

# Dimension-raising theorems for cohomological and extension dimensions

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## Abstract

We establish cohomological and extension dimension versions of the Hurewicz dimension-raising theorem.  
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## 1. Introduction

Consider a finite-to-one closed and surjective map  $f : X \rightarrow Y$ . The multiplicity function  $\mu : Y \rightarrow \mathbb{N}$  of the map  $f$  is defined by  $\mu(y) = \text{card}(f^{-1}(y))$ , and let  $\mu(f) = \{\mu(y) : y \in Y\}$ . W. Hurewicz [18] proved that  $\text{card}(\mu(f)) \geq k + 1$  provided  $\dim Y \geq \dim X + k$  with  $X$  and  $Y$  being separable metric spaces. Moreover, if  $\mu(f) = \{m(1) < m(2) < \dots < m(\ell)\}$ , where  $\ell \geq k + 1$ , then  $\dim Y_{m(s+1)}(f) \geq \dim Y - s$  for  $s = 0, 1, \dots, k$ . Here  $Y_m(f) = \{y : \mu(y) = \text{card}(f^{-1}(y)) \geq m\}$ . The last assertion is a generalization of the following result of H. Freudenthal [13]:  $\dim Y_{s+1}(f) \geq \dim Y - s$ ,  $s = 0, \dots, k$ , if  $\dim Y \geq \dim X + k$ . Hence,  $Y_{k+1}(f)$  is not empty which was established first in [17]. The theorem of W. Hurewicz was further generalized for metric spaces in [19,30], see also [14,29,31,25,11] for related results.

The present paper deals with cohomological and extension dimension versions of the Hurewicz theorem. Recall that the cohomological dimension  $\dim_G$  of the paracompact space  $X$  with coefficients in an Abelian group  $G$  is defined as follows:  $\dim_G X \leq n$  if and only if the Čech cohomology  $H^{n+1}(X, A, G) = 0$  for every closed set  $A$  in  $X$ . Equivalently,  $\dim_G X \leq n$  if and only if the homomorphism  $H^n(X, G) \rightarrow H^n(A, G)$  of the Čech cohomology groups induced by the inclusion  $A \subset X$  is an epimorphism for every closed  $A$  in  $X$ , see [20,3,7]. If  $Z$  is a subset of a space  $X$ , we define  $\text{rdim}_{X,G} Z = \max\{\dim_G F : F \subset Z \text{ is closed in } X\}$ .

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There exists also another way to define  $\dim_G$  for paracompact spaces. If  $L$  is a  $CW$ -complex, we say that a space  $X$  belongs to the class  $\alpha(K)$  if every map  $g : A \rightarrow K$ , where  $A \subset X$  is closed, can be extended to the whole space  $X$  provided  $g$  is extendable to some open neighborhood of  $A$  in  $X$ . Shvedov [27] has shown that if  $X$  is paracompact, then  $\dim_G X \leq n$  if and only if  $X \in \alpha(K(G, n))$ , where  $K(G, n)$  denotes the Eilenberg–MacLane complex of  $G$  in dimension  $n$  (for countable groups  $G$  this was established by Huber [16]).

Concerning the definition of extension dimension for paracompact spaces, we adopt Dydak's approach [8]. If  $X$  is a paracompact space and  $L$  a  $CW$ -complex, we write  $e\text{-dim } X \leq L$  if every map  $f : A \rightarrow L$ , where  $A \subset X$  is closed, extends over  $X$  up to homotopy. In such a case we say that  $L$  is an absolute extensor of  $X$  up to homotopy. Let us mention that, according to [8, Proposition 3.5], for every  $CW$ -complex  $L$  there exists a complete metrizable simplicial complex  $K(L)$  homotopy equivalent to  $L$  such that, for any paracompact space  $X$  we have  $K(L)$  is an absolute extensor of  $X$  if and only if  $L$  is an absolute extensor of  $X$  up to homotopy. In case  $X$  is compact or metrizable, the above definition of  $e\text{-dim}$  coincides with the original one introduced by Dranishnikov [4], i.e., every map  $f : A \rightarrow L$  with  $A \subset X$  being closed admits an extension over  $X$ . The notation  $e\text{-dim } X \leq e\text{-dim } Y$  means that  $e\text{-dim } X \leq L$  for every  $CW$ -complex  $L$  with  $e\text{-dim } Y \leq L$ .

The paper is organized as follows. A cohomological version of the Hurewicz theorem is established in Section 2. Section 3 is devoted to a cohomological version of the mentioned above Freudenthal's theorem. In the final Section 4 we deal with dimension-raising theorems for extension dimension. The first result in this section is an extension analogue of another dimension-raising theorem of Hurewicz stating that if  $f : X \rightarrow Y$  is a closed map with  $\text{card}(f^{-1}(y)) \leq n + 1$  for every  $y \in Y$ , then  $\dim Y \leq \dim X + n$ . This theorem, established first by Hurewicz [17] for separable metric spaces, has many generalizations. For the dimension  $\dim$  and normal spaces it was proved by Zarelua [32], Filippov [14] and Pasyukov [23]. Kuzminov [20, Theorem 14, 1), p. 24] was who first provided a cohomological version of this theorem for closed maps between finite-dimensional metrizable compact spaces and cohomological dimension  $\dim_G$  with respect to an Abelian group  $G$ . Kuzminov's proof was based on test spaces for  $\dim_G$ . With the same method he also established this theorem in the class of paracompact spaces with  $G$  being either a periodic group or a field [20, p. 39]. Zarelua [32] introduced a new technique for investigating closed finite-to-one maps of paracompact spaces. As a byproduct, he obtained another generalization of the Hurewicz theorem for paracompact spaces and  $\dim_G$ , where  $G$  is a commutative ring with a unity. Finally, for arbitrary Abelian groups  $G$ , paracompact spaces, and more general maps, this theorem was obtained by the first author [31, Corollary 4.1].

The introduction of extension dimension as a unification of the cohomological dimension  $\dim_G$  and the ordinary dimension  $\dim$  gave another way of possible generalizations of the Hurewicz results. Such a generalization was established by Dranishnikov and Uspenskij [6, Theorem 1.6] for metrizable compacta. Our Proposition 4.1 and Corollary 4.2 extend the Dranishnikov–Uspenskij theorem for more general spaces. Let us mention that, in case of finite-dimensional compacta (respectively metrizable spaces) and simply connected  $CW$ -complexes, Proposition 4.1 and Corollary 4.2 follow from the first author's result mentioned above [31, Corollary 4.1] combined with Dranishnikov's homological criterion for extensivity [4, Theorem 9] (respectively with Dydak's criterion [10, Theorems H and G]). We also provide an extension version of the Hurewicz theorem when  $\mu(f)$  has finitely many values.

Everywhere below, by a group we mean an Abelian group. Recall that a map  $f : X \rightarrow Y$  is perfect if  $f$  is a closed map having compact fibers  $f^{-1}(y)$ ,  $y \in Y$ . It is well known that this is equivalent to  $f$  being closed and  $f^{-1}(K)$  compact for all compact sets  $K \subset Y$ .

## 2. Hurewicz theorem for the cohomological dimension

Here, we shall prove a theorem for maps increasing the cohomological dimensions  $\dim_G$  with coefficients in a group  $G$ .

**Proposition 2.1.** *If  $f : X \rightarrow Y$  is a finite-to-one closed surjection between the metric space  $X$  and  $Y$  and  $\dim_G X + k \leq \dim_G Y < \infty$ , then  $\text{card}(\mu(f)) \geq k + 1$ . Moreover, if  $\mu(f) = \{m(1) < m(2) < \dots < m(\ell)\}$ , then  $\ell \geq k + 1$ , and  $\dim_G Y_{m(s+1)}(f) \geq \dim_G Y - s$  for  $s = 0, 1, \dots, k$ .*

*If  $\dim_G X < \dim_G Y = \infty$ , then  $\text{card}(\mu(f)) \geq \infty$  and  $\dim_G Y_m(f) = \infty$  for  $m = 2, 3, \dots$ .*

**Corollary 2.2.** *Let  $f : X \rightarrow Y$  be a finite-to-one closed surjection between metrizable spaces and  $\text{card}(\mu(f)) \leq k + 1$ . Then  $\dim_G Y \leq \dim_G X + k$ .*

**Remarks.** 1. Let  $f : X \rightarrow Y$  be a closed surjection between the metrizable spaces  $X$  and  $Y$ . Then the set  $Y_m(f) = \{y: \mu(y) = \text{card}(f^{-1}(y)) \geq m\}$  is an  $F_\sigma$  set in  $Y$ , see [11, Lemma 4.3.5, p. 243]. Therefore,  $\dim_G Y_m(f) = \text{rdim}_{Y,G} Y_m(f)$ .

2. Proposition 2.1 and Corollary 2.2 also hold in the more general situation when  $X$  is paracompact and sub-metrizable (recall that a space is sub-metrizable if it admits a continuous metric, or equivalently, it admits a bijective continuous map onto a metrizable space). For the class of sub-metrizable spaces and the ordinary dimension  $\dim$ , Proposition 2.1 and Corollary 2.2 were established in [30] (in this case the set  $Y_m(f)$  is also  $F_\sigma$  in  $Y$ ). Bredon [2, Theorem 8.15, p. 243] also provided a version of Corollary 2.2 for separable metric spaces  $X, Y$  and a principal ideal domain  $G$ .

The proof Proposition 2.1 is based on several lemmas. Before starting the proof, let us provide some notions and results from cohomology and sheaf theory, see [15,2,28], and some constructions and theorems of A.V. Zarelua [32–34]. Recall that the cohomological dimension  $\dim_{\mathcal{L}} Z$  of a space  $Z$  with coefficients in a sheaf of Abelian groups  $\mathcal{L}$  is defined by  $\dim_{\mathcal{L}} Z = \min\{n: H^{n+1}(Z, \mathcal{L}_U) = 0 \text{ for every open set } U \subset Z\}$ . Moreover,  $\dim_{\mathcal{L}} Z = \dim_G Z$  if  $\mathcal{L}$  is the constant sheaf  $Z \times G$ .

First, we recall the notion of a local system of sheaves of groups on a given space  $Y$ , see [32–34]. Let  $\Lambda$  be a partially ordered set. An open covering  $\Omega = \{U_\lambda: \lambda \in \Lambda\}$  of  $Y$  is said to be  $Y$ -directed [34, Definition 7] provided  $U_\mu \subset U_\lambda$  for  $\lambda \leq \mu$  and every index set  $\Lambda_y = \{\lambda \in \Lambda: y \in U_\lambda\}$ ,  $y \in Y$ , is directed, i.e., if  $y \in U_\lambda \cap U_\mu$ , then there exists  $\nu \in \Lambda$  such that  $\nu \geq \lambda, \nu \geq \mu$ , and  $y \in U_\nu \subset U_\lambda \cap U_\mu$ .

Let  $\Omega = \{U_\lambda: \lambda \in \Lambda\}$  be an  $Y$ -directed open covering. A local system of sheaves  $\Sigma = \{\Omega, \mathcal{L}_\lambda, \gamma_\mu^\lambda, \Lambda\}$  is a family of sheaves  $\mathcal{L}_\lambda$  on  $U_\lambda$  and homomorphisms  $\gamma_\mu^\lambda : \mathcal{L}_\lambda|_{U_\mu} \rightarrow \mathcal{L}_\mu$  with  $\mu \geq \lambda$  such that  $\nu \geq \mu \geq \lambda$  implies  $\gamma_\nu^\lambda = \gamma_\nu^\mu \gamma_\mu^\lambda$ , see [34, Definition 13]. A limit  $\text{lim ind } \Sigma$  of the local system of sheaves  $\Sigma$  is defined. This is a sheaf  $\mathcal{L}$  on  $Y$  with fibers  $\mathcal{L}_y = \text{lim ind}\{(\mathcal{L}_\lambda)_y, \lambda \in \Lambda_y\}$  and the topology of  $\mathcal{L}$  is induced by the topology of  $\mathcal{L}_\lambda$  [34, Definition 14].

We assume also that the homomorphisms  $\gamma_\mu^\lambda$  are monomorphism, and that for every  $U_{\lambda_1}, U_{\lambda_2} \in \Omega$  with  $U_{\lambda_1} \cap U_{\lambda_2} \neq \emptyset$  there is  $\mu \in \Lambda$  such that  $\mu \geq \lambda_1, \lambda_2$  and  $U_\mu = U_{\lambda_1} \cap U_{\lambda_2}$ . The local inductive systems of sheaves satisfying this condition are called regular [33, Definition 3.2].

A local system of sheaves on  $Y$ , may be considered as a collection of inductive systems of sheaves parameterized by the points of the spaces  $Y$ . Then the limit of local system of sheaves may be interpreted as a collection of inductive limits of groups parameterized by the points of the space  $Y$ .

**Lemma 2.3.** (See [32, Proposition 3.7].) *Let  $\Sigma = \{\Omega, \mathcal{L}_\lambda, \gamma_\mu^\lambda, \Lambda\}$  be a local system of sheaves on the space  $Y$ . If  $\text{rdim}(Y, \mathcal{L}_\lambda) \leq n$ , then  $\dim_{\mathcal{L}} Y \leq n$ , where  $\mathcal{L} = \text{lim ind } \Sigma$ .*

Here,  $\text{rdim}(Y, \mathcal{L}_\lambda) = \max\{\dim_{\mathcal{L}_\lambda} F: F \subset U_\lambda \text{ is a closed set in } Y\}$ . More generally, for a fixed open subset  $U$  of  $Y$  and a sheaf  $\mathcal{M}$  on  $U$ ,  $\text{rdim}(Y, \mathcal{M})$  denotes  $\max\{\dim_{\mathcal{M}} F: F \subset U, F \text{ is closed in } Y\}$ .

If  $f : X \rightarrow Y$  is a closed surjective map and  $G$  a group, then there is an exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow f_* f^* \mathcal{G} \rightarrow \mathcal{A} \rightarrow 0, \tag{1}$$

where  $\mathcal{G} = Y \times G$ , and  $f_*, f^*$  are the functors of direct and the inverse image of the sheaves with respect to the map  $f$ , see [2, Section I.3]. Observe also that  $f^* \mathcal{G} = X \times G$ .

Remind that a surjective map  $f : X \rightarrow Y$  is called zero-dimensional if  $\dim f^{-1}(y) = 0$  for every  $y \in Y$ . This is equivalent to  $\dim_G f^{-1}(y) = 0$  for every  $y \in Y$  and some (arbitrary) group  $G$ .

For closed zero-dimensional maps the sheaf  $\mathcal{A}$  is concentrated on the set  $Y_2(f)$ , i.e.,  $\mathcal{A}_y = 0$  for  $y \notin Y_2(f)$ . This fact was used in [29] and will be also exploited here.

**Lemma 2.4.** *Let  $f : X \rightarrow Y$  be a perfect 0-dimensional surjection on the paracompact space  $X$ . If  $\dim_G X < \dim_G Y < \infty$ , then  $\dim_{\mathcal{A}} Y = \dim_G Y - 1$ , where  $\mathcal{A}$  is the sheaf from the exact sequence (1). If  $\dim_G X < \dim_G Y = \infty$ , then  $\dim_{\mathcal{A}} Y = \infty$ .*

**Proof.** Since the map  $f$  is closed and 0-dimensional,  $f_*$  is an exact functor, see [32, Proposition 1.4]. Then, by [32, Proposition 1.5],  $H^i(X, \mathcal{G}_V) = H^i(Y, f_* f^* \mathcal{G}_U)$ , where  $V = f^{-1}(U)$ ,  $\mathcal{G}_V = (X \times G)_V = f^* \mathcal{G}_U$  and  $U$  is an open set in  $Y$ . Therefore,  $\dim_{f_* f^* \mathcal{G}} Y = \dim_G X$ .

First, consider the case  $\dim_G X < \dim_G Y < \infty$ . The exact sequence (1) implies  $\dim_{\mathcal{A}} Y \leq \dim_G Y - 1$ . If  $m = \dim_G Y$ , we can find an open set  $U$  in the space  $Y$  such that  $H^m(Y, \mathcal{G}_U) \neq 0$ . Then, according to (1), we have  $H^{m-1}(Y, \mathcal{A}_U) \neq 0$  because  $m > \dim_G X$ . Hence,  $\dim_{\mathcal{A}} Y \geq m - 1$ .

Now consider the case  $\dim_G X < \dim_G Y = \infty$ . Here,  $\dim_G Y = \infty$  means that there is a sequence of natural numbers  $n_1 < n_2 < \dots$  and open in  $Y$  sets  $U_k$  such that  $H^{n_k}(Y, \mathcal{G}_{U_k}) \neq 0$  for  $k = 1, 2, \dots$ . Since  $H^{n_k}(Y, f_* f^* \mathcal{G}_{U_k}) = H^{n_k}(X, \mathcal{G}_{V_k}) = 0$  for  $V_k = f^{-1}(U_k)$ , the exact cohomology sequence induced by (1) implies

$$H^{n_k-1}(Y, \mathcal{A}_{U_k}) \rightarrow H^{n_k}(Y, \mathcal{G}_{U_k}) \rightarrow 0$$

for every  $n_k > \dim_G X$ . Therefore,  $H^{n_k-1}(Y, \mathcal{A}_{U_k}) \neq 0$  for all these  $n_k$ , which yields  $\dim_{\mathcal{A}} Y = \infty$ .  $\square$

Now, we describe a construction of A.V. Zarelua from his proof of [33, Proposition 3.6]. This construction is given for closed finite-to-one and surjective maps  $f : X \rightarrow Y$  of paracompact spaces  $X$  and  $Y$ , but it also works for closed zero-dimensional surjective maps.

Local systems  $\Sigma_{\mathcal{L}}, \Sigma_{\mathcal{M}}, \Sigma_{\tilde{\mathcal{B}}}$  of sheaves of groups on a space  $Y$  are defined as follows. Let  $\sigma = \{U_1, \dots, U_{k(\sigma)}\}$  be a disjoint system of open sets in  $X$  such that  $f^{-1}(U) = U_1 \cup \dots \cup U_{k(\sigma)}$  for some open set  $U$  in  $Y$ . Consider the set  $\Lambda = \{(U, \sigma)\}$  of all pairs  $(U, \sigma)$  satisfying the above conditions and introduce a partial order on  $\Lambda$  determined by the inclusion:  $(U', \sigma') \geq (U, \sigma)$  if  $U' \subset U$  and  $\sigma'$  is a subdivision of  $\sigma$ . Then  $\Omega = \{U_\lambda = U : \lambda = (U, \sigma) \in \Lambda\}$  is an  $Y$ -directed open covering.

Let  $(U, \sigma) \in \Lambda, \sigma = \{U_1, \dots, U_{k(\sigma)}\}$  and

$$\Phi_j = \{x \in U_j : f^{-1}f(x) \setminus U_j \neq \emptyset\}, \quad j = 1, \dots, k(\sigma).$$

Each set  $\Phi_j$  is closed in  $U_j$ . Denote  $F_j = f(\Phi_j)$  and let  $F = F_1 \cup \dots \cup F_{k(\sigma)}$ . Obviously, all sets  $F$  and  $F_j, j = 1, \dots, k(\sigma)$ , are closed in  $U$ . Moreover,  $F_j \subset Y_2(f), j = 1, \dots, k(\sigma)$ .

Consider the sheaves  $\mathcal{L}(U, \sigma) = \mathcal{G}_F, \mathcal{M}(U, \sigma) = \bigoplus_1^{k(\sigma)} \mathcal{G}_{F_j}$  with  $\mathcal{G}$  being the constant sheaf  $U \times G$ . The natural projections  $\mathcal{G}_F \rightarrow \mathcal{G}_{F_j}$  induce a monomorphism  $\tilde{\alpha}(U, \sigma) : \mathcal{G}_F \rightarrow \bigoplus_1^{k(\sigma)} \mathcal{G}_{F_j}$ . The sheaf  $\tilde{\mathcal{B}}(U, \sigma)$  is defined by the exact sequence

$$0 \rightarrow \mathcal{G}_F \rightarrow \bigoplus_1^{k(\sigma)} \mathcal{G}_{F_j} \rightarrow \tilde{\mathcal{B}}(U, \sigma) \rightarrow 0, \tag{2}$$

i.e.,  $\tilde{\mathcal{B}}(U, \sigma) = \text{Coker}(\tilde{\alpha}(U, \sigma))$ .

If  $(U, \sigma), (U', \sigma') \in \Lambda$  with  $(U, \sigma) \leq (U', \sigma')$ , there are natural homomorphisms

$$\gamma_{(U', \sigma'), \mathcal{L}}^{(U, \sigma)} : \mathcal{L}(U, \sigma)|_{U'} \rightarrow \mathcal{L}(U', \sigma')$$

and

$$\gamma_{(U', \sigma'), \mathcal{M}}^{(U, \sigma)} : \mathcal{M}(U, \sigma)|_{U'} \rightarrow \mathcal{M}(U', \sigma').$$

The last two homomorphisms induce another one

$$\gamma_{(U', \sigma'), \tilde{\mathcal{B}}}^{(U, \sigma)} : \tilde{\mathcal{B}}(U, \sigma)|_{U'} \rightarrow \tilde{\mathcal{B}}(U, \sigma).$$

Then

$$\Sigma_{\mathcal{L}} = \{\Omega, \mathcal{L}(U, \sigma), \gamma_{(U', \sigma'), \mathcal{L}}^{(U, \sigma)}, \Lambda\},$$

$$\Sigma_{\mathcal{M}} = \{\Omega, \mathcal{M}(U, \sigma), \gamma_{(U', \sigma'), \mathcal{M}}^{(U, \sigma)}, \Lambda\}$$

and

$$\Sigma_{\tilde{\mathcal{B}}} = \{\tilde{\mathcal{B}}(U, \sigma), \gamma_{(U', \sigma'), \tilde{\mathcal{B}}}^{(U, \sigma)}, \Lambda\}$$

are local inductive systems of sheaves on the space  $Y$ .

Consider also the local inductive system

$$\Sigma_{\tilde{\mathcal{D}}} = \{\tilde{\mathcal{D}}(U, \sigma), \gamma_{(U', \sigma'), \tilde{\mathcal{D}}}^{(U, \sigma)}, \Lambda\},$$

where  $\tilde{\mathcal{D}}(U, \sigma) = \bigoplus_1^{k(\sigma)} \mathcal{G}_{U_j}$  and the homomorphisms  $\gamma_{(U', \sigma'), \tilde{\mathcal{D}}}^{(U, \sigma)}$  are defined by natural projections.

Let  $\mathcal{L} = \lim \text{ind } \Sigma_{\mathcal{L}}, \mathcal{M} = \lim \text{ind } \Sigma_{\mathcal{M}}$  and  $\tilde{\mathcal{B}} = \lim \text{ind } \Sigma_{\tilde{\mathcal{B}}}$ .

**Lemma 2.5.** (See [33, Proof of Proposition 3.6].) *Let  $f : X \rightarrow Y$  be a perfect zero-dimensional surjection between paracompact spaces. Then the sheaves  $\mathcal{A}$  and  $\tilde{\mathcal{B}}$  are isomorphic.*

**Proof.** In the case  $f$  is finite-to-one, the lemma was established by A.V. Zarelua in [33, Proof of Proposition 3.6] after the construction of the local inductive systems  $\Sigma_{\mathcal{L}}, \Sigma_{\mathcal{M}}, \Sigma_{\tilde{\mathcal{B}}}$  and their limits. The crucial point in that proof, where the assumption that  $f$  is finite-to-one is used, is to show that the sheafs  $\lim \text{ind } \Sigma_{\tilde{\mathcal{D}}}$  and  $f_* f^* \mathcal{G}$  are isomorphic. According to [31, Proposition 1.4], the last fact also holds when  $f$  a zero-dimensional map with all fibers  $f^{-1}(y), y \in Y$ , being compact. The remaining part of the Zarelua arguments work in the present situation.  $\square$

**Lemma 2.6.** *Let  $f : X \rightarrow Y$  be a perfect 0-dimensional surjection and  $X$  a paracompact space, then  $\dim_{\mathcal{A}} Y \leq \text{rdim}_{Y,G} Y_2(f)$ .*

**Proof.** The non-trivial case is  $\text{rdim}_{Y,G} Y_2(f) < \infty$ . According to Lemmas 2.3, 2.5 and the exact sequence (2), it suffices to prove the inequalities  $\text{rdim}(Y, \mathcal{L}(U, \sigma)) \leq \text{rdim}_{Y,G} Y_2(f)$  and  $\text{rdim}(Y, \mathcal{M}(U, \sigma)) \leq \text{rdim}_{Y,G} Y_2(f)$  for any  $(U, \sigma) \in \Lambda$ . On the other hand, since  $F = F_1 \cup \dots \cup F_{k(\sigma)}$  and the sets  $F, F_j, j = 1, \dots, k(\sigma)$ , are closed in  $U$ , the second inequality implies the first one. So, we need to prove only the second inequality.

Let  $C$  be a closed in  $Y$  subset of  $U$ . Then  $\dim_{\mathcal{G}_{F_i}} C = \dim_{\mathcal{G}_{F_i \cap C}} C = \dim_G C \cap F_i$ . Finally, since  $C \cap F_i$  is a closed in  $Y$  subset of  $Y_2(f)$ , we have  $\dim_G C \cap F_i \leq \text{rdim}_{Y,G} Y_2(f)$ . This completes the proof.  $\square$

**Corollary 2.7.** *Let  $X$  be a paracompact space and  $f : X \rightarrow Y$  a perfect 0-dimensional surjection. If  $\dim_G X < \dim_G Y < \infty$ , then  $\text{rdim}_{Y,G} Y_2(f) \geq \dim_G Y - 1$ . Moreover, if  $\dim_G X < \dim_G Y = \infty$ , then  $\text{rdim}_{Y,G} Y_2(f) = \infty$ .*

**Proof.** The proof follows from Lemmas 2.4 and 2.6.  $\square$

**Remark.** The second part of Corollary 2.7 also holds when  $f$  is a closed zero-dimensional surjection (not necessarily perfect), but  $X$  and  $Y$  being metrizable. Indeed, then  $\text{rdim}_{Y,G} Y_2(f) = \dim_G Y_2(f)$  and assume that  $\dim_G Y_2(f) < \infty$ . Let  $Y^1 = Y \setminus Y_2(f)$  and  $X^1 = f^{-1}Y^1$ . The map  $f_1 = f|_{X^1} : X^1 \rightarrow Y^1$  is a homeomorphism. Therefore,  $\dim_G Y^1 = \dim_G X^1$ . Moreover,  $\dim_G X^1 \leq \dim_G X$  [20, Theorem 16, 2]. Hence, by [9, Theorem B],  $\dim_G Y \leq \dim_G Y^1 + \dim_G Y_2(f) + 2 < \infty$ , which is a contradiction.

**Lemma 2.8.** *Let  $f : X \rightarrow Y$  be a closed 0-dimensional surjection between metrizable spaces and  $m \geq 2$ . If  $\dim_G X < \dim_G Y_m(f) < \infty$ , then  $\dim_{Y,G} Y_{m+1}(f) \geq \dim_{Y,G} Y_m(f) - 1$ . If  $\dim_G Y_m(f) = \infty$ , then  $\dim_G Y_{m+1}(f) = \infty$ .*

**Proof.** The set  $X_1 = \bigcup \{Bd(f^{-1}(y)) : y \in Y\}$ , where  $Bd(f^{-1}(y))$  is the boundary of  $f^{-1}(y)$  in  $X$ , is closed in  $X$  because the map  $f$  is closed [22, Lemma 23-2, p. 142]. So,  $f_1 = f|_{X_1} : X_1 \rightarrow Y$  is also closed, surjective and zero-dimensional. By the Veinstein’s lemma [11, 1.12.9, p. 111], all sets  $f_1^{-1}(y), y \in Y$ , are compact. Moreover,  $\dim_G X_1 \leq \dim_G X$ . Therefore, without restriction of the generality, we can assume that the original map  $f$  is perfect.

First consider the case  $\dim_G X < \dim_G Y_m(f) < \infty$ . Choose locally finite closed coverings  $\omega^j = \{F_s^j\}_s$  of  $X$  with  $\text{diam } F_s^j \leq 1/j, j = 1, 2, \dots$ . Since  $Y_m(f)$  is an  $F_{\sigma}$ -set in  $Y$ , there is a closed set  $C \subset Y$  with  $C \subset Y_m(f)$  and  $\dim_G C = \dim_G Y_m(f)$ . Let  $\tilde{C} = f^{-1}(C)$  and  $f_C = f|_{\tilde{C}} : \tilde{C} \rightarrow C$ . The map  $f_C$  is perfect, zero-dimensional and surjective. Moreover,

$$Y_m(f_C) = \{y \in C : \text{card}(f_C^{-1}(y)) \geq m\} = C. \tag{3}$$

Let  $y \in C$  and  $\{x_1, \dots, x_m\} \subset f^{-1}(y) = f_C^{-1}(y)$ . There exists  $j(y)$  such that the covering  $\omega^{j(y)}$  separates the points  $\{x_1, \dots, x_m\}$ , i.e., we can find disjoint sets  $F_{s_1}^{j(y)}, \dots, F_{s_m}^{j(y)} \in \omega^{j(y)}$  such that  $x_i \in F_{s_i}^{j(y)}, i = 1, \dots, m$ .

Consider the closed subsets  $F_{j(y)} = \bigcap_{i=1}^m f_C(F_{s_i}^{j(y)} \cap \tilde{C})$  of  $C$  and let  $C_j = \bigcup_{j(y)=j} F_{j(y)}$ . Since  $f$  is closed and the fibers  $f^{-1}(y), y \in Y$ , are compact, all systems  $\{F_{j(y)}\}_{j(y)=j}, y \in C$ , are locally-finite and consist of closed sets. Therefore, each set  $C_j$  is closed in  $C$ . Moreover, (3) implies  $C = \bigcup_j C_j$ . Then, there exist  $j_0$  and  $y_0 \in C$  with

$j_0 = j(y_0)$  and  $\dim_G Y_m(f) = \dim_G C = \dim_G C_{j_0} = \dim_G F_{j(y_0)}$ . The sets  $F_{s_1}^{j_0}, \dots, F_{s_m}^{j_0} \in \omega^{j_0}$  separate the points  $\{x_1, \dots, x_m\} \subset f^{-1}(y_0)$ . Let  $\Phi = F_{s_1}^{j_0} \cap f^{-1}(F_{j(y_0)})$  and  $f_\Phi = f|_\Phi : \Phi \rightarrow F_{j(y_0)}$ . The map  $f_\Phi$  is perfect, surjective and 0-dimensional and we have the following inclusion

$$Y_2(f_\Phi) = \{y \in F_{j(y_0)} : \text{card}(f_\Phi^{-1}(y)) \geq 2\} \subset Y_{m+1}(f). \tag{4}$$

Moreover,  $\dim_G \Phi \leq \dim_G X < \dim_G Y_m(f) = \dim_G F_{j(y_0)} < \infty$ . Then, by Corollary 2.2,  $\dim_G Y_2(f_\Phi) \geq \dim_G F_{j(y_0)} - 1 = \dim_G Y_m(f) - 1$ . Finally, (4) completes the proof in the case  $\dim_G Y_m(f) < \infty$ .

Now consider the case  $\dim_G Y_m(f) = \infty$ . Since  $Y_m(f)$  is an  $F_\sigma$  set in  $Y$ , we have two cases: (A) There is a sequence of natural numbers  $n_1 < n_2 < \dots$  converging to  $\infty$  and closed in  $Y$  sets  $C^k \subset Y_m(f)$  such that  $\dim_G C^k = n_k$  for  $k = 1, 2, \dots$ ; (B) There is a closed in  $Y$  set  $C \subset Y_m(f)$  such that  $\dim_G C = \infty$ .

Case (A). Let  $n_k > \dim_G X$ . Applying the construction from the proof of the lemma in the case  $\dim_G Y(f) < \infty$  (with  $C$  replaced by  $C^k$ ), we obtain  $\dim_G Y_{m+1} \geq n_k - 1$ . This implies  $\dim_G Y_{m+1} = \infty$ .

Case (B). We apply again the construction from the case  $\dim_G Y_m(f) < \infty$  for the set  $C$  to obtain  $C = \bigcup_j C_j$ , where  $C_j = \bigcup_{j(y)=j} F_{j(y)}$ ,  $y \in C$ . Since  $\dim_G C = \infty$ , we have again two cases: (B<sub>1</sub>) There exists a sequence of natural numbers  $n_1 < n_2 < \dots$  converging to  $\infty$  and closed sets  $C_{m_k} \subset C$  such that  $\dim_G C_{m_k} = n_k$ ; (B<sub>2</sub>) There exists  $j_0$  such that  $\dim_G C_{j_0} = \infty$  for some closed  $C_{j_0} \subset C$ .

In Case (B<sub>1</sub>) we argue as in Case (A) and obtain  $\dim_G Y_{m+1} = \infty$ .

Case (B<sub>2</sub>) splits again in two cases: (B<sub>21</sub>) There is a sequence of natural numbers  $n_1 < n_2 < \dots$  such that  $\lim n_k = \infty$  and  $\dim_G F_{j(y_k)} = n_k$  for  $y_k \in C_{j_0}$  and  $j(y_k) = j_0$ ; (B<sub>22</sub>) There exists  $y_0 \in C_{j_0}$  such that  $\dim_G F_{j(y_0)} = \infty$ .

The arguments from Case (A) applied under the hypotheses of Case (B<sub>21</sub>) imply  $\dim_G Y_{m+1} = \infty$ .

In Case (B<sub>22</sub>) we apply the construction from the case  $\dim_G Y_m(f) < \infty$  for the set  $C$  and consider the map  $f_\Phi : \Phi \rightarrow F_{j(y_0)}$ . This map is perfect, zero-dimensional and surjective. Moreover,  $\dim_G \Phi \leq \dim_G X$  and  $\dim_G F_{j(y_0)} = \infty$ . Then, by Corollary 2.7,  $\dim_G Y_2(f_\Phi) = \infty$ . Finally, inclusion (4) yields  $\dim_G Y_{m+1} = \infty$ .  $\square$

**Proof of Proposition 2.1.** Assume  $\dim_G X + k \leq \dim_G Y < \infty$ ,  $k \geq 1$ , and  $\mu(f) = \{m_1 < \dots < m_\ell\}$ . Then, by Corollary 2.7 and Lemma 2.8,

$$\dim_G Y_{m_{k+1}}(f) \geq \dim_G X.$$

So,  $Y_{m_{k+1}}(f)$  is not empty, which yields  $\ell \geq k + 1$ .

If  $\dim_G X < \dim_G Y = \infty$ , again by Corollary 2.7 and Lemma 2.8, we have  $\dim_G Y_m(f) = \infty$  for  $m \geq 2$ . Therefore,  $\text{card}(\mu(f)) = \infty$ .  $\square$

Another result of Hurewicz [18, Theorem II, pp. 74–76] is stating that if  $f : X \rightarrow Y$  is a closed surjective finite-to-one map between separable metrizable spaces,  $X$  has the property  $(\alpha)$  and  $\dim Y - \dim X \geq n$ , then  $\text{card}(\mu(f)) \geq n + 2$ .

Here, we say that a space  $X$  has the property  $(\alpha)$  if  $\dim A < \dim X$  for every nowhere dense closed set  $A$  in  $X$ . For example, every topological manifold  $X$  satisfies the property  $(\alpha)$ . In his proof Hurewicz used that the space  $Y$  can be assumed to satisfy the following condition  $(\beta)$ :  $\dim U = \dim Y$  for every open set  $U$  in  $Y$ .

A version of this theorem also holds for the cohomological dimension. We say that  $X \in (\alpha)_G$ , where  $G$  is a group, if  $\dim_G A < \dim_G X$  for every nowhere dense closed set  $A \subset X$ . Let us remind that every locally compact  $n$ -cohomological manifold  $M$  over the principal ideal domain  $L$ , notation  $M$  is  $n - cm_L$ , possesses the property  $(\alpha)_L$ , see [1, Proposition 4.9(a), p. 14]. We also consider the corresponding analogue of condition  $(\beta)$ : A space  $Y$  satisfies condition  $(\beta)_G$ , notation  $Y \in (\beta)_G$ , if  $\dim_G U = \dim_G Y$  for every open  $U \subset Y$ .

Now, we can state a version of the Hurewicz result mentioned above.

**Proposition 2.9.** *Let  $f : X \rightarrow Y$  be a closed, surjective and finite-to-one map on the metrizable space  $X$ . If  $X \in (\alpha)_G$ ,  $\dim_G X + n \leq \dim_G Y < \infty$  and  $Y \in (\beta)_G$ , then  $\text{card}(\mu(f)) \geq n + 2$ .*

Hurewicz derived his theorem from a lemma whose cohomological version is Lemma 2.10 below. Observe that Proposition 2.9 follows from Lemma 2.10 and the inequality (5).

**Lemma 2.10.** *Let  $f : X \rightarrow Y$  be a closed finite-to-one map between the metrizable spaces  $X$  and  $Y$ . Assume that  $X \in (\alpha)_G$  and  $Y \in (\beta)_G$ . If  $\dim_G X + k \leq \dim_G Y < \infty$  and  $m_0 = \max\{\mu(y) : y \in Y\}$ , then  $\dim_G Y_{m_0} < \dim_G X$ .*

**Proof.** The proof of this lemma adapts the arguments from the proof of the corresponding assertion for the dimension  $\dim$  [18, pp. 75–76]. Hurewicz worked with a countable basis of the space  $X$ . Instead of this, we take the collection of elements of the closed coverings  $\omega^j$ , see the proof of Lemma 2.8, Case (A). Then we can apply the constructions of W. Hurewicz with  $\dim$  replaced by  $\dim_G$ .  $\square$

### 3. Freudenthal’s theorem for the cohomological dimension

In this section we shall prove the following

**Proposition 3.1.** *Let  $f : X \rightarrow Y$  be a perfect 0-dimensional surjection between the paracompact spaces  $X$  and  $Y$ . Suppose  $\dim_G X + k \leq \dim_G Y < \infty$  for some natural number  $k$  and a countable group  $G$ . Then  $\text{rdim}_{Y,G} Y_{s+1}(f) \geq \dim_G Y - s$  for every  $s = 1, \dots, k$ .*

First, we need the following technical result.

**Lemma 3.2.** *Let  $f : X \rightarrow Y$  be a 0-dimensional closed and surjective map between the paracompact spaces  $X$  and  $Y$  and  $G$  be a countable group. Suppose  $F$  is a closed in  $Y$  subset of  $Y_2(f)$  and let  $\Phi = f^{-1}(F)$ . Then there exist families  $\{\Phi_{\alpha,g} : \alpha \in \Lambda, g \in G\}$  and  $\{F_{\alpha,g} : \alpha \in \Lambda, g \in G\}$  of closed sets in  $X$  and  $Y$ , respectively, such that:*

- (1)  $\Phi = \bigcup_{\alpha,g} \Phi_{\alpha,g}$ ,  $F = \bigcup_{\alpha,g} F_{\alpha,g}$  and  $f(\Phi_{\alpha,g}) = F_{\alpha,g}$ ;
- (2)  $\{F_{\alpha,g} : \alpha \in \Lambda, g \in G\}$  is locally countable in  $F$  and, for every  $g \in G$  the family  $\{F_{\alpha,g} : \alpha \in \Lambda\}$  is locally finite in  $F$ ;
- (3)  $Y_t(\varphi_{\alpha,g}) = \{y \in F_{\alpha,g} : \text{card}(\varphi_{\alpha,g}^{-1}(y)) \geq t\} \subset Y_{t+1}(f)$ , where  $\varphi_{\alpha,g} : \Phi_{\alpha,g} \rightarrow F_{\alpha,g}$  is the restriction map  $f|_{\Phi_{\alpha,g}}$ .

**Proof.** Recall that the support of the sheaf  $\mathcal{A}$  is the set  $Y_2(f)$ . For every  $y \in Y$  choose an open in  $Y$  set  $O_y$  and a section  $s_y \in \Gamma(\overline{O_y}, \mathcal{A})$  such that  $s_y(z) \neq 0$  for all  $z \in \overline{O_y} \cap F$ . Here  $\overline{O_y}$  is the closure of the set  $O_y$  in  $Y$ . Assume that  $O_y$  is so small that there exists a section  $\tilde{s}_y \in \Gamma(\overline{O_y}, f_*G)$  such that  $s_y$  corresponds to  $\tilde{s}_y$  by the natural projection  $f_*G \rightarrow \mathcal{A}$  from the exact sequence (1). Let  $t_y \in \Gamma(f^{-1}O_y, G)$  be the section which corresponds to the section  $\tilde{s}_y$  by the isomorphism of the groups of sections  $\Gamma(O_y, f_*G)$  and  $\Gamma(f^{-1}O_y, G)$ . Consider the covering  $\omega = \{O_y \cap F\}_{y \in F}$  and let  $\tilde{\omega} = \{F_\alpha : \alpha \in \Lambda\}$  be a closed locally finite covering of  $F$  refining  $\omega$ . Then, for every  $\alpha$  there is  $y(\alpha) \in F$  such that  $F_\alpha \subset \overline{O_{y(\alpha)}}$ . Let  $s_\alpha = s_{y(\alpha)}$ ,  $\tilde{s}_\alpha = \tilde{s}_{y(\alpha)}$ ,  $t_\alpha = t_{y(\alpha)}$  and  $\Phi_\alpha = f^{-1}(F_\alpha)$ .

Consider the sets  $\Phi_{\alpha,g} = \{x \in \Phi_\alpha : t_\alpha(x) = g\}$ ,  $\alpha \in \Lambda, g \in G$ . Each  $\Phi_{\alpha,g}$  is closed in  $X$  because the sheaf  $X \times G$  is a Hausdorff space. Moreover, since  $G$  is countable, the family  $\{\Phi_{\alpha,g} : g \in G\}$  is countable for every  $\alpha \in \Lambda$ . Obviously,  $\Phi_\alpha = \bigcup_{g \in G} \Phi_{\alpha,g}$ , and let  $F_{\alpha,g} = f(\Phi_{\alpha,g})$ . The family  $\{F_{\alpha,g} : \alpha \in \Lambda\}$  is locally finite in  $F$  for every  $g \in G$  because so is the family  $\{F_\alpha : \alpha \in \Lambda\}$ . This implies that  $\{F_{\alpha,g} : \alpha \in \Lambda, g \in G\}$  is a locally countable family in  $F$ . Obviously, every element of the last family is a closed subset of  $Y$ .

Denote by  $\varphi_{\alpha,g}$  the restriction of  $f$  on the set  $\Phi_{\alpha,g}$ . For every  $y \in F_{\alpha,g}$  we have  $s_\alpha(y) \neq 0$ . Hence, there exist points  $x_1, x_2 \in f^{-1}(y)$  such that  $t_\alpha(x_1) \neq t_\alpha(x_2)$ . Indeed, otherwise  $t_\alpha(x') = t_\alpha(x'')$  for all  $x', x'' \in f^{-1}(y)$  would imply  $s_\alpha(y) = 0$ , which is not possible. So,  $x_i \in \Phi_{\alpha,g_i}$ , where  $g_i = t_\alpha(x_i)$ ,  $i = 1, 2$ . Therefore,

$$Y_t(\varphi_{\alpha,g}) = \{y \in F_{\alpha,g} : \text{card}(\varphi_{\alpha,g}^{-1}(y)) \geq t\} \subset Y_{t+1}(f).$$

This completes the proof.  $\square$

**Proof of Proposition 3.1.** The proof goes by induction with respect to  $k$ . Corollary 2.7 provides the case  $k = 1$ . Assume the assertion has been proved for all  $k$  with  $1 \leq k \leq \ell - 1$  and let  $k = \ell$ .

Choose a closed in  $Y$  set  $F$  which is contained in  $Y_2(f)$  such that  $\dim_G F = \text{rdim}_{Y,G} Y_2(f) \geq \dim_G Y - 1 > \dim_G X$ . Consider the families  $\{\Phi_{\alpha,g} : \alpha \in \Lambda, g \in G\}$  and  $\{F_{\alpha,g} : \alpha \in \Lambda, g \in G\}$  from Lemma 3.2. Since  $F = \bigcup_{\alpha,g} F_{\alpha,g}$ ,  $G$  is countable and  $\{F_{\alpha,g} : \alpha \in \Lambda\}$  is locally finite in  $F$  for every  $g \in G$ ,  $\dim_G F_{\alpha,g} = \dim_G F$  for some

$\alpha \in \Lambda$  and  $g \in G$ . Then both  $F_{\alpha,g}$  and  $\Phi_{\alpha,g}$  are paracompact spaces and the map  $f_{\alpha,g} : \Phi_{\alpha,g} \rightarrow F_{\alpha,g}$  is perfect, surjective and zero-dimensional. Moreover,

$$\dim_G F_{\alpha,g} = \dim_G F \geq \dim_G Y - 1 \geq \dim_G X + k - 1 \geq \dim_G \Phi_{\alpha,g} + k - 1.$$

It follows from the inductive assumption that

$$\text{rdim}_{Y,G} Y_s(f_{\alpha,g}) = \text{rdim}_{\Phi_{\alpha,g},G} Y_s(f_{\alpha,g}) \geq \dim_G F_{\alpha,g} - s + 1$$

for  $s = 1, \dots, k - 1$ . According to Lemma 3.2(3), we have  $Y_s(f_{\alpha,g}) \subset Y_{s+1}(f)$ . Then

$$\text{rdim}_{Y,G} Y_{s+1}(f) \geq \text{rdim}_{Y,G} Y_s(f_{\alpha,g}) \geq \dim_G Y - s$$

for  $s = 1, \dots, k$ , which is exactly what we need.  $\square$

#### 4. Hurewicz theorem for extension dimension

In this section we establish extension dimensional analogues of the Hurewicz theorems when either  $\text{card}(f^{-1}(y)) \leq n + 1$  for every  $y \in Y$  or  $\text{card}(\mu(f)) \leq n + 1$ . A map  $f : X \rightarrow Y$  is said to have a metrizable kernel if there exist a metrizable space  $M$  and a map  $g : X \rightarrow M$  such that  $g$  is injective on each fiber of  $f$ , i.e.,  $g|_{f^{-1}(y)}$  is a one-to-one map for every  $y \in Y$ . Obviously, this is the case if  $X$  is submetrizable.

Proposition 4.1 below was established by Dranishnikov and Uspenskij [6, Theorem 1.6] in the case both  $X$  and  $Y$  are metrizable compacta. Our proof of this proposition is a modification of the Dranishnikov and Uspenskij arguments.

**Proposition 4.1.** *Let  $f : X \rightarrow Y$  be a closed surjective map admitting a metrizable kernel such that  $Y$  is a paracompact  $k$ -space and  $\text{card}(f^{-1}(y)) \leq n + 1$  for every  $y \in Y$ . Then  $e\text{-dim } Y \leq e\text{-dim}(X \times \mathbb{I}^n)$ .*

**Proof.** Let  $m = n + 1$  and  $P_m(X)$  be the space of all probability measures on  $X$  whose supports consist of at most  $m$  points. The map  $f$  has an extension  $P_m(f) : P_m(X) \rightarrow P_m(Y)$ . Actually,  $P_m(f)$  is the restriction of the map  $P_m(\beta f) : P_m(\beta X) \rightarrow P_m(\beta Y)$  on  $P_m(X)$ , where  $\beta f : \beta X \rightarrow \beta Y$  is the natural extension of  $f$  between the Čech–Stone compactifications of  $X$  and  $Y$ . Since  $f$  is a perfect map, so is  $P_m(f)$ . Hence,  $Z = P_m(f)^{-1}(Y)$  is a paracompact  $k$ -space as a perfect preimage of  $Y$ . If  $\Delta_n$  denotes the standard  $n$ -dimensional simplex in  $\mathbb{R}^n$ , there exists a map  $p : X^m \times \Delta_n \rightarrow P_m(X)$  assigning to each point  $(x_0, x_1, \dots, x_n, t_0, t_1, \dots, t_n) \in X^m \times \Delta_n$  the measure  $\mu = \sum_{i=0}^n t_i x_i$ . We consider another map  $\pi : T \rightarrow X \times \Delta_n$ , where  $T = p^{-1}(Z)$ , defined by  $\pi(x_0, x_1, \dots, x_n, t_0, t_1, \dots, t_n) = (x_0, t_0, t_1, \dots, t_n)$ . Obviously,  $\pi$  is surjective.

**Claim.** *The maps  $p|_T$  and  $\pi$  are perfect. Moreover  $T$  is closed in  $X^m \times \Delta_n$ .*

Since  $Z$  is a  $k$ -space, according to [12, Theorem 3.7.18], it suffices to show that  $p^{-1}(K)$  is a compact subset of  $T$  for every compact set  $K \subset Z$ . So, we fix such  $K$  and consider the compact set  $F = f^{-1}(P_m(f)(K)) \subset X$ . Then  $F^m \times \Delta_n$  is a compact subset of  $X^m \times \Delta_n$  containing  $p^{-1}(K)$ . On the other hand,  $p^{-1}(K)$  is closed in  $X^m \times \Delta_n$  because  $K$  is closed in  $P_m(X)$ . Therefore,  $p^{-1}(K)$  is compact. Obviously,  $X$  is  $k$ -space as a closed subset of  $Z$ , so is  $X \times \Delta_n$ . Consequently, to prove that  $\pi$  is perfect, again by [12, Theorem 3.7.18], it suffices to show that  $\pi^{-1}(K \times \Delta_n)$  is compact for any compact  $K \subset X$ . Since  $\pi^{-1}(K \times \Delta_n)$  is contained in  $F^m \times \Delta_n$ , the proof of the claim is reduced to establish that  $T$  is closed in  $X^m \times \Delta_n$ . The last assertion follows from the fact that  $Z$  is closed in  $P_m(X)$  (because  $Z = P_m(f)^{-1}(Y)$  and  $Y$  is closed in  $P_m(Y)$ ) and  $T = p^{-1}(Z)$ .

We are going to show that  $e\text{-dim } T \leq e\text{-dim}(X \times \mathbb{I}^n)$ . To this end, we need the following result of Dranishnikov and Uspenskij [6, Theorem 1.2]: If  $g : Z_1 \rightarrow Z_2$  is a perfect 0-dimensional surjection between paracompact spaces, then  $e\text{-dim } Z_1 \leq e\text{-dim } Z_2$  (this theorem was originally established for compact spaces, but the proof works for paracompact spaces as well). Since  $\pi$  is perfect and, obviously, each fiber of  $\pi$  is 0-dimensional, the above theorem yields  $e\text{-dim } T \leq e\text{-dim}(X \times \mathbb{I}^n)$ .

Next step is to show that  $e\text{-dim } Z \leq e\text{-dim}(X \times \mathbb{I}^n)$ . We fix a metric space  $(M, \rho)$  and a map  $g : X \rightarrow M$  such that  $g$  is injective on each fiber  $f^{-1}(y)$ ,  $y \in Y$ . Let  $d$  be the continuous pseudometric on  $X$  defined by  $d(x', x'') = \rho(g(x'), g(x''))$  and consider the corresponding pseudometric  $d_m$  on  $X^m$ ,  $d_m((x_i), (y_i)) = \max_{0 \leq i \leq m-1} d(x_i, y_i)$ . For every equivalence relation  $\mathfrak{R}$  on the set  $\{0, 1, \dots, n\}$  and every  $\varepsilon > 0$  let  $E_{\mathfrak{R}, \varepsilon} \subset X^m$  be the set of all  $m$ -tuples  $(x_0, \dots, x_n)$  satisfying the following conditions:

- if  $i \mathfrak{R} j$ , then  $x_i = x_j$ ;
- if  $i$  is not  $\mathfrak{R}$ -equivalent to  $j$ , then  $d(x_i, x_j) \geq \varepsilon$ .

Since  $d$  is a continuous pseudometric on  $X$ ,  $E_{\mathfrak{R}, \varepsilon}$  is a closed subset of  $X^m$ . We choose one element from every equivalence class of  $\mathfrak{R}$  and denote by  $S_{\mathfrak{R}}$  the set of these  $\mathfrak{R}$ -representatives. Let also

$$\Delta_{\mathfrak{R}} = \{(t_0, \dots, t_n) \in \Delta_n : t_i > 0 \text{ if and only if } i \in S_{\mathfrak{R}}\}.$$

As in the proof of Theorem 1.7 from [6], one can show that the restriction of  $p$  on the set  $C \times \Delta_{\mathfrak{R}}$  is one-to-one for every closed set  $C \subset E_{\mathfrak{R}, \varepsilon}$  with a  $d_m$ -diameter  $\leq \varepsilon/2$ . Since  $\Delta_{\mathfrak{R}}$  is an  $F_{\sigma}$ -subset of  $\Delta_n$ ,  $(C \times \Delta_{\mathfrak{R}}) \cap T$  is  $F_{\sigma}$  in  $T$ .

Now, for every  $\varepsilon > 0$  choose a locally finite closed covering  $\omega_{\varepsilon} = \{H_{\alpha, \varepsilon} : \alpha \in A_{\varepsilon}\}$  of  $X^m$  with each  $H_{\alpha, \varepsilon}$  having a  $d_m$ -diameter  $\leq \varepsilon/2$  (this can be done as follows: first, choose a locally finite closed covering  $\gamma_{\varepsilon}$  of  $M^m$  such that the  $\rho_m$ -diameter of each element of  $\gamma_{\varepsilon}$  is  $\leq \varepsilon/2$ , where  $\rho_m((a_i), (b_i)) = \max_{0 \leq i \leq n} \rho(a_i, b_i)$ , and then let  $\omega_{\varepsilon}$  to be  $(g^m)^{-1}(\gamma_{\varepsilon})$ ). Next, consider the locally finite in  $T$  families

$$\Theta_{\mathfrak{R}, k} = \{((H_{\alpha, 1/k} \cap E_{\mathfrak{R}, 1/k}) \times \Delta_{\mathfrak{R}}) \cap T : \alpha \in A_{1/k}\}$$

with  $k \in \mathbb{N}$  and  $\mathfrak{R}$  being an equivalence relation on  $\{0, 1, \dots, n\}$ . Each element of  $\Theta_{\mathfrak{R}, k}$  is an  $F_{\sigma}$ -subset of  $T$ , so  $((H_{\alpha, 1/k} \cap E_{\mathfrak{R}, 1/k}) \times \Delta_{\mathfrak{R}}) \cap T = \bigcup_{j=1}^{\infty} H_{\mathfrak{R}, k, j}(\alpha)$  such that all  $H_{\mathfrak{R}, k, j}(\alpha)$  are closed in  $T$ . Therefore, we obtain countably many families

$$\Omega_{\mathfrak{R}, k, j} = \{p(H_{\mathfrak{R}, k, j}(\alpha)) : \alpha \in A_{1/k}\}$$

of closed subsets of  $Z$ . Moreover, we already observed that  $p$  restricted to  $H_{\mathfrak{R}, k, j}(\alpha)$  is bijective. Because  $p|T$  is a closed map, this yields that  $H_{\mathfrak{R}, k, j}(\alpha)$  and  $p(H_{\mathfrak{R}, k, j}(\alpha))$  are homeomorphic. So,  $e\text{-dim } p(H_{\mathfrak{R}, k, j}(\alpha)) \leq e\text{-dim}(X \times \mathbb{I}^n)$ . On the other hand, since all  $\Theta_{\mathfrak{R}, k}$  are locally finite in  $T$  and  $p|T$  is perfect, the families  $\Omega_{\mathfrak{R}, k, j}$  are locally finite in  $Z$ . Therefore, each set  $Z_{\mathfrak{R}, k, j} = \bigcup \{p(H_{\mathfrak{R}, k, j}(\alpha)) : \alpha \in A_{1/k}\}$  is closed in  $Z$  and, according to the locally finite sum theorem for extension dimension [5, Proposition 1.18],  $e\text{-dim } Z_{\mathfrak{R}, k, j} \leq e\text{-dim}(X \times \mathbb{I}^n)$  for any  $\mathfrak{R}$  and  $k, j \in \mathbb{N}$ . Since  $d$  is a metric on each fiber of  $f$ , it is easily seen that  $Z$  is the union of the sets  $Z_{\mathfrak{R}, k, j}$ ,  $k, j \in \mathbb{N}$  and  $\mathfrak{R}$  being an equivalence relation on  $\{0, 1, \dots, n\}$ . Therefore, by the countable sum theorem for extension dimension,  $e\text{-dim } Z \leq e\text{-dim}(X \times \mathbb{I}^n)$ .

The last step of our proof is to show that  $e\text{-dim } Y \leq e\text{-dim}(X \times \mathbb{I}^n)$ . We are going to use the following result of Dranishnikov and Uspenskij [6, Proposition 2.3]: Every surjective perfect map  $g : Z_1 \rightarrow Z_2$  with convex fibers between paracompact spaces has the following property: for any CW-complex  $K$  and every closed subset  $B \subset Z_2$  the restriction  $g : g^{-1}(B) \rightarrow B$  induced a bijective map  $g^* : [B, K] \rightarrow [g^{-1}(B), K]$  between the homotopy classes (such maps are called hereditary shape equivalences). This result was established for compact spaces but its proof holds for paracompact spaces as well. Here, a map  $g : Z_1 \rightarrow Z_2$  has convex fibers if there exists a convex subset  $E$  of a locally convex linear space and a closed embedding  $j : Z_1 \subset Z_2 \times E$  such that the sets  $Z_1(y) = \{x \in E : (y, x) \in j(Z_1)\}$  are convex and compact for every  $y \in Z_2$ . The idea behind the proof of this result is the following simple fact: If  $\phi : \bar{Y} \rightarrow \bar{Z}$  is an upper semicontinuous set-valued map with compact and convex values, where  $\bar{Y}$  is paracompact and  $\bar{Z}$  is a convex subset of a locally convex linear space, then for every family  $\mathcal{U}$  of open in  $\bar{Z}$  sets with each  $\phi(y)$ ,  $y \in \bar{Y}$ , being a subset of some  $U \in \mathcal{U}$ , there exists a map  $h : \bar{Y} \rightarrow \bar{Z}$  such that  $h$  and  $\phi$  are  $\mathcal{U}$ -close, i.e., for every  $y \in \bar{Y}$  both  $h(y)$  and  $\phi(y)$  are contained in an element of  $\mathcal{U}$ .

In our situation, we consider the perfect map  $P_m(f) : Z \rightarrow Y$  which has convex fibers. Indeed, let  $P(\beta X)$  be the space of all probability measures on  $\beta X$ . It is well known that  $P(\beta X)$  is a compact and convex subset of  $\mathbb{R}^{C(\beta X)}$ ,  $C(\beta X)$  being the set of all continuous functions on  $\beta X$ . Moreover, there exists a natural map  $P(\beta f) : P(\beta X) \rightarrow P(\beta Y)$  extending  $P_m(f)$ . It is easily seen that the map  $j : Z \rightarrow Y \times P(\beta X)$ ,  $j(\mu) = (P(\beta f)(\mu), \mu)$ , is a closed embedding and all sets  $P_m(f^{-1}(y)) = \{\mu \in P(\beta X) : (y, \mu) \in j(Z)\}$ ,  $y \in Y$ , are convex and compact. So, by the mentioned above result [6, Proposition 2.3],  $P_m(f)$  is a hereditary shape equivalence. Since  $e\text{-dim } Z \leq e\text{-dim}(X \times \mathbb{I}^n)$ , this implies the required inequality  $e\text{-dim } Y \leq e\text{-dim}(X \times \mathbb{I}^n)$ .  $\square$

**Corollary 4.2.** *Let  $f : X \rightarrow Y$  be a surjective map between the compact spaces  $X$  and  $Y$  with  $\text{card}(f^{-1}(y)) \leq n + 1$  for every  $y \in Y$ . Then  $e\text{-dim } Y \leq e\text{-dim}(X \times \mathbb{I}^n)$ .*

**Proof.** Suppose  $L$  is a CW-complex such that  $e\text{-dim}(X \times \mathbb{I}^n) \leq L$  and  $m = n + 1$ . Obviously,  $X$  and  $Y$  have the same topological weight. If,  $w(X) = w(Y)$  is countable, the proof follows directly from Proposition 4.1. Otherwise,

using the notations from the proof of Proposition 4.1, consider the spaces  $Z = P_m(f)^{-1}(Y)$ ,  $T = p^{-1}(Z)$  and the diagram  $T \xrightarrow{p} Z \xrightarrow{P_m(f)} Y$ . According to the Ščepin spectral theorem [26], we can find continuous  $\omega$ -inverse systems  $S_X = \{X_\alpha, \xi_\alpha^\beta: \alpha, \beta \in A\}$  and  $S_Y = \{Y_\alpha, \varpi_\alpha^\beta: \alpha, \beta \in A\}$  consisting of metrizable compacta, and continuous maps  $f_\alpha: X_\alpha \rightarrow Y_\alpha, \alpha \in A$ , such that  $X = \varprojlim S_X, Y = \varprojlim S_Y$  and  $\varpi_\alpha \circ f = \xi_\alpha \circ f_\alpha$ . Here,  $\xi_\alpha: X \rightarrow X_\alpha$  and  $\varpi_\alpha: Y \rightarrow Y_\alpha$  are the projections of the inverse systems  $S_X$  and  $S_Y$ , respectively. Then  $P_m(X) = \varprojlim \{P_m(X_\alpha), P_m(\xi_\alpha^\beta): \alpha, \beta \in A\}$  and  $X^m \times \Delta_n = \varprojlim \{X_\alpha^m \times \Delta_n, (\xi_\alpha^\beta)^m \times \text{id}: \alpha, \beta \in A\}$ . This implies  $T = \varprojlim \{T_\alpha, (\xi_\alpha^\beta)^m \times \text{id}: \alpha, \beta \in A\}$  and  $Z = \varprojlim \{Z_\alpha, P_m(\xi_\alpha^\beta): \alpha, \beta \in A\}$ , where  $T_\alpha = (\xi_\alpha^m \times \text{id})(T)$  and  $Z_\alpha = P_m(\xi_\alpha)(Z)$ . We also consider the maps  $p_\alpha: T_\alpha \rightarrow Z_\alpha$  assigning to each point  $(\xi_\alpha(x_0), \xi_\alpha(x_1), \dots, \xi_\alpha(x_n), t_0, t_1, \dots, t_n) \in X_\alpha^m \times \Delta_n$  the measure  $\sum_{i=0}^{i=n} t_i \xi_\alpha(x_i) \in Z_\alpha$ . Therefore, the following diagrams are commutative for all  $\alpha$ :

$$\begin{array}{ccccccc}
 T & \xrightarrow{p} & Z & \xrightarrow{P_m(f)} & Y & \xleftarrow{f} & X & \xrightarrow{\text{id}} & P_m(X) \\
 \downarrow \xi_\alpha^m \times \text{id} & & \downarrow P_m(\xi_\alpha) & & \downarrow \varpi_\alpha & & \downarrow \xi_\alpha & & \downarrow P_m(\xi_\alpha) \\
 T_\alpha & \xrightarrow{p_\alpha} & Z_\alpha & \xrightarrow{P_m(f_\alpha)} & Y_\alpha & \xleftarrow{f_\alpha} & X_\alpha & \xrightarrow{\text{id}} & P_m(X_\alpha)
 \end{array}$$

Let us observe that the requirement for  $f$  in Proposition 4.1 to have a metrizable kernel was necessary only to prove that  $e\text{-dim } Z \leq e\text{-dim } T$ , the other two facts we established there remain valid in the present situation. So,  $e\text{-dim } T \leq e\text{-dim}(X \times \mathbb{I}^n)$  and  $e\text{-dim } Y \leq e\text{-dim } Z$ . Thus, we need to show that  $e\text{-dim } Z \leq e\text{-dim}(X \times \mathbb{I}^n)$ .

Since  $e\text{-dim } T \leq e\text{-dim}(X \times \mathbb{I}^n) \leq L$ , we can use the factorization theorem for extension dimension (see [24, Theorem 2], [21]) to construct the inverse system  $S_T = \{Z_\alpha, P_m(\xi_\alpha^\beta): \alpha, \beta \in A\}$  in such a way that  $e\text{-dim } T_\alpha \leq L, \alpha \in A$ . The equalities  $Z_\alpha = P_m(f_\alpha)^{-1}(Y_\alpha)$  and  $T_\alpha = p_\alpha^{-1}(Z_\alpha)$  may not be true for some  $\alpha$ , but we always have the inclusions  $Z_\alpha \subset P_m(f_\alpha)^{-1}(Y_\alpha)$  and  $T_\alpha \subset p_\alpha^{-1}(Z_\alpha)$ . Since each  $f_\alpha$  has a metrizable kernel, according to the arguments from the proof of Proposition 4.1,  $e\text{-dim } Z_\alpha \leq e\text{-dim } T_\alpha$ . Consequently,  $e\text{-dim } Z_\alpha \leq L$  for every  $\alpha$ . Finally,  $Z$  being the limit of the inverse system  $S_T$  yields  $e\text{-dim } Z \leq L$ .  $\square$

Next proposition provides the relation between  $e\text{-dim } Y$  and  $e\text{-dim } X$  when  $f: X \rightarrow Y$  is a finite-to-one closed map with  $\text{card}(\mu(f)) \leq n + 1$ . For any CW-complexes  $K$  and  $L$  their join is denoted by  $K \star L$ .

**Proposition 4.3.** *Let  $f: X \rightarrow Y$  be a closed surjection between metrizable spaces and  $\text{card}(\mu(f)) \leq n + 1$ . If  $e\text{-dim } X \leq L$ , then  $e\text{-dim } Y \leq \underbrace{L \star L \star \dots \star L}_{n+1}$ .*

**Proof.** Let  $\mu(f) = \{k_1, k_2, \dots, k_\ell\}$  and  $Y^i = \{y \in Y: \text{card}(f^{-1}(y)) = k_i\}$  for all  $i = 1, \dots, \ell$ . Obviously,  $Y = \bigcup_{i=1}^{\ell} Y^i$ . Since the extension dimension of a union of two subsets of any metrizable space is  $\leq$  the join of their extension dimensions [9, Theorem A], it suffices to prove that  $e\text{-dim } Y^i \leq L$  for any  $i$ .

Passing to the subsets  $Y^i \subset Y$  and  $X^i = f^{-1}(Y^i) \subset X$ , we can assume that all fibers  $f^{-1}(y), y \in Y$ , have cardinality exactly  $m$  for some  $m \geq 1$ . Now, fix a metric  $d$  on  $X$  and consider the space  $\text{exp}_m(X) = \{F \subset X: \text{card}(F) \leq m\}$  equipped with the Vietoris topology, or equivalently, with the Hausdorff metric generated by  $d$ . There exists a natural continuous map  $\text{exp}_m(f): \text{exp}_m(X) \rightarrow \text{exp}_m(Y)$  defined by  $\text{exp}_m(f)(F) = f(F)$ . Obviously,  $\text{exp}_m(f)$  is the restriction of the map  $\text{exp}_m(\beta f): \text{exp}_m(\beta X) \rightarrow \text{exp}_m(\beta Y)$  on  $\text{exp}_m(X)$ . Since  $f$  is perfect, the last observation implies that  $\text{exp}_m(f)$  is also perfect. So,  $Z = \text{exp}_m(f)^{-1}(Y)$  is a closed subset of  $\text{exp}_m(X)$  because  $Y$  is closed in  $\text{exp}_m(Y)$ .

There is also a map  $p: X^m \rightarrow \text{exp}_m(X), p((x_1, \dots, x_m)) = \{x_1, \dots, x_m\}$ . It is easily seen that  $p$  is perfect, so is  $p|_T$ , where  $T = p^{-1}(Z)$ . Moreover,  $Z$  being closed in  $\text{exp}_m(X)$  yields that  $T$  is closed in  $X^m$ . Let us consider the projection  $\pi: T \rightarrow X, \pi((x_1, \dots, x_m)) = x_1$ . Since  $(x_1, \dots, x_m) \in T$  if and only if  $f(x_1) = \dots = f(x_m)$ , we have  $\pi^{-1}(K) \subset f^{-1}(f(K))^m \cap T$  for every  $K \subset X$ . This implies that  $\pi$  is finite-to-one and  $\pi^{-1}(K)$  is compact provided  $K \subset X$  is compact. Hence,  $\pi$  is a perfect light map, and by [6, Theorem 1.2],  $e\text{-dim } T \leq e\text{-dim } X \leq L$ .

Next step is to show that  $e\text{-dim } U \leq L$ , where  $U$  is the subset of  $Z$  such that every  $F \in U$  consists of  $m$  points. Since  $U$  is open in  $Z$ , so is  $T_U = p^{-1}(U)$  in  $T$ . Consequently, by the countable sum theorem,  $e\text{-dim } T_U \leq e\text{-dim } T \leq L$ . Now, for every  $\varepsilon > 0$  let

$$E_\varepsilon = \{(x_1, \dots, x_m) \in T_U: d(x_i, x_j) \geq \varepsilon \text{ for } i \neq j\}.$$

If  $C$  is a closed subset of  $E_\varepsilon$  of  $d_m$ -diameter  $\leq \varepsilon/2$ , where  $d_m$  is the metric on  $X^m$  defined by the same equality as in Proposition 4.1, then  $p|_C$  is a homeomorphism. Indeed, suppose  $p((x_1, \dots, x_m)) = p((y_1, \dots, y_m))$  for some  $(x_1, \dots, x_m), (y_1, \dots, y_m) \in C$ . Then  $\{x_1, \dots, x_m\} = \{y_1, \dots, y_m\}$  and, since  $d(x_i, y_i) \leq \varepsilon/2$  for all  $i$ , it follows that  $x_i = y_i$ ,  $i \in \{1, \dots, m\}$ . So,  $p|_C$  is injective. On the other hand,  $p : T_U \rightarrow U$  is a perfect map and  $C$  is closed in  $T_U$  because  $E_\varepsilon \subset T_U$  is closed. Hence,  $p|_C$  is a homeomorphism with  $p(C)$  being a closed subset of  $U$ . Obviously, the sets  $E_{1/k}$ ,  $k \geq 1$ , form a closed covering of  $T_U$ . For each  $k$  consider a locally finite in  $T_U$  covering  $\Theta_k$  of  $E_{1/k}$  with each  $H \in \Theta_k$  being a closed set in  $T_U$  of  $d_m$ -diameter  $\leq 1/2k$ . Therefore,  $\{p(H) : H \in \Theta_k\}$  is a locally finite family in  $U$  covering  $p(E_{1/k})$  and consisting of closed sets in  $U$  with  $e\text{-dim } p(H) \leq L$ . Consequently,  $D_k = \bigcup \{p(H) : H \in \Theta_k\}$  is closed in  $U$  and  $e\text{-dim } D_k \leq L$ . Finally, since  $U = \bigcup_{k=1}^{\infty} D_k$ , we obtain  $e\text{-dim } U \leq L$ .

The last step is to show that  $e\text{-dim } Y \leq L$ . We define the sets

$$Y_\varepsilon = \{y \in Y : d(x', x'') > \varepsilon \text{ for every } x' \neq x'' \in f^{-1}(y)\},$$

$\varepsilon > 0$ . Using that  $f$  is a perfect map and any fibers of  $f$  contains exactly  $m$  points, one can show that each  $Y_\varepsilon$  is open in  $Y$  and all maps  $g_\varepsilon : Y_\varepsilon \rightarrow U$ ,  $g_\varepsilon(y) = f^{-1}(y)$ , are embeddings. Hence,  $e\text{-dim } Y_\varepsilon \leq L$  because  $e\text{-dim } U \leq L$  yields  $e\text{-dim } A \leq L$  for every subset  $A \subset U$ . To complete the proof, observe that  $Y = \bigcup_{k=1}^{\infty} Y_{1/k}$  with each  $Y_{1/k}$  being an  $F_\sigma$  subset of  $Y$  implies  $e\text{-dim } Y \leq L$ .  $\square$

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