pi^d \subset \mathbb{R}^m$ *with* $m - \dim f \le d \le m$ *and any* $y \in Y$*.*

Theorem 3.4 implies that Conjecture 2 is true for 0-dimensional Y. As in Theorem 3.4, it suffices to prove Conjecture 2 in the special case of $d = m - \dim f$. Conjecture 2 is also true when $\dim f = 0$ by Uspenskij's theorem about light maps [136] (see also Theorem 2.15).

Suppose that $f : X \to Y$, $\varphi \in C(X, \mathbb{R}^m)$, and t, d, and T are integers satisfying the inequalities $0 \le t \le d \le T \le m$ and $d - t + 1 \le q$. Consider the set $B^f_{q,d,t,T}(\varphi)$ consisting of all points $(y, y_1, \dots, y_q) \in Y \times (\mathbb{R}^m)^q$ satisfying the following condition: there exist points $x_1, \dots, x_q \in f^{-1}(y)$ such that $x_i \ne x_j$ for $i \ne j$, $y_i = \varphi(x_i)$, and all y_i with $i = 1, \dots, q$ belong to a d-plane in $\mathbb{R}^m$ parallel to some coordinates planes $\Pi^t \subset \Pi^T \subset \mathbb{R}^m$.

Conjecture 3 (see [10]). *Let* $f : X \to Y$ *be a map of finite-dimensional compact metrizable spaces. Then* $C(X, \mathbb{R}^m)$ *contains a dense* G_δ*-set* $\mathcal{H}$ *consisting of maps* φ *such that*

$$\dim B^f_{q,d,t,T}(\varphi) \le \dim Y + \dim f + (T-d)(d-t) - (q-1)(m - \dim f - d)$$

for any integers d*,* t*,* T*, and* q *satisfying the inequalities* $0 \le t \le d \le T \le m$*,* $d - t + 1 \le q$*, and* $\dim f + d + 1 \le m$*.*

If the right-hand side of the inequality from Conjecture 3 is ≤ -1, then the set $B^f_{q,d,t,T}(\varphi)$ is empty. Conditions under which $B^f_{q,d,t,T}(\varphi)$ is empty are considered in Conjecture 1. If $d = 0$, then the set $B^f_{q,0,0,T}(\varphi)$ does not depend on T. In this case, it is homeomorphic to the set

$$B^f_q(\varphi) = \{(y, z) \in Y \times \mathbb{R}^m : |f^{-1}(y) \cap \varphi^{-1}(z)| \ge q\}.$$

For $d = 0$ and one-point Y, Conjecture 3 coincides with the Hurewicz theorem on regularly branched maps. By Theorem 2.22, the inequality from Conjecture 3 holds for $d = 0$ and all $q \ge 1$ if and only if φ is an f-regularly branched map. Note that for $d \ge 1$, the conjecture is open even in the case of one-point Y.

For maps $f_i : X_i \to \mathbb{R}^m$, where $i = 1, \dots, q$, and integers $0 \le t \le d \le T \le m$, let $B_{d,t,T}(f_1, \dots, f_q)$ be the set of all points $(y_1, \dots, y_q) \in (\mathbb{R}^m)^q$ such that the points $y_i = f(x_i)$, where $i = 1, \dots, q$, belong to a d-plane in $\mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T$. By $C_{d,t,T}(f_1, \dots, f_q)$ we denote the set

$$\bigl\{(x_1, \dots, x_q) \in X_1 \times \dots \times X_q : \bigl(f(x_1), \dots, f(x_q)\bigr) \in B_{d,t,T}(f_1, \dots, f_q)\bigr\}.$$

Conjecture 4 (see [10]). *Suppose that numbers* $n_1, \dots, n_q, m, d, t, T$ *satisfy the inequalities*

$$0 \le t \le d \le T \le m,\ 0 \le n_1,\ \dots,\ 0 \le n_q,\ n_1 + 1 + d \le m,\ \dots,\ n_q + 1 + d \le m,\ d - t + 1 \le q,$$

and

$$n_1 + \dots + n_q \ge (m-d)(q-1) - (T-d)(d-t).$$

Then there exists an $\varepsilon > 0$ *and maps* $f_i : \Delta^{n_i} \to \mathbb{R}^m$*, where* $i = 1, \dots, q$*, such that the set* $B_{d,t,T}(g_1, \dots, g_q)$ *is nonempty for any maps* $g_i : \Delta^{n_i} \to \mathbb{R}^m$ ε*-close to* f_i *for* $i = 1, \dots, q$*.*

In the simplest case of $q = 1$, $n_1 = 0$, $n_2 = m$, and $d = 0$, Conjecture 4 coincides with Alexandroff's theorem that the identity map of the ball is essential. For $d = 0$, Conjecture 4 is also true (see, e.g., [5, Corollary 3] and [13, Lemma 6.3, p. 65]). For $d = q - 1$, $t = 0$, and $T = m$, Conjecture 4 was proved by Boltyanskii [12, Lemma 9] (this is the main ingredient in his proof that $m \geq nk + n + k$ if the set of k-regular maps from X to $\mathbb{R}^m$ is dense in $C(X, \mathbb{R}^m)$ for some n-dimensional polyhedron X).

Conjecture 5 (see [10])**.** *Suppose that numbers $n_1, \dots, n_q$, m, d, t, and T satisfy the inequalities*

$$0 \leq t \leq d \leq T \leq m,\ 0 \leq n_1,\ \dots,\ 0 \leq n_q,\ n_1 + 1 + d \leq m,\ \dots,\ n_q + 1 + d \leq m,\ d - t + 1 \leq q.$$

Then there exists an $\varepsilon > 0$ and maps $f_i : \Delta^{n_i} \to \mathbb{R}^m$, where $i = 1, \dots, q$, such that

$$\dim C_{d,t,T}(g_1, \dots, g_q) \geq n_1 + \dots + n_q - (m - d)(q - 1) + (T - d)(d - t)$$

for any maps $g_i : \Delta^{n_i} \to \mathbb{R}^m$ ε-close to f_i for $i = 1, \dots, q$.

For $d = 0$, Conjecture 5 was proved in [5, Corollary 3].

We conclude this section with the following problem, where Δ^N_n denotes the n-skeleton of the N-simplex Δ^N.

Problem 6 (see [10])**.** *Find all integers n, m, q, d, t, and T satisfying the inequalities*

$$0 \leq t \leq d \leq T \leq m, \quad n + 1 + d \leq m, \quad d - t + 1 \leq q,$$

and

$$n \geq (m - n - d)(q - 1) - (T - d)(d - t) \qquad (n, m, q, d, t, T)$$

and the following condition: there exists a number N such that, for any map $f : \Delta^N_n \to \mathbb{R}^m$, there exist pairwise disjoint simplices $\sigma_1, \dots, \sigma_q \subset \Delta^N_n$ whose images under f meet a d-plane $\Pi^d \subset \mathbb{R}^m$ parallel to some coordinate planes $\Pi^t \subset \Pi^T$.

Note that, if numbers n, m, q, d, t, and T satisfying the conditions of the problem exist, then Theorem 3.9 cannot be improved not only at the level of a dense set of maps but also at the level of the *existence* of one map with small preimages, even in the class of polyhedra. Most of the results on the existence of a number N mentioned in the problem were obtained for $d = 0$; they were obtained by van Kampen and Flores [66, 79] (for $q = 2$ and $N = 2n + 2$); Sarkaria [118] (for prime q and $N = qn + 2q - 2$); Volovikov [137] (for prime power q and $N = qn + 2q - 2$); and Bogatyi [5, Corollary 11] (for $q = n + 1$ and $N \leq 2n^2 + 5n$). Another result of Bogatyi [11] provides such an N for $d = q - 1$, $t = 0$, and $T = m$, where q is odd. Živaljević [146] proved an embedding theorem for the graph $K_{6,6}$ into $\mathbb{R}^3$, which implies a positive solution of the problem for $n = 1$, $m = 3$, $t = 0$, $T = 3$, $q = 4$, and $N = 11$.

For $d = q - 1$, $q = 2$, $t = 0$, and $T = m$, there exists no number N satisfying the condition in the problem (see [12]).

The above problem can be formulated differently: *Given a family of integers n, q, d, t, and T, find the largest number m for which a number N satisfying the above condition exists.* Then the problem of determining the smallest number N arises. In this case, the description of minimal subpolyhedra in Δ^N_n is more complicated. For $n = 1$, $d = 0$, and $m = 2$, the Kuratowski graphs K_5 and $K_{3,3}$ are minimal subpolyhedra.

The above problem is related to conjectures about different forms of the Tverberg theorem [8, 146]. It is also well known that k-regular maps are closely related to interpolation and approximation problems. For this reason, it is important to find applications of the maps described by Theorem 3.9 and Theorem 3.13 to interpolation and approximation problems.

4. Theorems on Dimension-Lowering and Dimension-Raising Maps

This section collects some generalizations of the Hurewicz dimension-lowering and dimension-raising theorems to extension dimension. The "dimensional scale" corresponding to the extension dimension is much finer than the usual integer-valued one and allows us to discover new properties of spaces. Moreover, a variety of known facts can be considered from a more general point of view.

The Hurewicz dimension-lowering theorem states that if $f : X \to Y$ is a closed map, then $\dim X \leq \dim f + \dim Y$, where $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$. It was first proved by Hurewicz [76] for compact metrizable spaces. This theorem was extended to separable metrizable spaces by Hurewicz and Wallman [78] and to metrizable spaces by Morita [100] and Nagami [101]. Sklyarenko [119] proved it for paracompact spaces (see also Filippov [64, 65] and Pasynkov [107, 108] for other generalizations of the dimension-lowering theorem).

One of the first extensional generalizations of the Hurewicz theorem was obtained by Dranishnikov and Uspenskij in [49].

Theorem 4.1 (see [49]). *If $f : X \to Y$ is a 0-dimensional map between compact spaces, then* $\text{e-dim}\, X \leq \text{e-dim}\, Y$.

Theorem 4.1 combined with Theorem 2.10 gives yet another generalization of the Hurewicz formula: If $\dim f \leq n$, X and Y are compact metrizable spaces, and Y is a C-space, then $\text{e-dim}\, X \leq \text{e-dim}(Y \times \mathbb{I}^n)$.

Extensional versions of the dimension-lowering theorem for compact spaces were also obtained by Levin and Lewis [93].

Theorem 4.2 (see [48, 93] for finite-dimensional Y). *Let $f : X \to Y$ be a map of compact metrizable spaces such that X is finite-dimensional and Y is full-valued, and let K be a CW-complex such that* $\text{e-dim}(f^{-1}(y) \times Y) \leq K$ *for every $y \in Y$. Then* $\text{e-dim}\, X \leq K$.

Theorem 4.3 (see [93]). *Suppose that $f : X \to Y$ is a map of compact metrizable spaces, X is finite-dimensional, K and L are CW-complexes, K is countable,* $\text{e-dim}\, f \leq K$, *and* $\text{e-dim}\, Y \leq L$. *Then* $\text{e-dim}\, X \leq K \wedge L$.

Recall that a compact space Y is said to be dimensionally full-valued if $\dim_G Y = \dim_{\mathbb{Z}} Y$ for any Abelian group G, where $\dim_G Y$ is the cohomological dimension of Y with respect to the group G and $\mathbb{Z}$ is the additive group of integers. If Y is finite-dimensional, then Y is full-valued if and only if $\dim_G Y = \dim Y$ for any group G. Moreover, the inequality $\text{e-dim}\, f \leq K$ for a map $f : X \to Y$ means that $\text{e-dim}\, f^{-1}(y) \leq K$ for any $y \in Y$. In particular, $\dim_G f$ denotes $\max\{\dim_G f^{-1}(y) : y \in Y\}$ for any Abelian group G.

Theorem 4.3 is a parametric version of the Dranishnikov–Dydak theorem [44] that $\text{e-dim}(X \times Y) \leq K \wedge L$ for finite-dimensional compact metrizable spaces X and Y with $\text{e-dim}\, X \leq K$ and $\text{e-dim}\, Y \leq L$.

Dimension-lowering type theorems for cohomological dimension were proved by Kuz'minov [88] and Dranishnikov [41].

Theorem 4.4 (see [41]). *Let $f : X \to Y$ be a map between finite-dimensional compact metrizable spaces. Then the following assertions are valid*:

(a) $\dim_G X \leq \dim f + \dim_G Y$ *for any Abelian group G*;
(b) $\dim_G X \leq \dim_G f + \dim Y$ *for any Abelian group G*;
(c) $\dim_G X \leq \dim_G f + \dim_G Y$ *for any principal ideal domain G with unit*;
(d) $\dim_G X \leq \dim_G f + \dim_G Y + 1$ *for any Abelian group G.*

In comparison with the case of compact spaces, extensional dimension-lowering type theorems for noncompact spaces are more complicated. First, not all CW-complexes are absolute neighborhood extensors even for the class of paracompact spaces. This suggests various modifications of the definition of extension dimension for noncompact spaces. One of the possible approaches is to define extension dimension for the class of CW-spaces introduced by Dranishnikov and Dydak [40]. This class consists of all spaces X such that every contractible CW-complex is an $AE(X)$, or, equivalently, every CW-complex is an $ANE(X)$. Another approach was developed by Chigogidze in [25, 27]. The third approach (see Kuz'minov [88] and Rubin [117]) is as follows: for any CW-complex K, we consider the class $\alpha(K)$ of spaces X such that any map from a closed set $A \subset X$ to K can be extended to the entire X, provided that it can be extended to some neighborhood of A in X. Dydak [57] also suggested an approach based on absolute extensors up to homotopy.

The most general extensional version of the dimension-lowering theorem is the following result due to Brodskii and Chigogidze [15] (a special case of this theorem, where $\dim Y \leq m$ and $Z = \mathbb{I}^m$, was proved earlier in [31]).

Theorem 4.5 (see [15]). *Let $f : X \to Y$ be a closed map from a k-space X onto a paracompact C-space Y. Suppose that $M \in AE(Y)$ for a quasi-finite CW-complex M and L is a CW-complex such that $L \in ANE(Z \times X)$ and $L \in AE(f^{-1}(y) \times Z)$ for any point $y \in Y$ and any compact space Z with* $\operatorname{e-dim} Z \leq M$. *Then $L \in AE(X)$.*

Theorem 4.5 was proved for finite complexes M, but the proof works also for quasi-finite complexes. Recall that a complex M is quasi-finite [81, 82] if every finite subcomplex K of M is contained in a finite subcomplex P such that any map $g : A \to K$, where A is a closed subset of a normal space Z with $M \in AE(Z)$, can be extended to a continuous map $\bar{g} : Z \to P$ (see also Secs. 5.5 and 7 in this paper).

Corollary 4.6 (see [15]). *Suppose that $f : X \to Y$ is a map of finite-dimensional compact spaces and* $\operatorname{e-dim} Y \leq M$ *for a finite CW-complex M. If* $\operatorname{e-dim}(f^{-1}(y) \times Y) \leq L$ *for some CW-complex L and any point $y \in Y$, then* $\operatorname{e-dim} X \leq L$.

We complete the discussion of the dimension-lowering theorem with the following open problem.

Problem 4.1. *Suppose that $f : X \to Y$ is a perfect surjective map between metrizable spaces such that* $\dim Y \times f^{-1}(y) \leq n$ *for all $y \in Y$. Is it true that* $\dim X \leq n$*?*

The dimension-raising map theorem asserts that if $f : X \to Y$ is a closed map and there exists an integer k such that $|f^{-1}(y)| \leq k+1$ for every $y \in Y$, then $\dim Y \leq \dim X + k$. For separable metrizable spaces, it was proved by Hurewicz [76] and extended by Morita [99] to arbitrary metrizable spaces. Zarelua [144] generalized this theorem to normal spaces (see also Filippov [65] for an elementary proof of the same theorem). Another generalizations of the dimension-raising map theorem were obtained by Keesling [85], Skordev [120, 121], and Filippov [64].

Zarelua's proof of the dimension-raising theorem for paracompact spaces is based on cohomological methods. He proved in [144] that if $f : X \to Y$ is a closed map between paracompact spaces and each fiber $f^{-1}(y)$ contains at most $k+1$ points, then $\dim_L Y \leq \dim_L X + k$, where L is any commutative ring with unit. The result of Zarelua was generalized by Skordev [122] as follows.

Theorem 4.7 (see [122]). *Suppose that $f : X \to Y$ is a closed map between paracompact spaces and each fiber $f^{-1}(y)$ has at most $k+1$ components. If $\max\{d_n + n : n > 0\} \leq m - 1$, then $\dim_L Y \leq \max(m, \dim_L X) + k$ for any module L over a commutative ring with unit, where*

$$d_n = \max\{\dim F : F \text{ is closed in } Y \text{ and } F \subset M_n\}, \quad M_n = \{y \in Y : \dim_L f^{-1}(y) \geq n\}.$$

An extensional version of the dimension-raising map theorem was obtained by Dranishnikov and Uspenskij [49].

Theorem 4.8 (see [49]). *If $f : X \to Y$ is a surjective map between compact metric spaces and k is an integer such that $|f^{-1}(y)| \leq k+1$ for every $y \in Y$, then* $\operatorname{e-dim} Y \leq \operatorname{e-dim}(X \times \mathbb{I}^k)$.

It is easy to show that the Levin–Rubin–Shapiro extensional dimension factorization theorem [94] and the Ščepin spectral theorem imply Theorem 4.8 for all compact (not necessarily metrizable) spaces.

5. Universal Spaces

The problem of the existence of a universal object plays a central role in many fields of mathematics. In topology, a universal object is defined as follows. Let $\mathcal{C}$ be a class of topological spaces with certain properties. A space U is said to be universal for $\mathcal{C}$ if $U \in \mathcal{C}$ and any space X from $\mathcal{C}$ can be embedded in U.

In classical dimension theory, the properties defining the class $\mathcal{C}$ usually refer to Lebesgue dimension. In cohomological and, more generally, extension dimension theory, conditions are imposed on the cohomological (extension) dimension of spaces. Dimensional conditions may be supplemented by other topological properties. For instance, we can consider compact spaces of given weight, separable metrizable spaces, or compact metrizable spaces. Universal spaces were considered in many classical papers on dimension theory, including [4, 95, 102, 103, 106, 113, 143].

In this section, we assume that all spaces are separable and metrizable, unless otherwise specified.

5.1. Universal spaces in extension dimension theory. One of the most famous unsolved problems related to universal spaces in dimension theory is as follows.

Problem 5.1 (see [138]). *Does there exist a universal compact metric space of given integral cohomological dimension?*

In this problem, $\mathcal{C}$ is the class of all compact metrizable spaces X with $\dim_{\mathbb{Z}} X \leq n$. For extension dimension, it can be generalized as follows.

Problem 5.2. *Let $\mathcal{C}_L$ denote the class of all compact metrizable spaces X such that* $\text{e-dim}\, X \leq [L]$, *where L is a countable locally finite complex. Characterize all complexes L for which $\mathcal{C}_L$ contains a universal space.*

Indeed, for L we can take the Eilenberg–MacLane complex $K(\mathbb{Z}, n)$. In what follows, we refer to a universal object for the class $\mathcal{C}_L$ as an $[L]$-universal compactum.

Problem 5.2 has partial solutions. First of all, results of Chigogidze [27, Theorem 2.5] and Dydak [54] imply that a universal compactum exists in the case where L is finite or, more generally, finitely dominated. A standard way of obtaining such a universal compactum is to construct an $[L]$-invertible map $f : X_L \to \mathbb{I}^\omega$, where X_L is a compact metric space with $\text{e-dim}\, X \leq [L]$ (in fact, $\text{e-dim}\, X_L = [L]$, because X_L is universal).

Definition 5.1 (see [27]). *A map $f : X \to Y$ is said to be $[L]$-invertible if, for any space Z with* $\text{e-dim}\, Z \leq [L]$ *and any map $g : Z \to Y$, there exists a map $h : Z \to X$ such that $f \circ h = g$.*

Note that the $[L]$-invertibility of L and the universality of $\mathbb{I}^\omega$ for all compact metrizable spaces guarantees the $[L]$-universality of X_L. In turn, the map f can be obtained as a limit projection of a certain L-resolvable inverse sequence. We recall the definition below.

Definition 5.2 (see [45]). *An inverse sequence $\{X_n, p_n^{n+1}\}$ of compact metrizable spaces is said to be L-resolvable if, for any n, any closed subset A of X_n, and any map $g : A \to L$, there exists an $m \geq n$ such that the map $(g \circ p_n^m) : (p_n^m)^{-1} A \to L$ can be extended over X_m.*

This property of the sequence is used to ensure that $\text{e-dim}\, X_L \leq [L]$. Indeed, it can be shown that if an inverse sequence of compact metrizable spaces is L-resolvable, then the extension dimension of the limit space does not exceed $[L]$ (see [45, Lemma 2.1]).

Moreover, the sequence is constructed so that all bonding maps are $[L]$-invertible (which implies the $[L]$-invertibility of the limit projection f) and the nontrivial preimages of points under the bonding maps are homeomorphic to L. The latter condition is crucial, as it ensures the compactness of each space in the sequence and hence of X_L for finite L. On the other hand, the construction does not yield a compact space when applied to countable L, for example, to $K(\mathbb{Z}, n)$.

Nevertheless, a universal object for any (countable locally finite) complex L does exist in a larger class of spaces, namely, in the class of separable metrizable spaces. Dydak and Mogilski [59] proved that, for any n, there exists a Polish space X of integral cohomological dimension n which contains a topological copy of any separable metric space Y with $\dim_{\mathbb{Z}} Y \le n$. This result was generalized by Olszewski [104], who proved the existence of a universal separable metric space of given extension dimension $[L]$. As a corollary, this implies the existence of a universal separable metrizable space of given cohomological dimension with respect to any countable Abelian group of coefficients.

5.2. Projective resolution of extension problems. In this section, we briefly outline the construction of universal spaces as limits of inverse spectra in the case of finite L (see [27, 45]).

The following basic idea is due to Dranishnikov [42]; it found further applications in the papers [45, 46] of Dranishnikov and Repovš. Given a space X, its closed subset A, and a map $f : A \to L$, we can guarantee the existence of an extension of f over X only if $\text{e-dim}\, X \le [L]$. We can, however, always extend f *projectively* as follows (see [45]; a similar terminology was introduced in [17] in relation to another problem). Let $q : L \times \mathbb{I} \to \text{Cone}(L)$ denote the natural quotient map. Note that f can be extended to a map $\overline{f} : X \to \text{Cone}(L)$. Consider the pullback diagram

$$\begin{array}{ccc} R(X, A, f) & \longrightarrow & L \times \mathbb{I} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle q} \\ X & \xrightarrow{\ \overline{f}\ } & \text{Cone}(L) \end{array} .$$

We say that the space $R(X, A, f)$ *projectively resolves the extension problem* (X, A, f). The crucial property of the space $R(X, A, f)$ and the map p is that the map $(f \circ p) : p^{-1}(A) \to L$ can be extended to a map $F : R(X, A, f) \to L$ (see [45, Lemma 2.2]).

Suppose given an arbitrary compact metrizable space X_0. We construct an inverse sequence as follows. Let us represent the set of all positive integers as the union of disjoint cofinal subsets $\mathbb{N} = \cup\{N_i \mid i = 0, 1, 2, \dots\}$. Suppose that the compact space X_i is already constructed. We want to projectively resolve all extension problems $f : A \to L$, where A is a closed subset of X_i. It turns out that it is sufficient to resolve only *countably* many extension problems. First, consider any countable family $\mathcal{A}$ of closed subsets of X_i such that, for any closed subset F of X_i and any open neighborhood U of F, there exists an A in $\mathcal{A}$ such that $F \subset A \subset U$. Next, for each $A \in \mathcal{A}$, consider a countable family $\mathcal{F}_A$ of maps from A to L dense in the space of all maps from A to L. Since L is an ANR, it suffices to resolve the extension problems from the set $\bigcup\{\mathcal{F}_A \mid A \in \mathcal{A}\}$. Let us enumerate this set by the elements of N_i. Suppose that $f : A \to L$ is the extension problem with least number in the set $\bigcup\{N_k \mid k = 0, 1, 2, \dots, i\}$ not yet resolved. Note that A is a closed subset of X_j for $j \le i$. We set $X_{i+1} = R(X_i, (p^i_j)^{-1}(A), f \circ p^i_j)$ and define p^{i+1}_i as the natural projection.

It can be shown that the sequence thus obtained is $[L]$-resolvable (see [45, Lemma 2.3]). Thus, the limit compact space is of dimension at most $[L]$. If we need it to be universal, then the construction of the resolution $p : R(X, A, f) \to X$ must be modified as follows [27]. Consider the product $\text{Cone}(L) \times L$ and its subset $T = \{((l, t), l) : (l, t) \in L \times [0, 1)\}$. Let U be an open neighborhood of the vertex of the cone in $\text{Cone}(L)$. We set $\widetilde{L} = (\overline{U} \times L) \cup T$ and define the map $\widetilde{q} : \widetilde{L} \to \text{Cone}(L)$ as the restriction of the projection $\text{Cone}(L) \times L \to \text{Cone}(L)$ to $\widetilde{L}$. It can be shown that the map $\widetilde{q}$ is $[L]$-invertible and approximately $[L]$-soft (see [27, Lemma 2.6]).

Definition 5.3 (see [27])**.** *A map $f : X \to Y$ is said to be $[L]$-soft (approximately $[L]$-soft) if, for any space Z with* $\text{e-dim}\, Z \le [L]$*, any closed subset A of Z (for any open cover ω of Y), and any two maps $g : A \to X$ and $h : Z \to Y$ such that $f \circ g = h|_A$, there exists a map $\overline{g} : Z \to X$ such that it extends g over Z and $f \circ \overline{g} = h$ (respectively, $f \circ \overline{g}$ is ω-close to h).*

The map $\widetilde{q}$ can be used to projectively resolve the extension problem $f : A \to L$. Namely, we extend f to a map $\overline{f} : X \to \text{Cone}(L)$ and define $\widetilde{R}(X, A, f) \to X$ to be the pullback in the diagram (see [27,

p. 324])

$$\begin{array}{ccc} \widetilde{R}(X,A,f) & \longrightarrow & \widetilde{L} \\ {\scriptstyle \widetilde{p}}\downarrow & & \downarrow{\scriptstyle \widetilde{q}\ \ ([L]\text{-invertible})} \\ X & \xrightarrow{\ \overline{f}\ } & \mathrm{Cone}(L) \end{array}$$

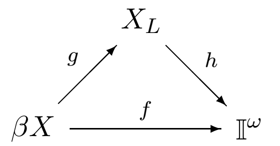
It is easy to show that the map $\widetilde{p} : \widetilde{R}(X, A, f) \to X$ is $[L]$-invertible as well. Thus, in the inverse sequence constructed by using the resolutions $\widetilde{R}(X, A, f)$ instead of $R(X, A, f)$, all bonding maps are $[L]$-invertible. This guarantees the $[L]$-invertibility of the limit projection. As a consequence, if X_0 is the Hilbert cube, then the limit compactum is $[L]$-universal (even in the class of separable metrizable spaces).

If the compact space $\widetilde{R}(X, A, f)$ is an ANR, then $\widetilde{p}$ is approximately $[L]$-soft (because of the approximate $[L]$-softness of $\widetilde{q}$). If X_0 is a Hilbert cube or a finite polyhedron, then the above construction can be modified (by a more careful choice of the sets in $\mathcal{A}$ and maps in $\mathcal{F}_A$ at each step) so as to make all compact spaces in the inverse sequence into ANRs. In this case, all bonding maps and, therefore, the limit projection are approximately $[L]$-soft. Moreover, if X_0 is a polyhedron, then all compact spaces in the sequence can be made into polyhedra and all bonding maps, into simplicial maps. More details can be found in [45, Lemma 2.7] and [27, Proposition 2.23].

5.3. Compacta of given extension dimension. An approach similar to that described in Sec. 5.2 was used by Dranishnikov and Repovš [45, 46] to obtain, for given complexes L and M, a compact metrizable space X such that $\text{e-dim}\, X \le [L]$ and $\text{e-dim}\, X > [M]$. Of course, such a compact space does not always exist; L and M must satisfy certain conditions. The required compactum is the limit of the inverse sequence constructed as above starting with the suspension ΣM. The inverse sequence is constructed by using resolutions of type $R(X, A, f)$, which can be improved so that all spaces in the sequence are polyhedra and all bonding maps are simplicial. In particular situations, some modifications may be required. A map $p : X \to \Sigma M$ is obtained as the limit projection [46]. The key idea is to find a continuous cohomology theory h^* (or a homology theory h_*) for which $\widetilde{h}^*(L) \approx 0$ and $\widetilde{h}^*(\Sigma M)$ is nontrivial [45, Theorem 2.4] (see also [46, Theorem 1.3]). The assumption that h^* is continuous is necessary to obtain an inverse sequence consisting of *compact* spaces in the case where L or M is not finite. Suppose that such a cohomology theory h^* exists. Since all fibers of the bonding maps are either homeomorphic to L or trivial, it follows that all bonding maps are acyclic with respect to $\widetilde{h}^*$, and it can be shown by using a Vietoris–Begle-type theorem that all bonding maps induce isomorphisms in cohomology. The nontriviality of $\widetilde{h}^*(M)$ implies that the map p is not null-homotopic. Therefore, the restriction $p|_{p^{-1}(M)}$ of p to the preimage of M, considered as the equator of ΣM, cannot be extended to a map from X to M. Hence $\text{e-dim}\, X > [M]$. Detailed proofs can be found in [45, 46]; in [45], the role of h_* was played by the classical homology. It was shown in [46] that truncated cohomology can be used in place of h^*.

The idea to distinguish between two CW-complexes by using a suitable cohomology theory is due to Dranishnikov. In [37], he used K-theory (which distinguishes between the sphere S^4 and $K(\mathbb{Z}, 3)$) to construct a compact space of finite integral cohomological dimension and infinite covering dimension, thus solving the famous problem of Alexandroff. After Dydak and Walsh discovered truncated cohomology [61], the method was developed further and found new applications. A similar approach was used by Levin in [90, 91].

5.4. Compactifications and $[L]$-invertible maps. Chigogidze suggested a different construction of $[L]$-universal compact space [25]. Suppose that L is a complex such that, for any space X, $\text{e-dim}\, X \le [L]$ implies $\text{e-dim}\, \beta X \le [L]$. Consider the set of all Polish spaces X_t with $\text{e-dim}\, X_t \le [L]$ and their maps $f_t : X_t \to \mathbb{I}^\omega$ (note that the spaces X_t may coincide for different t if the corresponding maps f_t differ). Let X be the discrete sum of all X_t, and let $f : X \to \mathbb{I}^\omega$ be the map coinciding with f_t on each X_t. Then $\text{e-dim}\, X \leq [L]$, and it can be shown that f is $[L]$-invertible. Consider $\beta f : \beta X \to \mathbb{I}^\omega$. Note that βf is $[L]$-invertible as well. Since $\text{e-dim}\, X \leq [L]$, it follows from the assumption about L that $\text{e-dim}\, \beta X \leq [L]$. It can be shown that the map $\beta f : \beta X \to \mathbb{I}^\omega$ can be factored through a metrizable compactum X_L of extension dimension not exceeding $[L]$ as



Finally, it is easy to show that $h : X_L \to \mathbb{I}^\omega$ is $[L]$-invertible.

This construction suggests the following question [26].

Problem 5.3. *Characterize the complexes L such that $L \in AE(X)$ if and only if $L \in AE(\beta X)$ for any space X.*

From works of Dranishnikov [36], Dydak [53], Dydak and Walsh [60], and Levin [91], we know that, for each $n \geq 2$, there exists a (metrizable separable) space X for which $\dim_{\mathbb{Z}} X \leq n$ and $\dim_{\mathbb{Z}} \beta X > n$. Therefore, in the above problem, L cannot be the Eilenberg–MacLane complex $K(\mathbb{Z}, n)$ for $n \geq 2$. In fact, results of Levin [91] imply that L cannot coincide with the complex $K(G, 2)$ for any nontrivial Abelian group G.

Chigogidze [26] showed that Problem 5.3 is equivalent to the following problem.

Problem 5.4. *Characterize the complexes L for which there exists an $[L]$-invertible map $f : X \to \mathbb{I}^\omega$, where X is a compact metrizable space such that* $\text{e-dim}\, X \leq [L]$.

It is seen from the above considerations that the following problem is closely related to Problems 5.3 and 5.4 (see [28, 54]).

Problem 5.5. *Let L be a complex such that an $[L]$-universal compactum exists. Is it true that the extension type of L contains a finitely dominated complex?*

In the next section, we show that the answer to this question is negative [81] and give solutions to Problems 5.3 and 5.4.

5.5. Quasi-finite complexes. We begin by introducing the notion of a quasi-finite complex. Let X be a space. We say that a pair of spaces $V \subset U$ is *X-connected* if, for every closed subspace $A \subset X$, any map from A to V can be extended to a map from X to U. If $\mathcal{C}$ is a class of spaces, then a pair of spaces $V \subset U$ is said to be *$[L]$-connected with respect to $\mathcal{C}$* if, for every space $X \in \mathcal{C}$ with $\text{e-dim}\, X \leq [L]$, the pair $V \subset U$ is X-connected. It follows from [16, Proposition A.1] that if a pair $V \subset U$ is $[L]$-connected with respect to Polish spaces, then it is $[L]$-connected with respect to normal spaces.

Definition 5.4. *We say that a complex L is quasi-finite if, for every finite subcomplex P of L, there exists a finite subcomplex eP of L containing P such that the pair $P \subset eP$ is $[L]$-connected for Polish spaces.*

The following theorem [26, 81] shows that the class of quasi-finite complexes provides a solution to Problem 5.3 (and 5.4). The equivalences (ii)–(vi) in this theorem were proved by Chigogidze [26].

Theorem 5.1. *Let L be a locally finite countable CW-complex. Then the following conditions are equivalent*:

(i) *L is quasi-finite*;

(ii) $\text{e-dim}\, \beta X \leq [L]$ *whenever X is a (Tychonoff) space with* $\text{e-dim}\, X \leq [L]$;

(iii) e-dim $\beta X \le [L]$ *whenever* X *is a normal space with* e-dim $X \le [L]$*;*
(iv) e-dim $\beta(\oplus\{X_t \mid t \in T\}) \le [L]$ *whenever* T *is an arbitrary index set and* X_t*, where* $t \in T$*, is a separable metrizable space with* e-dim $X_t \le [L]$;
(v) e-dim $\beta(\oplus\{X_t \mid t \in T\}) \le [L]$ *whenever* T *is an arbitrary indexing set and* X_t*, where* $t \in T$*, is a Polish space with* e-dim $X_t \le [L]$;
(vi) *there exists an* $[L]$*-invertible map* $f : X \to \mathbb{I}^\omega$*, where* X *is a compact metrizable space with* e-dim $X \le [L]$.

Remark. *For a normal space* X*, the inequality* e-dim $X \le [L]$ *means that* $L \in AE(X)$*. The definition of extension dimension for arbitrary (Tychonoff) spaces can be found in [25] or in Sec. 7 of this paper.*

The above theorem and the remarks after Problem 5.3 imply that none of the Eilenberg–MacLane complexes $K(G, n)$, where G is a countably generated nontrivial Abelian group, is quasi-finite for all $n \ge 2$. The following statement is an easy corollary of Theorem 5.1.

Theorem 5.2. *If* L *is a quasi-finite complex, then there exists an* $[L]$*-universal compact space.*

The natural question that arises first is: Are there quasi-finite complexes which are not finite (or finitely dominated)? It is easy to show that the wedge product of any number of finite complexes is always a quasi-finite complex. Nevertheless, such a wedge product may be equivalent to a finite (finitely dominated) complex. Our immediate goal is to construct an example of a quasi-finite complex which is not equivalent to a finitely dominated complex [81]. This example and Theorem 5.2 give a negative solution to Problem 5.5.

5.6. An example of a quasi-finite complex. Let $M_p = M(\mathbb{Z}_p, 2)$ denote a Moore space of type $(\mathbb{Z}_p, 2)$. Note that M_p is a finite complex. Consider a locally finite countable CW-complex M homotopy equivalent to the wedge product $\vee\{M_p : p\text{-prime}\} \vee S^3$. Clearly, M is quasi-finite (because it is equivalent to a wedge of finite complexes). We claim that the extension type of M does not contain a finitely dominated complex and, therefore, $[S^2] < [M] < [S^3]$ (obviously, $[S^2] \le [M] < [S^3]$).

Suppose that, on the contrary, L is a finitely dominated countable complex for which $[M] = [L]$. Then $[S^2] \le [L] < [S^3]$. The inequality $[S^2] \le [L]$ implies that L is an absolute extensor for the disk. Therefore, $[L]$ is simply connected, and hence $H_2(L) \approx \pi_2(L)$. Since L is finitely dominated, it follows that the groups $H_2(L) \approx \pi_2(L)$ are finitely generated; moreover, $\pi_2(L)$ is nontrivial, because $[L] < [S^3]$. The further argument depends on whether or not $H_2(L)$ is torsion.

First, suppose that the free part of $H_2(L)$ is nontrivial. Then $H_2(L) \approx \mathbb{Z} \oplus H$, where H is finitely generated. According to the universal coefficient formula, the groups $H^1(M; \mathbb{Q})$ and $H^2(M; \mathbb{Q})$ are trivial. On the other hand, the suspension isomorphism and the Hurewicz theorem imply $H_3(\Sigma L) \approx \mathbb{Z} \oplus H$; hence, again by the universal coefficient formula, the group $H^3(\Sigma L; \mathbb{Q})$ is nontrivial. Applying construction of Dranishnikov and Repovš (see [45, Theorem 2.4] and [46, Theorem 1.3]) and using the idea from the proof of Theorem 1.4 in [46, p. 351], we obtain a metrizable compactum X for which e-dim $X \le [M]$ and $L \notin AE(X)$. This construction was outlined in Sec. 5.3; instead of h^*, the rational Čech cohomology should be used. Since $L \notin AE(X)$, it follows that $[L] \ne [M]$.

Now suppose that $H_2(L)$ is torsion. Then $H_2(L) \approx \bigoplus\{\mathbb{Z}_{n_i} : i = 1, \cdots, m\}$. Choose a prime $p > \max\{n_i : i = 1, \dots, m\}$. The universal coefficient formula shows that $H^1(L; \mathbb{Z}_p) \approx 0$, $H^2(L; \mathbb{Z}_p) \approx 0$, and $H^3(\Sigma M_p; \mathbb{Z}_p)$ is nontrivial. As above, we can find a map $f : X \to \Sigma M_p$, where X is a compact metrizable space for which e-dim $X \le [L]$ and e-dim $X > [M_p]$. The inequality $[M] \le [M_p]$ contradicts the equality $[M] = [L]$.

We conclude this section with two questions related to the above considerations. The first question is similar to Problem 5.5 but refers to quasi-finite complexes.

Problem 5.6. *Let* L *be a complex for which an* $[L]$*-universal compact space exists. Is it true that* L *is quasi-finite?*

The above example of a quasi-finite complex is a wedge product of finite complexes. What are other possible examples?

Problem 5.7. *Does there exist a quasi-finite complex which is not equivalent to a wedge product of finite complexes?*

Recently, Cencelj, Dydak, Smrekar, Vavpetič, and Virk obtained new results on quasi-finite complexes, but they are beyond the scope of this paper. The interested reader can find them in [22, 23].

6. Quasi-Finite Complexes and Absolute Extensors in Dimension [L]

Not only the general problem of the existence of a universal compact space of given extension dimension is of interest but also the question of how "nice" this compact space may be. For example, since absolute extensors in dimension n play an important role in the study of universal spaces in classical dimension theory, it is natural to consider the following modification of Problem 5.2 in Sec. 5.1.

Problem 6.1. *Characterize all complexes L for which $\mathcal{C}_L$ contains a universal object which is an absolute extensor in dimension $[L]$ for Polish spaces (or compact metrizable spaces).*

Recall that a space Y is an absolute (neighborhood) extensor in dimension $[L]$ (or, briefly, $Y \in A(N)E([L])$) for a given class of spaces $\mathcal{C}$ if $Y \in A(N)E(X)$ for all X from $\mathcal{C}$ such that $\text{e-dim}\, X \leq [L]$.

For a finite complex L, the existence of an $[L]$-universal compact absolute extensor in dimension L follows from the existence of an $[L]$-invertible approximately $[L]$-soft map $f : X_L \to \mathbb{I}^\omega$ (see [27]). Here X_L is a compact metrizable space with $\text{e-dim}\, X_L = [L]$. Note that $X_L \in AE([L])$ for Polish spaces, because f is approximately $[L]$-soft. The idea of the construction of f was outlined in Sec. 5.2. It should also be mentioned that there always exists an $AE([L])$-space universal for the class of all Polish at most $[L]$-dimensional spaces. Indeed, for any countable complex L, there exists an $[L]$-soft map $f : X_L \to \mathbb{I}^\omega$ [25]. On the other hand, Zarichnyi [145] proved that there exists no universal compact space of given integral cohomological dimension which is an absolute extensor with respect to compact metrizable spaces of given cohomological dimension. Thus, in Problem 6.1, the complex L cannot be the Eilenberg–MacLane complex $K(\mathbb{Z}, n)$ for $n \geq 2$.

We shall see that quasi-finite complexes may be candidates to provide a solution to Problem 6.1. Namely, we shall show that if there exists an $[L]$-universal compact space which is an absolute extensor in dimension $[L]$ for Polish spaces, then L must be quasi-finite.

6.1. The connected-pair property.

Let $\mathcal{B}$ be a class of spaces. We say that a complex L possesses the *connected-pair property with respect to* $\mathcal{B}$ if, for any compact metrizable space K with $\text{e-dim}\, K \leq [L]$, there exists a compact metrizable space C containing K such that $\text{e-dim}\, C \leq [L]$ and the pair $K \subset C$ is $[L]$-connected with respect to $\mathcal{B}$ (see Sec. 5.5). The following technical lemma describes the behavior of connected pairs of compact spaces with respect to the Stone–Čech compactification.

Lemma 6.1 (see [83, Lemma 3.1])**.** *Suppose that T is an arbitrary index set and $\{X_t \mid t \in T\}$ is a family of Polish spaces such that* $\text{e-dim}\, X_t \leq [L]$ *for each $t \in T$. Let $X = \bigoplus\{X_t \mid t \in T\}$. Suppose that $K \subset C$ is a pair of compact metrizable spaces such that* $\text{e-dim}\, K \leq [L]$*. If the pair $K \subset C$ is $[L]$-connected with respect to Polish spaces, then this pair is βX-connected.*

The following lemma establishes an important relationship between the connected-pair property and the property of being an absolute extensor for a given space.

Lemma 6.2 (see [83, Lemma 3.2])**.** *Let L be a complex with the connected-pair property with respect to Polish spaces, and let X be a compact space. Suppose that each pair $K \subset C$ of compact metrizable spaces with* $\text{e-dim}\, C \leq [L]$ *is X-connected provided that $K \subset C$ is $[L]$-connected with respect to Polish spaces. Then any metrizable space Y such that $Y \in AE([L])$ for metrizable compacta and* $\text{e-dim}\, Y \leq [L]$ *is an $AE(X)$.*

To investigate the relationship between the connected-pair property, quasi-finite complexes, and Problem 6.1, we apply a technique similar to that used to construct approximations of multivalued maps. For this purpose, we need simplices and skeletons associated with an open cover of a space [16].

6.2. Simplices and skeletons associated with covers. A standard way to construct an approximation of a multivalued map is as follows. First, we consider a sufficiently fine open cover of a space and a canonical map to the nerve of this cover. The second step is to define a map on the nerve by induction on the dimensions of skeletons. The required approximation is the composition of these two maps. The construction can be implemented provided that, for example, the set-valued map is UV^n-valued and the domain X has Lebesgue dimension n. However, when we try to obtain extension-dimensional versions of various approximations and selections theorems, the situation changes. Namely, if the extension dimension of the space X does not coincide with the Lebesgue dimension of X, the required map on the nerve of a cover cannot be defined.

Thus, approximations must be constructed directly from the space X. The following notation, which proved very useful for constructing approximations, was introduced in [16].

For an open cover $\Sigma \in \operatorname{cov}(X)$, let $\Sigma^{(k)}$ denote its "k-skeleton," i.e., the set of all points in X at which the order of Σ is at most $k+1$. Thus, we set $\Sigma^{(k)} = \{x \in X \mid \operatorname{ord}_\Sigma x \le k+1\}$. For elements $s_0, s_1, \dots, s_n \in \Sigma$ with nonempty intersection $\bigcap\limits_{i=0}^{n} s_i$, we define the "closed n-simplex"

$$[s_0, s_1, \dots, s_n] = \bigcup_{i=0}^{n} s_i \setminus \bigcup \{s \in \Sigma \mid s \ne s_i, i = 0, 1, \dots, n\}$$

and its "interior" $\langle s_0, s_1, \dots, s_n\rangle = \bigcap\limits_{i=0}^{n} s_i \cap \Sigma^{(n)}$. It is easy to check that the n-skeleton consists of the n-simplices

$$\Sigma^{(n)} = \bigcup \{[s_{i_0}, s_{i_1}, \dots, s_{i_n}] \mid \bigcap_{k=0}^{n} s_{i_k} \ne \varnothing\}$$

and that any simplex consists of "boundary" and "interior," i.e.,

$$[s_0, s_1, \dots, s_n] = \bigcup_{m=0}^{n} [s_0, \dots, \widehat{s}_m, \dots, s_n] \cup \langle s_0, s_1, \dots, s_n\rangle.$$

Clearly, $\Sigma^{(k)}$ is closed in X and $\Sigma^{(n)} = X$ if the cover Σ has order $n+1$. Note also that the interiors of distinct k-simplices are mutually disjoint, and

$$\Sigma^{(k)} = \bigcup \{\langle s_{i_0}, s_{i_1}, \dots, s_{i_n}\rangle \mid \bigcap_{k=0}^{n} s_{i_k} \ne \varnothing\} \cup \Sigma^{(k-1)}.$$

6.3. Characterization of quasi-finite complexes. The following lemma is crucial for the characterization which we obtain below. It allows us to construct approximate liftings with respect to a map whose fibers have nice extensional properties for a given space.

Lemma 6.3 (see [83, Lemma 3.3])**.** *Suppose that X is a compact space and Z is a paracompact space such that any compact subspace of Z has finite Lebesgue dimension. Let $g : Y \to Z$ be a surjection such that, for every $z \in Z$ and any neighborhood $U(z)$ of z in Z, there exists a smaller neighborhood $V(z)$ of z such that $g^{-1}(V(z)) \in AE(X)$. Then, for any $\omega \in \operatorname{cov}(Z)$ and any map $f : X \to Z$, there exists a map $\tilde{f} : X \to Y$ such that the maps f and $g \circ \tilde{f}$ are ω-close.*

Proof. Note that $f(X) \subset Z$ is compact; therefore, $\dim f(X) = n < \infty$ for some n. We set $\omega_0 = \omega$ and construct a sequence of covers $\omega_1, \omega_2, \dots, \omega_n$ by induction as follows. Suppose that $\omega_i \in \operatorname{cov}(Z)$ is already constructed and let ν be a strong star refinement of ω_i. For each $z \in Z$, we choose $U(z) \in \nu$ containing z and take a smaller neighborhood $V(z) \subset U(z)$ of z such that

$$g^{-1}(V(z)) \in AE(X). \qquad (\dagger)$$

We set $\omega_{i+1} = \{V(z) \mid z \in Z\}$. Obviously, ω_{i+1} is a strong star refinement of ω_i.

Let $\Sigma \in \operatorname{cov}(f(X))$ be a finite strong star refinement of ω_n restricted to $f(X)$ for which $\operatorname{ord}\Sigma \le n+1$. We set $\widehat{\Sigma} = \{f^{-1}(U) \mid U \in \Sigma\}$. Clearly, $\widehat{\Sigma}$ is a finite open cover of X of order $\le n+1$. Let us construct a sequence of maps $f_0, f_1, \dots, f_n$ such that $f_k : \widehat{\Sigma}^{(k)} \to Y$ and

$$g(f_k(x)) \in \operatorname{St}(f(x), \omega_{n-k}) \qquad (*)$$

for all k. To construct f_0, we choose a point $P_s \in g^{-1}(f(s))$ for each $s \in \widehat{\Sigma}$ and put $f_0|_{[s]} = P_s$ for every closed 1-simplex $[s]$. Suppose that f_k is already constructed. Let us construct f_{k+1}. It suffices to define it on the interior $\langle\sigma\rangle$ of each simplex $[\sigma] = [s_0, s_1, \dots, s_{k+1}]$. Since $\widehat{\Sigma}$ is finite and the interiors of closed k-simplices are mutually disjoint, we can consider each simplex independently. Let $[\sigma]' = [\sigma] \cap \widehat{\Sigma}^{(k)}$. The cover Σ being a strong star refinement of ω_n (and, consequently, of ω_{n-k}) and inclusion $(*)$ imply the existence of a $W_\sigma \in \omega_{n-k}$ such that $g(f_k([\sigma]')) \subset \operatorname{St}(W_\sigma, \omega_{n-k})$. Since ω_{n-k} is a strong star refinement of ω_{n-k-1} and by the construction of ω_{n-k-1}, there exists an element $V_\sigma \in \omega_{n-k-1}$ having property $(\dagger)$ and such that $\operatorname{St}(W_\sigma, \omega_{n-k}) \subset V_\sigma$. We have $g^{-1}(V_\sigma) \in AE(X)$; therefore, the map $f_k|_{[\sigma]'}$ can be extended to a map $f_{k+1} : [\sigma] \to g^{-1}(V_\sigma) \subset Y$. It is easy to see that condition $(*)$ holds for f_{k+1}. Finally, we set $\widetilde{f} = f_n$. □

Now we can characterize quasi-finite complexes in terms of the connected-pair property.

Theorem 6.4 (see [83, Theorem 3.4])**.** *A locally finite countable complex L has the connected-pair property with respect to Polish spaces if and only if L is quasi-finite.*

Proof. The "if" part follows from the existence of an $[L]$-invertible map. Suppose that K is a compact metrizable space with $\operatorname{e-dim} K \le L$. We embed K into the Hilbert cube $\mathbb{I}^\omega$ and consider an $[L]$-invertible map $f : Y \to \mathbb{I}^\omega$ such that Y is a compact metrizable space with $\operatorname{e-dim} Y \le L$. The existence of such a map is guaranteed by Theorem 5.1 (vi). Consider the adjunction space $Y \cup_f K$, that is, the disjoint union of $Y - f^{-1}(K)$ and K with the topology consisting of the usual open subsets of $Y - f^{-1}(K)$ and all sets of the form $f^{-1}(U - K) \cup (U \cap K)$, where U are open subsets of $\mathbb{I}^\omega$. There are two associated maps $p_K : Y \to Y \cup_f K$ and $f_K : Y \cup_f K \to \mathbb{I}^\omega$ such that $f = f_K \circ p_K$. Since f is $[L]$-invertible, so is f_K. Moreover, the set $Y - f^{-1}(K)$ is open in Y and, therefore, can be represented as a countable union of compact sets, each with e-dim $\le L$. By the countable-sum theorem, we have $\operatorname{e-dim} Y \cup_f K \le L$.

It remains to show that the pair $K \subset Y \cup_f K$ is L-connected. Suppose that Z is a Polish space with $\operatorname{e-dim} Z \le L$, A is a closed subset of Z, and $g : A \to K$ is a map. We can consider g as a map from A to $K \subset \mathbb{I}^\omega$. Since $\mathbb{I}^\omega$ is an absolute extensor, it follows that g has an extension $\overline{g} : Z \to \mathbb{I}^\omega$. Finally, since f_K is L-invertible, we can lift $\overline{g}$ to a map $h : Z \to Y \cup_f K$ with $f_K \circ h = \overline{g}$. Clearly, h extends g.

To prove the "only if" part, we show that L satisfies condition (v) in Theorem 5.1. Let $\{X_t \mid t \in T\}$ be a family of Polish spaces (where T is an arbitrary index set) such that $\operatorname{e-dim} X_t \le [L]$ for all $t \in T$. Consider $X = \bigoplus\{X_t \mid t \in T\}$. We must show that $\operatorname{e-dim} \beta X \le [L]$. Take a closed subset A of βX and a map $f : A \to L$. Consider an $[L]$-soft map $g : Y \to L$, where Y is a Polish space with $\operatorname{e-dim} Y \le [L]$. The existence of such a map follows from [25, Proposition 5.9]. We claim that the map g satisfies the conditions of Lemma 6.3 (with L instead of Z). Indeed, take $z \in L$ and its open neighborhood $U(z)$. There exists a neighborhood $V(z) \subset U(z)$ of z in L which is an absolute extensor. Since g is $[L]$-soft, we have $g^{-1}(V(z)) \in AE([L])$ for Polish spaces. Successively applying Lemmas 6.1 and 6.2 (for the pair $g^{-1}(V(z)) \subset g^{-1}(V(z))$), we conclude that $g^{-1}(V(z)) \in AE(\beta X)$. This proves the claim. The same argument shows that $Y \in AE(\beta X)$.

Since L is an ANR-space, it follows that there exists an open cover $\omega \in \operatorname{cov}(L)$ such that any two maps ω-close to L are homotopic. Applying Lemma 6.3 to the maps $g : Y \to L$ and $f : A \to L$ and the cover ω, we obtain a map $\tilde{f} : A \to Y$ such that f and $g \circ \tilde{f}$ are ω-close. Since $Y \in AE(\beta X)$, $\tilde{f}$ extends to a map $\overline{\tilde{f}} : \beta X \to Y$. Let $f' = g \circ \overline{\tilde{f}} : \beta X \to L$. Note that $f'|_A$ is ω-close to f and, therefore, f admits an extension over βX, as required. □

The following theorem, which is the central result of this section, is a direct corollary of Theorem 6.4.

Theorem 6.5 (see [83, Theorem 3.5])**.** *Suppose that there exists an $[L]$-universal compact metrizable space which is an $AE([L])$ for Polish spaces. Then L is quasi-finite.*

Note that none of the Eilenberg–MacLane complexes $K(G, n)$, where G is a nontrivial Abelian group and $n \geq 2$, is quasi-finite (see the remark after Theorem 5.1). Thus, this theorem implies the following assertion.

Theorem 6.6 (see [83, Theorem 3.6])**.** *Suppose that G is a countably generated nontrivial Abelian group and $n \geq 2$ is an integer. Then there exists no universal compact space of given cohomological dimension n with respect to the coefficient group G which is an absolute extensor with respect to Polish spaces of cohomological dimension $\leq n$.*

The following question naturally arises; a positive answer to this question would provide a complete solution to Problem 6.1 in the case of Polish spaces.

Problem 6.2. *Let L be a quasi-finite complex. Does there exist a compact metrizable space L which is an $[L]$-universal absolute extensor in dimension $[L]$ for Polish (or compact metrizable) spaces?*

7. Further Properties of Quasi-Finite Complexes

In this section, we extend the definition of quasi-finite CW-complexes to arbitrary complexes, not necessarily countable and locally finite [82].

We shall see that quasi-finite complexes are similar to finite complexes in many respects [82]. We characterize quasi-finite complexes in terms of L-invertible maps and the dimensional properties of compactifications. We also prove a version of the factorization theorem and construct universal spaces of arbitrary weight.

7.1. Extension dimension and quasi-finite CW-complexes: The general case. First, we need a more careful definition of an absolute extensor. Suppose that X and Y are spaces, $A \subset X$, and $g : A \to Y$ is a map. We write $Y \in ANE(g, A, X)$ if g has a continuous extension $\bar{g} : U \to Y$ to a neighborhood U of A in X for which there exists a function $h : X \to [0, 1]$ such that $h^{-1}((0, 1]) = U$ and $h(A) = 1$. If $U = X$ in the above definition, then we write $Y \in AE(g, A, X)$. Note that, according to [55, Lemma 2.8], $Y \in ANE(g, A, X)$ if and only if g extends to a map $\overline{g} : X \to \operatorname{Cone}(Y)$.

We say that L is an absolute extensor for X (and write, as usual, $L \in AE(X)$) if $L \in AE(g, A, X)$ for any closed set $A \subset X$ and any map $g : A \to L$ with $L \in ANE(g, A, X)$. We also say that the extension dimension of X is at most L ($\text{e-dim}\, X \leq L$) if $L \in AE(X)$. Dydak's version of the homotopy extension theorem [55, Theorem 13.7] implies that if L_1 is homotopy equivalent to L_2, then $\text{e-dim}\, X \leq L_1$ is equivalent to $\text{e-dim}\, X \leq L_2$ for any space X. Moreover, our definition of e-dim coincides with Chigogidze's definition [25] for countable L and with the original Dranishnikov's definition [40] for compact spaces.

The definition of connected pairs required a similar revision.

Definition 7.1. *A pair of spaces $K \subset P$ is said to be L-connected if, for any closed subset $A \subset X$ of a space X with $\text{e-dim}\, X \leq L$, every map $g : A \to K$ has an extension $\overline{g} : X \to P$ provided A is normally placed in X with respect to (g, P). The latter means that, for any continuous function h on P, the function $h \circ g$ can be continuously extended over X.*

The notion of a normally placed set was introduced by Chigogidze [25]. Obviously, if X is a normal space, then every closed subspace A of X is normally placed with respect to any map $g : A \to K$. We say that a pair $K \subset P$ is L-connected with respect to a given class of spaces $\mathcal{C}$ if, in the above definition, X satisfies the additional requirement $X \in \mathcal{C}$.

Now, we can extend the definition of a quasi-finite complex to arbitrary complexes.

Definition 7.2. *A CW-complex L is quasi-finite if every finite subcomplex P of L is contained in a finite subcomplex $eP \subset L$ such that the pair $P \subset eP$ is L-connected in the sense of Definition 7.1.*

It can be shown that this definition coincides with Definition 5.4 for locally finite countable L.

Slightly modifying Definition 5.1, we say that a map $f : X \to Y$ is L-invertible for a given class of spaces $\mathcal{C}$ if, for any map $g : Z \to Y$, where Z is in $\mathcal{C}$ and $\text{e-dim}\, Z \le L$, there exists a map $h : Z \to X$ such that $g = f \circ h$. If the class $\mathcal{C}$ consists of all (Tychonoff) spaces, then we say that f is L-invertible.

The following theorem characterizes quasi-finite CW-complexes in terms of Stone–Čech compactification, L-invertible maps, and extension of connected pairs; it can be considered as a generalization of Theorem 5.1.

Theorem 7.1 (see [82, Theorem 2.1]). *The following conditions are equivalent for any CW-complex L and any infinite cardinal τ:*

1. *L is quasi-finite*;
2. $\text{e-dim}\, \beta X \le L$ *for any (Tychonoff) space X with* $\text{e-dim}\, X \le L$;
3. *there exists an L-invertible map $f : Y_\tau \to \mathbb{I}^\tau$ such that Y_τ is a compact space of weight $\le\tau$ and* $\text{e-dim}\, Y_\tau \le L$;
4. *for every L-connected pair $P \subset M$, where P is a compact space of weight $\le\tau$ and M an arbitrary space, there exists a compact space $eP \subset M$ containing P for which $w(eP) \le \tau$ and the pair $P \subset eP$ is L-connected.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\text{e-dim}\, X \le L$. Take a closed subset A of βX and consider $f : A \to L$. It is well known that every CW-complex is an absolute neighborhood extensor for the class of compact spaces; hence $L \in ANE(f, A, \beta X)$. Consequently, there exists a closed neighborhood B of A in βX and a map $g : B \to L$ extending f. The set $g(B)$, being compact, is contained in a finite subcomplex P of L. Since L is quasi-finite, there exists a finite subcomplex eP of L such that the pair $P \subset eP$ is L-connected. We can assume that B is a zero-set in βX, i.e., the preimage of zero under some continuous real-valued function defined on βX. Then $B \cap X$, being a nonempty zero-set in X, is normally placed in X with respect to (g, eP). Therefore, the map $g : B \cap X \to P$ extends to a map $h : X \to eP$, because $\text{e-dim}\, X \le L$ and the pair $P \subset eP$ is L-connected. Finally, let $\overline{h} : \beta X \to eP$ be the unique extension of h. Then $\overline{h}$ is an extension of f, and $\text{e-dim}\, \beta X \le L$.

(2) $\Rightarrow$ (3) Consider the family of all maps $\{h_\alpha : X_\alpha \to \mathbb{I}^\tau\}_{\alpha\in\Lambda}$ such that every X_α is a closed subset of $\mathbb{I}^\tau$ with $\text{e-dim}\, X_\alpha \le L$. Let X be the disjoint sum of all X_α, and let $h : X \to \mathbb{I}^\tau$ be the map coinciding with h_α on every X_α. Clearly, $\text{e-dim}\, X \le L$. Therefore, $\text{e-dim}\, \beta X \le L$. Consider the extension $\overline{h} : \beta X \to \mathbb{I}^\tau$. By the factorization theorem from [94], there exists a compact space Y_τ of weight $\le\tau$ and maps $r : \beta X \to Y_\tau$ and $f : Y_\tau \to \mathbb{I}^\tau$ such that $\text{e-dim}\, Y_\tau \le L$ and $f \circ r = \overline{h}$.

Let us show that f is L-invertible. Take a space Z with $\text{e-dim}\, Z \le L$ and a map $g : Z \to \mathbb{I}^\tau$. Considering βZ and the extension $\overline{g} : \beta Z \to \mathbb{I}^\tau$ of g, we can assume that Z is compact. We can also assume that the weight of Z is $\le\tau$ (otherwise, we again apply the factorization theorem from [94] to find a compact space T of weight $\le\tau$ and maps $g_1 : Z \to T$ and $g_2 : T \to \mathbb{I}^\tau$ for which $\text{e-dim}\, T \le L$ and $g_2 \circ g_1 = g$ and consider the space T and the map g_2 instead of Z and g, respectively). Thus, without loss of generality, we can assume that Z is a closed subspace of $\mathbb{I}^\tau$. According to the definition of X and h, there exists an index $\alpha \in \Lambda$ for which $Z = X_\alpha$ and $g = h_\alpha$. The restriction $r|Z : Z \to Y_\tau$ is a lifting of g; i.e., $f \circ (r|Z) = g$.

(3) $\Rightarrow$ (4) Suppose that P is a compact subset of the space M such that $w(P) \le \tau$ and the pair $P \subset M$ is L-connected. We embed P in $\mathbb{I}^\tau$ and consider an L-invertible map $f : Y_\tau \to \mathbb{I}^\tau$ such that Y_τ

is compact and $\operatorname{e-dim} Y_\tau \le L$. We set $\tilde{P} = f^{-1}(P)$ and $h = f|\tilde{P}$. Obviously, $\tilde{P}$ is normally placed in Y_τ with respect to (h, M). Consequently, h extends to a map $\overline{h} : Y_\tau \to M$. Let $eP = \overline{h}(Y_\tau)$. Clearly, $w(eP) \le \tau$; thus, it only remains to show that $P \subset eP$ is L-connected. Consider $g : A \to K$, where $A \subset X$ is a closed normally placed subset of X with respect to (g, eP) and $\operatorname{e-dim} X \le L$. This implies that A is normally placed in X with respect to $(g, \mathbb{I}^\tau)$. Since $\mathbb{I}^\tau$ is an absolute extensor, the map g has an extension $g_1 : X \to \mathbb{I}^\tau$. We lift g_1 to a map $g_2 : X \to Y_\tau$ such that $f \circ g_2 = g_1$ (recall that f is L-invertible) and set $\overline{g} = \overline{h} \circ g_2$. Clearly, $\overline{g}$ is a map from X to eP extending g.

(4) ⇒ (1) Take a finite subcomplex P of L. First, let us show that the pair $P \subset L$ is L-connected. Suppose that Z is a space with $\operatorname{e-dim} Z \le L$, $A \subset Z$ is closed, and $g : A \to P$ is a map such that A is normally placed in Z with respect to (g, L). Since P is C-embedded in L, it follows that A is normally placed in Z with respect to (g, P). This condition, together with the fact that P is an absolute neighborhood extensor for all separable metric spaces, implies $P \in ANE(g, A, Z)$. Indeed, consider an embedding of P in $\mathbb{R}^\omega$ and take a retraction $r : U \to P$, where U is a neighborhood of P in $\mathbb{R}^\omega$. Since A is normally placed in Z with respect to (g, P), it follows that g has an extension $h : Z \to \mathbb{R}^\omega$. The set $h^{-1}(U)$ is a cozero neighborhood of A in Z which contains the zero-set $h^{-1}(P)$, and $r \circ h : h^{-1}(U) \to P$ extends g. Hence $P \in ANE(g, A, Z)$, which implies $L \in ANE(g, A, Z)$. Since $\operatorname{e-dim} Z \le L$, it follows that g can be extended to a map $\overline{g} : Z \to L$. Thus, $P \subset L$ is an L-connected pair. Hence there exists a compact set $E \subset L$ containing P such that the pair $P \subset E$ is L-connected. Finally, we take a finite subcomplex eP of L which contains E and observe that the pair $P \subset eP$ is also L-connected. Hence L is quasi-finite. □

7.2. Quasi-finite complexes and universal spaces. An immediate consequence of the existence of $[L]$-invertible maps (Theorem 7.1(3)) and the universality of $\mathbb{I}^\tau$ is the existence of universal compact spaces in dimension $[L]$ of arbitrary weight for a quasi-finite L.

Theorem 7.2. *Let L be a quasi-finite complex, and let τ be an infinite cardinal. Then the class of all compact spaces of weight $\le\tau$ with extension dimension $\le L$ contains a universal object.*

Another consequence of Theorem 7.1(3) is the existence of a compact metrizable space universal for the class of all separable metrizable spaces with e-dim $\le L$ for every quasi-finite complex L. In particular, any space from this class has a metrizable compactification of extension dimension at most L. Indeed, let Y_ω be the space from Theorem 7.1(3). Then, for every X from the above class, there exists an embedding $i : X \to \mathbb{I}^\omega$ which can be lifted to a map $j : X \to Y_\omega$. The required compactification of X is the closure of $j(X)$ in Y_ω. The following observation provides a characterization of countable quasi-finite complexes in terms of compactifications.

Proposition 7.3 (see [82, Corollary 2.3]). *For any countable complex L, the following conditions are equivalent*:

(a) *L is quasi-finite*;
(b) *for every separable metrizable space X with* $\operatorname{e-dim} X \le L$ *and its metrizable compactification $c(X)$, there exists a metrizable compactification $c^*(X)$ such that* $\operatorname{e-dim} c^*(X) \le L$ *and $c^*(X) \ge c(X)$ (i.e., there exists a map from $c^*(X)$ onto $c(X)$ which is the identity on X).*

Proof. (*a*) ⇒ (*b*) Let $c(X)$ be a metric compactification of X. Consider a map $f : \beta X \to c(X)$ such that $f(x) = x$ for every $x \in X$. By Theorem 7.1(2), $\operatorname{e-dim} \beta X \le L$; hence f can be factored through a compact metrizable space Z with $\operatorname{e-dim} Z \le L$. Clearly, Z is a compactification of X which is $\ge c(X)$.

(*b*) ⇒ (*a*) According to [59, Corollary 3.4], there exists a compact metrizable space Y with $\operatorname{e-dim} Y \le L$ and a surjective map $f : Y \to \mathbb{I}^\omega$ such that, for any map $g : X \to \mathbb{I}^\omega$, where X is a separable metrizable space with $\operatorname{e-dim} X \le L$, there exists an embedding $i : X \to Y$ lifting g, i.e., such that $f \circ i = g$. Hence f is L-invertible with respect to separable metric spaces. By Theorem 7.1(3), it suffices to show that f is L-invertible. Consider $g : Z \to \mathbb{I}^\omega$, where $\operatorname{e-dim} Z \le L$. According to [25, Proposition 4.9], there exists a Polish space P with $\operatorname{e-dim} P \le L$ and maps $h : Z \to P$ and $q : P \to \mathbb{I}^\omega$

for which $g = q \circ h$. Let $\overline{q} : P \to Y$ be a lifting of q such that $f \circ \overline{q} = q$. Then $\overline{q} \circ h$ is the required lifting of g. □

It is well known that every metric space admits a uniformly 0-dimensional map to l_2. The following theorem establishes the existence of a universal L-dimensional metrizable space of an arbitrary weight for quasi-finite L.

Theorem 7.4 (see [82, Proposition 2.7]). *Let L be a quasi-finite CW-complex. Then, for every $\tau \geq \omega$, there exists a perfect L-invertible surjection $f_{(L,\tau)} : Y_{(L,\tau)} \to l_2(\tau)$ such that*

(a) *$Y_{(L,\tau)}$ is a completely metrizable space of weight τ with* $\operatorname{e-dim} Y_{(L,\tau)} \leq L$*;*
(b) *every (completely) metrizable space of weight $\leq\tau$ and extension dimension $\leq L$ can be embedded as a (closed) subspace in $Y_{(L,\tau)}$.*

Proof. By Theorem 7.1(3), there exists an L-invertible map $f : Y \to \mathbb{I}^\omega$, where Y is a compact metrizable space with $\operatorname{e-dim} Y \leq L$. Suppose that l_2 is embedded in $\mathbb{I}^\omega$. We set $Y_{(L,\omega)} = f^{-1}(l_2)$ and $f_{(L,\omega)} = f|Y_{(L,\omega)}$. Then $\operatorname{e-dim} Y_{(L,\omega)} \leq L$ and f is L-invertible; hence so is $f_{(L,\omega)}$.

If $\tau > \omega$, we take a metric d_1 on $l_2(\tau)$ and a uniformly 0-dimensional map $g : l_2(\tau) \to l_2$ with respect to d_1. Let $Y_{(L,\tau)}$ denote the fiber product of $l_2(\tau)$ and $Y_{(L,\omega)}$ with respect to the maps g and $f_{(L,\omega)}$. Consider the projections $f_{(L,\tau)} : Y_{(L,\tau)} \to l_2(\tau)$ and $h : Y_{(L,\tau)} \to Y_{(L,\omega)}$. Since $f_{(L,\omega)}$ is a perfect L-invertible surjection, it follows that so is $f_{(L,\tau)}$. If d_2 is any metric on $Y_{(L,\omega)}$, then h is uniformly 0-dimensional with respect to the metric $d = \sqrt{d_1^2 + d_2^2}$ on $Y_{(L,\tau)}$ (see [2, Lemma 4, p. 379]). Thus, $Y_{(L,\tau)}$ admits a uniformly 0-dimensional map to the space $Y_{(L,\omega)}$ of extension dimension $\leq L$. Therefore, by [92, Theorem 1.2], $\operatorname{e-dim} Y_{(L,\tau)} \leq L$. Note that $Y_{(L,\tau)}$ is completely metrizable as a perfect preimage of the completely metrizable space $l_2(\tau)$.

Assertion (b) follows directly from the L-invertibility of $f_{(L,\tau)}$ and the universality of $l_2(\tau)$. □

Corollary 7.5 (see [82, Corollary 2.9]). *Suppose that L is a quasi-finite complex and X is a metrizable space. Then* $\operatorname{e-dim} X \leq L$ *if and only if X admits a uniformly 0-dimensional map to a separable metrizable space of extension dimension $\leq L$.*

Proof. Sufficiency follows from the above-mentioned result of Levin [92, Theorem 1.2]. Suppose that X is a metrizable space of weight τ with $\operatorname{e-dim} X \leq L$. By Theorem 7.4, X can be embedded in the space $Y_{(L,\tau)}$. It follows from the construction of $Y_{(L,\tau)}$ that the map $h : Y_{(L,\tau)} \to Y_{(L,\omega)}$ is uniformly 0-dimensional; hence so is the restriction $h|X$. This completes the proof of the corollary. □

Olszewski proved a completion theorem for L-dimensional metric spaces, where L is an arbitrary countable CW-complex [105]. It follows from Theorem 7.4 that such a theorem is also valid for quasi-finite (not necessarily countable) complexes L.

Theorem 7.6 (see [82, Corollary 2.8]). *Let L be a quasi-finite complex. Then every metrizable space X with* $\operatorname{e-dim} X \leq L$ *has a completion of extension dimension $\leq L$.*

Levin, Rubin, and Shapiro proved a general factorization theorem for L-dimensional compact spaces, where L is an arbitrary complex [94]. We give a factorization theorem for L-dimensional metrizable spaces for quasi-finite L (in [92, Theorem 1.5], a similar result for countable L was obtained).

Theorem 7.7 (see [82, Proposition 2.10]). *Suppose that L is a quasi-finite complex, Y is a metrizable space, and $f : X \to Y$ is a map. If* $\operatorname{e-dim} X \leq L$*, then f can be factored through a metrizable space Z such that* $\operatorname{e-dim} Z \leq L$ *and $w(Z) \leq w(Y)$.*

Proof. First, let us show how to reduce this theorem to the case of separable Y. This reduction is well known (see, e.g., [2, p. 388]), but we present it here for the reader's convenience. Suppose that the required assertion is true for separable metrizable Y. Taking a uniformly 0-dimensional map $g : Y \to l_2$ and applying the "separable factorization theorem" to the map $g \circ f : X \to l_2$, we obtain a separable metrizable space M and maps $q : X \to M$ and $h : M \to l_2$ such that $\text{e-dim}\, M \le L$ and $h \circ q = g \circ f$. Let $p_M : Z \to M$ and $p_Y : Z \to Y$ be the pullbacks of g and h, respectively. Clearly, Z is a metrizable space of weight $w(Z) \le w(Y)$. Since g is uniformly 0-dimensional, so is p_M. According to [92, Theorem 1.2], we have $\text{e-dim}\, Z \le L$.

Now let us prove the separable case. Suppose that $\tilde{Y}$ is a metrizable compactification of Y and $\tilde{f} : \beta X \to \tilde{Y}$ is the Stone–Čech extension of f. Since L is quasi-finite, it follows that $\text{e-dim}\, \beta X \le L$. Therefore, we can apply the factorization theorem of Levin, Rubin, and Shapiro from [94] and obtain a compact metrizable space $\tilde{Z}$ and maps $\tilde{f}_1 : \beta X \to \tilde{Z}$ and $\tilde{f}_2 : \tilde{Z} \to \tilde{Y}$ such that $\tilde{f}_2 \circ \tilde{f}_1 = \tilde{f}$ and $\text{e-dim}\, \tilde{Z} \le L$. The space $Z = \tilde{f}_1(X)$ and the maps $f_1 = \tilde{f}_1|X$ and $f_2 = \tilde{f}_2|Z$ form the required factorization. □

We say that a map $f : X \to Y$ is L-soft if, for any space Z with $\text{e-dim}\, Z \le L$, any closed set $A \subset Z$, and any two maps $h : Z \to Y$ and $g : A \to X$, where A is normally placed in Z with respect to (g, X) and $f \circ g = h|A$, there exists an extension $\overline{g} : Z \to X$ of g such that $f \circ \overline{g} = h$ (cf. Definition 5.3). If, in addition, Z belongs to a given class of spaces $\mathcal{C}$, then we say that f is L-soft with respect to the class $\mathcal{C}$. It was established in [30] that, for any countable complex L and any metric space Y, there exists an L-soft map $f : X \to Y$ such that X is a metric space of extension dimension $\le L$ and $w(X) = w(Y)$. Quasi-finite complexes also have this property.

Theorem 7.8 (see [82, Proposition 2.11]). *Let L be a quasi-finite CW-complex. Then, for every $\tau \ge \omega$, there exists an L-soft map $p_{(L,\tau)} : X_{(L,\tau)} \to l_2(\tau)$ such that*

(a) *$X_{(L,\tau)}$ is a completely metrizable space of weight τ with $\text{e-dim}\, X_{(L,\tau)} \le L$;*
(b) *$X_{(L,\tau)}$ is an absolute extensor for all metrizable spaces of extension dimension $\le L$;*
(c) *$p_{(L,\tau)}$ is a strongly (L,τ)-universal map, i.e., for any open cover ω of $X_{(L,\tau)}$, any (complete) metrizable space Z of weight $\le \tau$ with $\text{e-dim}\, Z \le L$, and any map $g : Z \to X_{(L,\tau)}$, there exists a (closed) embedding $h : Z \to X_{(L,\tau)}$ ω-close to g such that $p_{(L,\tau)} \circ g = p_{(L,\tau)} \circ h$.*

8. Homotopy

Methods of algebraic topology are extensively used in general and geometric topology, in particular, in dimension theory. For example, we use algebraic invariants to define the cohomological dimension of a topological space and study its properties. Moreover, classical homotopy theory is indispensable for studying $ANE(n)$-spaces. On the other hand, the progress made in dimension theory, namely, the appearance of extension dimension, has given rise to a new concept in homotopy theory. In [27], Chigogidze initiated $[L]$-homotopy theory, intended to be applied to $[L]$-dimensional spaces. The notion of $[L]$-homotopy is defined for any finite complex L. The [point]-homotopies coincide with the usual homotopies, and the $[S^n]$-homotopies coincide with the n-homotopies suggested by Fox [67] and studied by Whitehead [141]. However, $[L]$-homotopies differ substantially from classical homotopies in the general case. Below, we define them and mention some of their properties. Following Chigogidze [27], we define $[L]$-homotopy groups and compute them in some (very special) cases. We also cite the main result of [29] that $[L]$-homotopies are indeed homotopies in the sense of Quillen [114]. This result shows that $[L]$-homotopy theory is an adequate homotopy theory from the categorical point of view.

Throughout this section, L denotes a finite CW-complex.

8.1. $[L]$-Homotopy.

We begin by defining $[L]$–homotopic maps. The following definition differs from the classical definition of homotopy in that it requires the existence of a "path" between two maps modulo $[L]$-dimensional spaces.

Definition 8.1 (see [27, 29]). *Suppose that A is a subspace of a Polish space X and $f_0, f_1 : X \to Y$ are two maps such that $f_0(x) = f_1(x)$ for each $x \in A$. We say that f_0 and f_1 are $[L]$–homotopic relative to A (and write $f_0 \overset{[L]}{\simeq} f_1 \,\text{rel}\, A$) if, for any map $h : Z \to X \times [0,1]$, where Z is a Polish space* *of extension dimension* e-dim $Z \le [L]$, *there exists a map* $H : Z \to Y$ *such that*

$$H(z) = \begin{cases} h(f_0(z)) & \text{if } z \in h^{-1}(X \times \{0\} \cup A \times [0,1]), \\ h(f_1(z)) & \text{if } z \in h^{-1}(X \times \{1\}). \end{cases}$$

If $A = \varnothing$, *then we say that* f_0 *and* f_1 *are* $[L]$-*homotopic* (*and write* $f_0 \overset{[L]}{\simeq} f_1$).

Obviously, any map to an $AE([L])$-space is $[L]$-homotopic to a constant map. The following statement establishes a relationship between homotopy and the order on complexes.

Proposition 8.1 (see [27]). *Let* L *and* K *be finite connected complexes. If* $[K] \le [L]$, *then any* $[L]$-*homotopic maps are also* $[K]$-*homotopic. Moreover, if* $[K] < [L]$, *then the identity map* id_K *is not* $[L]$-*homotopic to a constant map.*

The usual homotopies and n-homotopies coincide with the [singleton]-homotopies and $[S^n]$-homotopies, respectively. However, even for relatively simple complexes different from spheres, a new homotopy theory arises. For instance, let $n \ge 1$ be an integer, and L be a connected complex such that $[S^n] < [L] < [S^{n+1}]$ (for example, $L = \mathbb{R}P^n \vee S^{n+1}$). Since $L \in AE([L])$, we have (see [27, p. 328] and Proposition 8.1) $\mathrm{id}_L \overset{[L]}{\simeq} \text{const}$ but $\mathrm{id}_L \overset{[S^{n+1}]}{\not\simeq} \text{const}$, and $\mathrm{id}_{S^n} \overset{[S^n]}{\simeq} \text{const}$ but $\mathrm{id}_{S^n} \overset{[L]}{\not\simeq} \text{const}$.

One might expect that $[L]$-homotopies behave nicely with respect to $ANE([L])$-spaces. This is indeed the case. First of all, they have the following extension property.

Proposition 8.2. *[27, Proposition 2.28] Let* X *be a Polish* $ANE([L])$-*space. Suppose that* A *is closed in a separable metrizable space* B *with* e-dim$(B) \le [L]$. *If maps* $f, g : A \to X$ *are* $[L]$-*homotopic and* f *has an extension* $F : B \to X$, *then* g *has an extension* $G : B \to X$, *and it can be assumed that* $F \overset{[L]}{\simeq} G$.

In addition, any two sufficiently close maps to an $ANE([L])$ space are $[L]$-homotopic.

Proposition 8.3 (see [27, Proposition 2.26]). *Let* X *be a Polish* $ANE([L])$-*space. Then there exists an open cover* ω *of* X *such that any two* ω-*close maps from any separable metrizable space to* X *are* $[L]$-*homotopic.*

Next, there is a natural relation of $[L]$-homotopy equivalence.

Definition 8.2 (see [27]). *A map* $f : X \to Y$ *is said to be an* $[L]$-*homotopy equivalence if there exists a map* $g : Y \to X$ *such that the compositions* $g \circ f$ *and* $f \circ g$ *are* $[L]$-*homotopic to* id_X *and* id_Y, *respectively.*

The following proposition provides a source of examples of $[L]$-homotopy equivalences and highlights the relationship between $[L]$-homotopies and approximately $[L]$-soft maps.

Proposition 8.4 (see [27, Proposition 2.29]). *Suppose that* $f : X \to Y$ *is a map between* $ANE([L])$-*compacta and* e-dim $Y \le [L]$. *Then* f *is approximately* $[L]$-*soft if and only if* f *is an* $[L]$-*homotopy equivalence.*

It is well known that a metric space Y is $ANE(n)$ if and only if Y is locally $(n-1)$-connected (see [50, 87]). The $ANE([L])$-spaces have a similar local characterization. Namely, employing $[L]$-homotopies, we can define the notion of $[L]$-contractibility by analogy with the classical case.

Definition 8.3. *Let* L *be a finite CW-complex. A space* X *is said to be locally* $[L]$-*contractible* (*notation:* $X \in LC^{[L]}$) *if, for any point* $x \in X$ *and any neighborhood* U *of* x, *there exists a smaller neighborhood* V *of* x *such that the inclusion of* V *into* U *is* $[L]$-*homotopic in* U *to a constant map.*

Theorem 8.5 (see [16, Theorem 4.1]). *Let* L *be a finite CW-complex such that* $[L] \le [S^n]$ *for some* n. *A Polish space* Y *is an* $ANE([L])$ *if and only if it is locally* $[L]$-*contractible.*

Note that the condition $[L] \le [S^n]$ cannot be removed because of the celebrated Borsuk's example of a contractible locally contractible space which is not an absolute extensor [14, p. 124]. Note also that the local $[S^n]$-contractibility of a Polish space is equivalent to the local $(n-1)$-connectedness of this space. To see this, it suffices to remember that the $ANE([S^n])$-spaces coincide with the $ANE(n)$-spaces and combine Theorem 8.5 with the classical characterization of $ANE(n)$-spaces.

8.2. $[L]$-Complexes. The most natural and best studied class of spaces in classical homotopy theory is that of CW-complexes. The class of $[L]$-complexes intended to play a similar role in $[L]$-homotopy theory was introduced in [29]. First, we need the following resolution theorem. It shows how to obtain $ANE([L])$-spaces as preimages of usual CW-complexes under special $[L]$-invertible and approximately $[L]$-soft maps.

Theorem 8.6 (see [29, Proposition 3.6]; cf. [27, Proposition 2.23])**.** *Let L be a finite complex, and let X be a locally finite complex. Then there exists a locally compact metrizable space $\mu_X^{[L]}$ and an $[L]$-invertible approximately $[L]$-soft proper map $f_X^{[L]} : \mu_X^{[L]} \to X$ satisfying the following conditions*:

(i) $\mu_X^{[L]} \in ANE([L])$;

(ii) $\text{e-dim}(\mu_X^{[L]}) = [L]$;

(iii) *for any map $f : B \to \mu_X^{[L]}$, where B is a compact space with* $\text{e-dim}(B) \le [L]$, *and any open cover $\omega \in \operatorname{cov}(\mu_X^{[L]})$, there exists an embedding $g : B \to \mu_X^{[L]}$ such that it is ω-close to f and $f_X^{[L]} \circ g = f_X^{[L]} \circ f$*;

(iv) *if, in addition, X is a locally finite polyhedron and τ is its triangulation, then it can be assumed that, for any subpolyhedron $X_0 \subset X$ of X with respect to τ, the inverse image $\left(f_X^{[L]}\right)^{-1}(X_0)$ is a locally compact $ANE([L])$ and the restriction of $f_X^{[L]}$ to $\left(f_X^{[L]}\right)^{-1}(X_0)$ is approximately $[L]$-soft.*

This theorem is proved by using a modified construction of Dranishnikov and Repovš [45] (see also Sec. 5.2).

Before defining $[L]$-homotopy groups, we introduce the simplest $[L]$-complexes, namely, the $[L]$-spheres and $[L]$-disks, which are analogs of the usual spheres and disks. We shall see in what follows that they can also be regarded as "building blocks" for $[L]$-complexes. For each $n \ge 0$, consider $[L]$-dimensional compact $AE([L])$-spaces $D^n_{[L]}$ which admit ($[L]$-invertible) approximately $[L]$-soft maps onto the n-disk D^n (see Theorem 8.6). Theorem 8.6(iv) allows us to treat the pairs $(S^n_{[L]}, D^{n+1}_{[L]})$ as approximately $[L]$-soft preimages of the standard pairs (S^n, D^{n+1}) consisting of the $(n+1)$-disks and their boundaries $\partial D^{n+1} = S^n$ for $n \ge 0$. We call $D^n_{[L]}$ an n-$[L]$-*disk*. The n-$[L]$-*spheres* are defined to be compact spaces admitting approximately $[L]$-soft maps onto the n-sphere S^n; we denote them by $S^n_{[L]}$.

Importantly, for any given n, there may exist many n-$[L]$-disks and n-$[L]$-spheres (see the remark after Theorem 2.7 in [27, p. 327]). Nevertheless, being $AE([L])$-compacta, all of the n-$[L]$-disks are $[L]$-homotopy equivalent to the singleton. The following proposition shows that all of the n-$[L]$-spheres are also $[L]$-homotopy equivalent.

Proposition 8.7 (see [29, Proposition 3.5])**.** *Suppose that $p : X \to Y$ is an approximately $[L]$-soft map between Polish spaces and $f_1, f_2 : Z \to X$ are maps defined on a Polish space Z and such that*

(a) *$f_1(z) = f_2(z)$ for some point $z \in Z$*;

(b) *$p \circ f_1 \overset{[L]}{\simeq} p \circ f_2 \operatorname{rel} z$.*

Then $f_1 \overset{[L]}{\simeq} f_2 \operatorname{rel} z$.

At this point, we may encounter another difficulty, which is caused by the nonuniqueness of an $[L]$-homotopy equivalence between two $[L]$-spheres. Nevertheless, we can distinguish a certain class of $[L]$-homotopy equivalences between $[L]$-spheres, namely, the *$[L]$-homotopy equivalences of canonical type* [29, p. 138], which allows us to give a correct definition of $[L]$-homotopy groups.

Next, we define *$[L]$-polyhedra* and *$[L]$-complexes* (see [29, Sec. 3.1]).

Definition 8.4. *We say that a space X of extension dimension* $\text{e-dim}\, X \leq [L]$ *is a (finite) $[L]$-polyhedron if it admits a proper approximately $[L]$-soft map $f : X \to Y$ onto a locally finite polyhedron Y satisfying condition (iv) in Theorem 8.6 for some (finite) triangulation τ of Y. The $[L]$-complexes are defined to be spaces $[L]$-homotopy equivalent to $[L]$-polyhedra. Recall that a map is said to be proper if the preimage of any compact subspace is compact.*

Any $[L]$-polyhedron is built in a standard way, namely, by attaching "cells" (i.e., $[L]$-disks) along their "boundaries" (i.e., $[L]$-spheres) via inclusion maps. We can also define analogs of n-skeletons and show that the topology of $[L]$-polyhedra is similar to the standard CW topology [29]. The classical polyhedra are obtained by setting $L = \{\text{point}\}$ (or $L = S^n$ for n-dimensional polyhedra). However, the class of $[S^n]$-polyhedra is obviously larger than that of n-dimensional polyhedra.

8.3. $[L]$-Homotopy groups. Consider a pointed space (X, x_0). Take $n \geq 1$ and a point $s \in S^n$. Let $S^n_{[L]}$ be an n-$[L]$-sphere, and let $f : S^n_{[L]} \to S^n$ be the corresponding approximately $[L]$-soft map. Take a point $s_{[L]} \in f^{-1}(s)$. By $\pi_n^{[L]}(X, x_0) = \Big[(S^n_{[L]}, s_{[L]}), (X, x_0)\Big]_{[L]}$ we denote the set of relative $[L]$-homotopy classes of maps of pointed spaces. Let us describe a group structure on this set (see [29]).

First, we define a group operation. Take two elements α and β in $\pi_n^{[L]}(X, x_0)$ with representatives $a, b : (S^n_{[L]}, s_{[L]}) \to (X, x_0)$. Let S^n_+ and S^n_- denote the upper and lower hemispheres, respectively, and take an equator E of S^n containing s. Let $h : S^n \to S^n \vee S^n$ be the homotopy comultiplication defining the standard H-cogroup structure of the sphere S^n (see, e.g., [125, Definition 2.16]), and let $f_- : S^n_{[L]} \to S^n$ and $f_+ : S^n_{[L]} \to S^n$ be two copies of the map f.

Given maps a and b, we need to construct a new map $a * b$ from a pointed n-$[L]$-sphere to (X, x_0); its $[L]$-homotopy class will represent the product of α and β. Since S^n is an ANE-compactum, there exists an open cover ω such that any two ω-close maps from any compact space to S^n are homotopic (as maps to S^n). Since f_+ is approximately $[L]$-soft, there exists a map $\widetilde{h}_+ : f^{-1}(S^n_+) \to S^n_{[L]}$ such that $\widetilde{h}_+(f^{-1}(E)) = s_{[L]}$ and the composition $f_+ \circ \widetilde{h}_+$ is ω-close to the composition $h \circ f|_{f^{-1}(S^n_+)}$. Similarly, there exists a map $\widetilde{h}_- : f^{-1}(S^n_-) \to S^n_{[L]}$ such that $\widetilde{h}_-(f^{-1}(E)) = s_{[L]}$ and the composition $f_- \circ \widetilde{h}_-$ is ω-close to the composition $h \circ f|_{f^{-1}(S^n_-)}$.

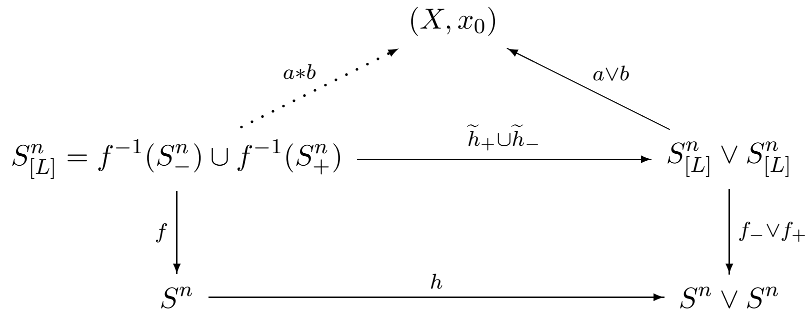
Now we can define the composition

$$a * b = ((a \vee b) \circ (\widetilde{h}_+ \cup \widetilde{h}_+)) : (S^n_{[L]}, f^{-1}(E)) \to (X, x_0)$$

by

$$(a * b)(x) = \begin{cases} a(\widetilde{h}_+(x)) & \text{if } x \in f^{-1}(S^n_+), \\ b(\widetilde{h}_-(x)) & \text{if } x \in f^{-1}(S^n_-). \end{cases}$$

The following diagram (which is commutative up to $[L]$-homotopy) illustrates the situation:



The choice of the cover ω ensures that the relative $[L]$-homotopy class $[a * b]_{[L]}$ of this composition does not depend on the choice of the representatives a and b (and of the maps $\widetilde{h}_+$ and $\widetilde{h}_-$). This allows us to define the product of α and β by setting $\alpha * \beta = [a * b]_{[L]}$.

The $[L]$-homotopy class of the map sending $S^n_{[L]}$ to the point x_0 is the identity element with respect to this operation. To complete the construction, we need to define inverse elements. Let $a : (S^n_{[L]}, s_{[L]}) \to (X, x_0)$ be a representative of $\alpha \in \pi_n^{[L]}(X, x_0)$, and let $g : S^n \to S^n$ be the map defined by $g(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_{n+1})$ (the points of the equator E are fixed under this map). Since f is approximately $[L]$-soft, there exists a map $\widetilde{g} : S^n_{[L]} \to S^n_{[L]}$ such that $\widetilde{g}(s_{[L]}) = s_{[L]}$ and the composition $f \circ \widetilde{g}$ is ω-close to the composition $g \circ f$. We set $\alpha^{-1} = [a \circ \widetilde{g}]_{[L]}$. It is routine to show that α^{-1} has the desired property and does not depend on the choice of the representative a.

As mentioned above, the n-$[L]$-sphere is defined not uniquely. Thus, we must show that the groups $\pi_n^{[L]}(X, x_0)$ are well-defined. It was mentioned in Sec. 8.2 that different n-$[L]$-spheres are $[L]$-homotopy equivalent, and the $[L]$-homotopy equivalence between them is of canonical type. Importantly, any two $[L]$-homotopy equivalences of canonical type are $[L]$-homotopic. Using them, we can show that the group $\pi_n^{[L]}(X, x_0)$ does not depend on the choice of the $[L]$-sphere.

Finally, let $f : (X, x_0) \to (Y, y_0)$ be a map of pointed spaces, and let $[f]_{[L]} \in [(X, x_0), (Y, y_0)]_{[L]}$ be its relative $[L]$-homotopy class. Then we can define a natural homomorphism $[f]_{[L]} : \pi_n^{[L]}(X, x_0) \to \pi_n^{[L]}(Y, y_0)$ in a standard way.

8.4. Computations. Obviously, the [singleton]-homotopy groups of a space coincide with the usual homotopy groups. On the other hand, it can be shown that the system of $[S^n]$-homotopy groups of a space X has the form

$$\pi_1(X),\ \pi_2(X),\ \ldots,\ \pi_{n-1}(X),\ 0,\ 0,\ \ldots,\ 0,\ \ldots.$$

This suggests the conjecture that, generally, the $[L]$-homotopy groups may differ substantially from the usual homotopy groups. Thus, the following problem arises.

Problem 8.1. *Describe the $[L]$-homotopy groups of a space X in terms of the classical algebraic invariants of X and L (in particular, in terms of their homotopy and (co)homology groups).*

In the rest of this section, we make a first step toward solving this problem. Namely, we show that the n-$[L]$-homotopy group of S^n is isomorphic to $\mathbb{Z}$ for certain class of complexes L [80]. First, we obtain some information about the cohomology groups of complexes L satisfying the condition $[L] \leq [S^m]$.

Proposition 8.8 (see [80, Proposition 3.1]). *Suppose that L is a finite complex such that $[L] \leq [S^{n+1}]$ and n is the minimal integer with this property. Then, for any $q \leq n$, the groups $H_q(L)$ and $H^q(L)$ are torsion.*

Proof. First, note that both groups $H_q(L)$ and $H^q(L)$ are finitely generated. Suppose that there exists a $q \leq n$ such that $H_q(L) \approx \mathbb{Z} \oplus G$. To obtain a contradiction, we shall show that $[L] \leq [S^q]$. Since L is finite, it suffices to compare L and S^q by using compact metrizable spaces. Consider a compact metrizable space X for which $L \in AE(X)$. Note that $\dim X$ is finite, because $[L] \leq [S^{n+1}]$. We set $H = H_q(L)$. By Theorem 1.1, we have $\dim_H X \leq q$. Hence $H^{q+1}(X, A; H) = 0$ for any closed subspace A of X. Applying the universal coefficient formula to the group H^{q+1} and using the assumption $H^q(L) \approx \mathbb{Z} \oplus G$, we see that $H^{q+1}(X, A) = 0$ for all closed $A \subset X$. Hence $\dim_{\mathbb{Z}} X \leq q$ and, therefore, $\dim X \leq q$, because X is finite-dimensional. This implies $S^q \in AE(X)$.

Using the universal coefficient formula, which establishes a relation between homology and cohomology, and taking into account the fact that the homology and cohomology groups of L are finitely generated, we obtain the required assertion for $H^q(L)$. □

Proposition 8.8, the universal coefficient formula, and the structure theorem for finitely generated Abelian groups imply the following assertion.

Proposition 8.9 (see [80, Proposition 3.3]). *Suppose that L is a finite complex such that $[L] \leq [S^{n+1}]$ and n is the minimal integer with this property. Then there exists an arbitrarily large integer p such that $H^q(L; \mathbb{Z}_p)$ is trivial for any $q \leq n$.*

In what follows, we consider complexes L satisfying the inequalities $[S^n] < [L] \leq [S^{n+1}]$ for some integer n. In this case, $\text{e-dim}\, S^n = [S^n] < [L]$ and, therefore, we can take S^n for the n-$[L]$-sphere $S^n_{[L]}$. It follows immediately that, for any space X, the group $\pi_n^{[L]}(X)$ is the quotient of the classical homotopy group $\pi_n(X)$ by the $[L]$-homotopy relation. Thus, for the group $\pi_n^{[L]}(S^n)$, there are three possibilities: $\pi_n^{[L]}(S^n) \approx \mathbb{Z}$, $\pi_n^{[L]}(S^n) \approx \mathbb{Z}_m$, and $\pi_n^{[L]}(S^n) \approx 0$. We characterize the (hypothetical) equality $\pi_n^{[L]}(S^n) \approx \mathbb{Z}_m$ in terms of extensions of maps.

Proposition 8.10 (see [80, Proposition 4.3]). *Let L be a finite complex satisfying $[S^n] < [L] \leq [S^{n+1}]$. If $\pi_n^{[L]}(S^n) \cong \mathbb{Z}_m$, then, for any Polish space X such that $\text{e-dim}\, X \leq [L]$, any closed subspace A of X, and any map $f : A \to S^n$, there exists an extension $\overline{h} : X \to S^n$ of the composition $h = z_m f$, where $z_m : S^n \to S^n$ is a map of degree m.*

The proof of the following statement is standard.

Proposition 8.11 (see [80, Proposition 5.3]). *Suppose that X is a compact metrizable space, A is a closed subspace of X, $f : A \to S^n$ is a map, k is an integer, and ζ is a generator of the group $H^n(S^n; \mathbb{Z}_k) \approx \mathbb{Z}_k$. If f can be extended to a map from the entire space X to S^n, then $\delta^*_{X,A}(f^*(\zeta)) = 0$ in the group $H^{n+1}(X, A; \mathbb{Z}_k)$, where $\delta^*_{X,A} : H^n(A; \mathbb{Z}_k) \to H^{n+1}(X, A; \mathbb{Z}_k)$ is the connecting homomorphism.*

Now we are ready to show that $\pi_n^{[L]}(S^n) \approx \mathbb{Z}$.

Theorem 8.12 (see [80, Theorem 5.4]). *If L is a finite complex such that $[S^n] < [L] \leq [S^{n+1}]$ for some n, then $\pi_n^{[L]}(S^n) \approx \mathbb{Z}$.*

Proof. Suppose that, on the contrary, $\pi_n^{[L]}(S^n) \approx \mathbb{Z}_m$ or $\pi_n^{[L]}(S^n) \approx 0$. We consider only the former case (the latter is treated similarly). Using the technique described in Sec. 5.2 (which follows the ideas of [45]), we obtain an L-resolvable inverse sequence $\mathcal{S} = \{X_i, p_i^{i+1}\}$ consisting of compact polyhedra, in which $X_0 = D^{n+1}$ and the preimages of all bonding maps are either homeomorphic to L or trivial. We set $X = \lim \mathcal{S}$.

Note that $\operatorname{e-dim} X \le [L]$, because $\mathcal{S}$ is L-resolvable. Let $p_0 : X \to X_0 = D^{n+1}$ denote the limit projection. Take an integer $p \ge m+1$ as in Proposition 8.9. The Vietoris–Begle theorem and the choice of p imply that, for every index i and any $X'_i \subset X_i$, the homomorphism $(p_i^{i+1})^* : H^k(X'_i;\mathbb{Z}_p) \to H^k((p_i^{i+1})^{-1}X'_i;\mathbb{Z}_p)$ is an isomorphism for $k \le n$ and a monomorphism for $k = n+1$. Therefore, for each $D' \subset X_0 = D^{n+1}$, the homomorphism $p_0^* : H^k(D';\mathbb{Z}_p) \to H^k((p_0)^{-1}D';\mathbb{Z}_p)$ is an isomorphism for $k \le n$ and a monomorphism for $k = n+1$. In particular, $H^n(X;\mathbb{Z}_p) \approx 0$.

Let $A = (p_0)^{-1}S^n$. Take a generator $\zeta \in H^n(S^n;\mathbb{Z}_p) \approx \mathbb{Z}_p$. Since $p_0^* : H^n(S^n;\mathbb{Z}_p) \to H^n(A;\mathbb{Z}_p)$ is an isomorphism, it follows that $p_0^*(\zeta) \in H^n(A;\mathbb{Z}_p)$ is an element of order p. The exact sequence of the pair (X, A)

$$\ldots \to H^n(X;\mathbb{Z}_p) \approx 0 \xrightarrow{i_{X,A}} H^n(A;\mathbb{Z}_p) \xrightarrow{\delta^*_{X,A}} H^{n+1}(X,A;\mathbb{Z}_p) \to \ldots$$

implies that $\delta^*_{X,A}$ is a monomorphism, and hence

$$\delta^*_{X,A}(p_0^*(\zeta)) \in H^{n+1}(X,A;\mathbb{Z}_p)$$

is an element of order p.

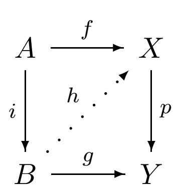
Now consider the composition $h = z_m \circ p_0$, where $z_m : S^n \to S^n$ is a map of degree m. By assumption and by Proposition 8.10, this map can be extended to a map from X to S^n. Combining this with Proposition 8.11, we see that $\delta^*_{X,A}(h^*(\zeta)) = 0$ in $H^{n+1}(X,A;\mathbb{Z}_p)$. On the other hand, $\delta^*_{X,A}(h^*(\zeta)) = m \cdot \delta^*_{X,A}(p_0^*(\zeta))$. This contradiction shows that $\pi_n^{[L]}(S^n) \cong \mathbb{Z}$. □

8.5. Model categories. An axiomatic approach to homotopy theory was suggested by Quillen [114], who introduced the notion of model category. Suppose that $\mathcal{C}$ is a category. To specify a model category structure on $\mathcal{C}$, we must distinguish between the following three classes of morphisms:

(i) weak equivalences;
(ii) fibrations;
(iii) cofibrations,

each of which is closed under compositions and contains all identity morphisms of all objects from $\mathcal{C}$. A morphism which is both a fibration (cofibration) and a weak equivalence is called an *acyclic* fibration (respectively, acyclic cofibration). The following axioms must be satisfied.

(1) Finite limits and colimits exist in $\mathcal{C}$.
(2) If f and g are maps in $\mathcal{C}$ such that gf is defined and two of the three maps f, g, and gf are weak equivalences, then so is the third.
(3) If f is a retract of g and g is a fibration, a cofibration, or a weak equivalence, then so is f.
(4) Given a commutative diagram



(solid arrows), a lift (dotted arrow) exists in either of the following two situations:
  (i) i is a cofibration and p is an acyclic fibration,
  (ii) i is an acyclic cofibration and p is a fibration.
(5) Any map f can be factored in two ways:
  (i) $f = pi$, where i is a cofibration and p is an acyclic fibration,
  (ii) $f = qj$, where j is an acyclic cofibration and q is a fibration.

In any abstract category $\mathcal{C}$ with a model-category structure, a homotopy theory can be developed (see [114]). This was done, e.g., in [18] and [74]. Moreover, if a homotopy theory satisfying the above axioms already exists, then powerful category methods can be applied to study it. See [51] for more information on model categories.

The classical homotopy structure satisfies Quillen's axioms. Moreover, it gives rise to two different model-category structures on the category of topological spaces, Quillen's [114] and Strøm's [124]. The former uses Serre fibrations as fibrations and weak homotopy equivalences as weak equivalences, while the latter employs Hurewicz fibrations and homotopy equivalences. A model category associated to n-homotopies with a structure based on weak n-homotopy equivalences and n-fibrations was constructed by Elvira-Donazar and Hernandez-Paricio in [62].

Now let us construct a closed model-category structure associated to $[L]$-homotopy on the category of topological spaces. It is convenient to describe it in terms of lifting properties. We say that a map $i : A \to B$ has the *left lifting property* (LLP) with respect to a map $p : X \to Y$ and p has the *right lifting property* (RLP) with respect to i if, for every commutative square diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

(solid arrows), there exists a lift (dotted arrow) $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$. We describe the model-category structure by specifying the required morphisms.

Definition 8.5 (see [29, Definition 4.2])**.** *Let $f : X \to Y$ be a map of topological spaces. We say that f is*

(i) *a weak equivalence if it is a weak $[L]$-homotopy equivalence;*
(ii) *a fibration if it has an RLP with respect to inclusions of finite $[L]$-polyhedra $A \subset B$ such that both A and B are $AE([L])$-spaces;*
(iii) *a cofibration if it has an LLP with respect to acyclic fibrations.*

In the above definition, *weak $[L]$-homotopy equivalences* are maps that induce isomorphisms of $[L]$-homotopy groups [29, Definition 4.1], as in the classical case.

The following result shows that the morphisms from Definition 8.5 satisfy the axioms of model category. Thus, $[L]$-homotopy is indeed a homotopy in the sense of Quillen.

Theorem 8.13 (see [29, Theorem 4.7])**.** *Let L be a finite complex. Then the category of topological spaces with weak equivalences, fibrations, and cofibrations introduced in Definition* 8.5 *is a model category.*

We emphasize that each finite complex L determines a closed model category structure. Note that this structure differs from all classical structures [62, 114, 124] even for $L = S^n$ (see the concluding remarks in [29, p. 148]). Thus, since $[S^n]$-homotopies coincide with n-homotopies, we obtain a new model category structure describing n-homotopies. For one-point L, it is unclear whether the structure coincides with that described in [114].

The following generalization of classical Whitehead's theorems [139, 140] can be obtained as an application of Theorem 8.13.

Theorem 8.14 (see [29, Corollary 4.8])**.** *A map of $[L]$-complexes is an $[L]$-homotopy equivalence if and only if it induces isomorphisms of all $[L]$-homotopy groups.*

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A. Karasev
Department of Computer Science and Mathematics, Nipissing University, Canada
E-mail: alexandk@nipissingu.ca
V. Valov
Department of Computer Science and Mathematics, Nipissing University, Canada
E-mail: veskov@nipissingu.ca