

## ON METRIZABILITY OF SELECTORS

S. J. Nedev, V. M. Valov

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A special case of Michael's theorem [8] asserts that if  $X$  is a complete metric space, then every lower semicontinuous closed valued mapping from a paracompact Hausdorff space  $Y$  to  $X$  admits an upper semicontinuous compact valued selection. A possible inversion of this theorem is considered in this paper.

**1. Introduction and Notations.** The following definition is motivated by the Michael's result, cited above.

**Definition 1.** Let  $\mathfrak{R}$  denote an arbitrary class of topological spaces. The space  $Y$  is called  $\mathfrak{R}$ -selector if whenever  $\Phi: X \rightarrow F(Y)$  is l. s. c. with  $X \in \mathfrak{R}$ , there is a u. s. c. selection  $\psi: X \rightarrow C(Y)$  for  $\Phi$ .

Here  $F(Y)$  stands for the family of all nonvoid closed subsets of  $Y$  and  $C(Y) = \{F \in F(Y) : F \text{ is compact}\}$ . Recall that a mapping  $\Phi: X \rightarrow F(Y)$  is said to be l. s. c. (u. s. c.) if  $\{x \in X : \Phi(x) \cap \Gamma \neq \emptyset\} = \Phi^{-1}(\Gamma)$  is open (closed) in  $X$  for every open (closed)  $\Gamma$  in  $Y$ .

Our aim in this paper is to present sufficient conditions for the metrization of  $\mathfrak{R}$ -selectors.

Next, let  $\mathfrak{P}$  ( $\mathfrak{P}_0$ ) denote the class of paracompact (zero-dimensional) spaces,  $\mathfrak{C}$  - the class of compact spaces and  $\mathfrak{C}_0 = \mathfrak{C} \cap \mathfrak{P}_0$  (the covering dimension  $\dim$  is meant everywhere in this paper).

### 2. The General Case.

A). Every closed subspace of a  $\mathfrak{R}$ -selector is a  $\mathfrak{R}$ -selector.

B). Every  $\mathfrak{P}_0$ -selector is a  $\mathfrak{P}$ -selector and every  $\mathfrak{C}_0$ -selector is a  $\mathfrak{C}$ -selector.

A) is obvious and B) is easily seen by passing to the absolutes in the sense of Ponomarev [11] and Gleason [4].

**Theorem 1.** Every compact  $\mathfrak{C}_0$ -selector is metrizable.

**Proof.** Let  $Y$  be a compact  $\mathfrak{C}_0$ -selector. By the proposition B) if  $Y \subset X$  and  $X$  is compact, then there is a u. s. c. retraction  $r: X \rightarrow C(Y)$  (i. e.  $r(y) = y$  for each  $y \in Y$ ). One can get this retraction as a u. s. c. selection of the mapping  $\Phi: X \rightarrow F(Y)$ , defined by letting  $\Phi(y) = \{y\}$  for each  $y \in Y$  and  $\Phi(x) = Y$  for each  $x \in X \setminus Y$ . Thus, by a result of Dranishnikov [oral communication] based on a Shirokov's [13] technique,  $Y$  is a Dudundji space in the sense of Pelczynski [9]; hence  $Y$  is a dyadic space by a result of Haydon [5]. By A) each closed subspace of  $Y$  is a dyadic space and consequently  $Y$  is metrizable by a result of Efimov [2].

**Theorem 2.** If a  $\mathfrak{P}_0$ -selector  $X$  is  $p$ -paracompact in the sense of Arhangel'skii [1], then  $X$  is completely metrizable.

The common proof of the Theorems 2 and 2' is presented at the end of the paper. A part of this proof is covered by the following

**Proposition 1.** Let  $X$  be  $p$ -paracompact with the following property: whenever a paracompact  $Y$  contains  $X$  as a closed subset (with each point of  $Y \setminus X$  isolated), there is a neighbourhood  $O$  of  $X$  in  $Y$  and a u. s. c. retraction  $r: O \rightarrow F(Y)$ . Then  $X$  is Čech-complete.

This proposition was actually proved in the paper of Przymusiński [12].

### 3. The Convex Case.

**Definition 2.** A space  $Y$  is called a convex- $\mathfrak{R}$ -selector if the following holds: (i)  $Y$  is a convex closed subspace of a certain locally convex vector space and (ii) whenever  $\Phi: X \rightarrow F(Y)$  is l. s. c. and convex valued with  $X \in \mathfrak{R}$ , there is a u. s. c. selection  $\psi: X \rightarrow C(Y)$  for  $\Phi$ .

**Theorem 1'.** Every compact convex  $\mathfrak{C}_0$ -selector is metrizable.

Theorem 1' generalizes a result of Magerl [7], the last requiring the existence of single-valued continuous selections. The proof of Theorem 1' is based on the following

**Proposition 2** (V. Valov). Let  $X$  be a compact convex subspace of a locally convex vector space. If no compact convex subspace of  $X$  contains an uncountable disjoint family of open subsets, then  $X$  contains no one-point compactification of an uncountable discrete space.

**Proof.** Suppose this is not the case and let  $\alpha T$  be the one-point compactification of the uncountable discrete subset  $T$  of  $X$ , with the unique nonisolated point  $x_0$  of  $\alpha T \subset X$ . Denote  $P = \overline{\text{conv}(\alpha T)}$  — the closed convex hull of  $\alpha T$  and let  $K$  be the set of all extreme points of  $P$ . By virtue of the Krein-Milman theorem,  $K \subset \alpha T$ . We will get a contradiction by showing that  $\{x_0\}$  is a  $G_\delta$ -set in  $P$ ;  $T$  must therefore be countable. First, suppose  $K$  is countable. For a finite subset  $F \subset K$  put  $P(F) = \overline{\text{conv}(K \setminus F)}$ . Obviously,  $\bigcap \{P(F) : F \text{ is a finite subset of } K \setminus \{x_0\}\} = \{x_0\}$ . Let  $k \in K \setminus \{x_0\}$ . Then,  $k \notin P(k)$ . Indeed, let  $H$  be a closed convex neighbourhood of  $x_0$  with  $k \notin H$ , and let  $K \setminus H = k_1, k_2, \dots, k_n, k$ . For a point  $p \in P \cap H$ ,  $k \notin \Delta_p = \text{conv}(p, k_1, \dots, k_n)$ , because  $k$  is an extreme point of  $P$ . Thus, there are disjoint convex open sets  $U_p$  and  $V_p$  in  $X$ , such that  $k \in U_p$  and  $\Delta_p \subset V_p$ . Let  $P \cap H \subset V_{p_1} \cap \dots \cap V_{p_s} = V$ . Then,  $\text{conv}(K \setminus \{k\}) \subset V$  and hence  $k \notin P(k)$ , because  $U \cap V = \emptyset$  where  $U = \bigcap_{i=1}^s U_{p_i}$ . Thus,  $P$  is the cone  $\text{conv}(P(k), k)$  and hence  $P(k)$  is a  $G_\delta$ -set in  $P$ . Analogously, if  $k_1, k_2 \in K \setminus \{x_0\}$ , then  $P(k_1)$  is the cone  $\text{conv}(P(k_1, k_2), k_1)$ ; thus,  $P(k_1, k_2)$  is a  $G_\delta$ -set in  $P(k_1)$  and hence in  $P$  and so on; by induction we see that  $P(F)$  is a  $G_\delta$ -set in  $P$  for every finite subset  $F$  of  $K \setminus \{x_0\}$ .

Suppose now that  $K$  is uncountable. Note that  $T$  may be assumed to be linearly independent; if not, we replace  $T$  by the maximal linearly independent subset  $T'$  of  $T$ ; the only thing to check is that  $\text{Card}(T') > \aleph_0$ . But in the opposite case (i. e.  $\text{Card}(T') \leq \aleph_0$ )  $T$  will be contained in a countable union of finite-dimensional vector subspaces which is impossible. Now, for each  $k \in K' = K \setminus \{x_0\}$  define mapping  $f_k: \text{conv } K' \rightarrow [0, 1]$  by letting  $f_k(p) = \lambda$ , where

$$p = \lambda k + \sum_{i=1}^n \lambda_i k_i$$
 is the unique representation of  $p \in \text{conv } K'$  as a convex linear

combination of elements of  $K'$ . It is easy to see that  $f_k$  is continuous at  $k$  for every  $k \in K'$ . Let  $U_k = \{p \in \text{conv } K' : f_k(p) > 2^{-1}\}$ . The family  $\{U_k : k \in K'\}$  is open and disjoint in  $\text{conv } K'$  which contradicts the assumption that  $P$  contains no uncountable disjoint family of open sets (note that  $\text{conv } K'$  is dense in  $P$ ).

**Remark.** Since every nonmetrizable dyadic space contains the one-point compactification of an uncountable discrete space (see Efimov [3]), proposition 2 answers a question of Magerl [7] positively.

**Theorem 2'.** Every  $p$ -paracompact convex- $\mathfrak{B}_0$ -selector  $X$  is completely metrizable.

**Lemma 1.** Let  $Z = \Pi\{X_\alpha : \alpha \in A\}$ , where  $X_\alpha$  is a separable metric space and let  $r: Z \rightarrow C(X)$  be u. s. c. Suppose each point of  $X$  is a  $G_\delta$ -set. Then there is a countable subset  $B$  of  $A$  such that, if  $\text{Card}(r(z_i)) = 1$ ,  $i = 1, 2$  and  $\pi_B(z_1) = \pi_B(z_2)$ , then  $r(z_1) = r(z_2)$  (here  $\pi_B$  is the natural projection  $\pi_B: Z \rightarrow \Pi\{X_\alpha : \alpha \in B\}$ ). This lemma may be proved as in the case when  $r$  is single valued, by using some results of R. P o l a. E. P o l in [10].

**Lemma 2.** Every  $p$ -paracompact Lindeloff convex- $\mathfrak{B}_0$ -selector  $X$  is metrizable.

**Proof.** It is well-known that  $X$  is contained as a closed subspace in a product  $Z = M \times \Pi\{X_\alpha : \alpha \in A\}$ , where  $M$  is a separable metric space and  $X_\alpha = [0, 1]$  for each  $\alpha \in A$ . Since  $Z$  is paracompact, there is a u. s. c. retraction  $r: Z \rightarrow C(X)$ . Next, note that in virtue of Proposition 1 and Theorem 1' each one-point set of  $X$  is a  $G_\delta$ -set. By Lemma 1, there is a countable subset  $B$  of  $A$ , such that if  $x_1, x_2 \in X$  and  $\pi(x_1) = \pi(x_2)$ , then  $x_1 = x_2$  (here  $\pi$  stands for the projection  $\pi: Z \rightarrow M \times \Pi\{X_\alpha : \alpha \in B\}$ ). Thus,  $\pi/X$  is a one-to-one continuous mapping from  $X$  to a metric space and hence  $X$  is metrizable by a result of Arhangel'skii [1].

**Lemma 3.** Let  $X$  be a  $p$ -paracompact closed convex subset of a locally convex vector space. Then the closed convex hull of each compact subset of  $X$  is Lindeloff.

**Proof.** Let  $F$  be a compact subset of  $X$ . Denote  $\text{conv}_n(F) = \left\{ x \in X : x = \sum_{i=1}^n \lambda_i x_i : x_i \in F, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, i = 1, 2, \dots, n \right\}$ ;  $\Delta_n = \left\{ (\lambda_1, \dots, \lambda_n) : \sum_{i=1}^n \lambda_i = 1, 0 \leq \lambda_i \leq 1, i = 1, 2, \dots, n \right\}$ . Define  $f: F^n \times \Delta_n \rightarrow \text{conv}_n(F)$  by the formula  $f(x_1, \dots, x_n; \lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i x_i$ . Obviously  $f$  is a continuous mapping onto. Hence  $\text{conv}_n(F)$

is a compact and thus  $\text{conv } F = \bigcup_{n=1}^{\infty} \text{conv}_n(F)$  is Lindeloff. Since  $\overline{\text{conv } F}$  is a  $p$ -paracompact space, there is a perfect onto mapping  $g: \overline{\text{conv } F} \rightarrow M$ , where  $M$  is a metric space (see [1]). But  $g(\text{conv } F)$  is Lindeloff, hence separable metric space which is dense in  $M$ . Thus  $M$  is separable and hence  $\overline{\text{conv } F}$  is Lindeloff.

**Corollary 1.** Every compact subset of a  $p$ -paracompact convex- $\mathfrak{C}_0$ -selector is metrizable.

**Proof of Theorem 2 and Theorem 2'.** Let  $X$  be closed subset of the product  $Z = M \times \Pi\{X_\alpha : \alpha \in A\}$ , where  $M$  is a metric space and  $X_\alpha = [0, 1]$  for each  $\alpha \in A$ . Take a perfect surjection  $f: M_0 \rightarrow M$  with  $M_0$  — a zero-dimensional metric space. Put  $Z_0 = M_0 \times \Pi\{X_\alpha : \alpha \in A\}$  and define the mapping  $g: Z_0 \rightarrow Z$  by the formula

$g(m, x) = (f(m), x)$ , where  $m \in M_0$  and  $x \in \Pi\{X_\alpha : \alpha \in A\}$ . Note that  $g$  is perfect. Following Klebanov [6], the proof of Theorem 7, we fix a sequence  $\{\omega_n\}_{n=1}^{\infty}$  of discrete open coverings  $\omega_n$  of  $M_0$  with the properties: (i)  $\omega_{n+1}$  refines  $\omega_n$  for  $n = 1, 2, \dots$  and (ii)  $\{St\omega_n(m)\}_{n=1}^{\infty}$  is a local base for the topology of  $M_0$  at the point  $m$  for each  $m \in M_0$  (here  $St\omega_n(m) = \bigcup \{U \in \omega_n : m \in U\}$ ). Fix a point  $m(U) \in U$  for every  $U \in \omega_n$  and  $n = 1, 2, \dots$ . For an arbitrary  $m \in M_0$  let  $U_n^m$

be the only element of  $\omega_n$  which contains  $m$ . Put  $P(m) = \{m\} \times \prod \{X_\alpha : \alpha \in A\}$ ,  $Y = g^{-1}(X)$  and let  $r: Z_0 \rightarrow C(X)$  be u. s. c. with  $r(y) = g(y)$  for every  $y \in Y$ . Note that  $r(P(m)) = \bigcup \{r(z) : z \in P(m)\}$  is compact and hence — metrizable subset of  $X$  (in the case of Theorem 2 this follows from Theorem 1 and in the case of Theorem 2' this follows by Corollary 1). This fact, together with the u. s. continuity of  $r$ , allows us to fix a countable subset  $B(m) \subset A$  such that, if  $x, y \in P(m)$  and  $\pi_{B(m)}(x) = \pi_{B(m)}(y)$ , then  $r(x) \cap r(y) \neq \emptyset$  ( $\pi_{B(m)}: Z_0 \rightarrow \prod \{X_\alpha : \alpha \in B(m)\}$  is the natural projection). Next, let  $\rho_m$  be a metric on  $\prod \{X_\alpha : \alpha \in B(m)\}$  taking values in  $[0, 1]$ . Define a continuous pseudometric  $d^m$  on  $P(m)$  by letting  $d^m(x, y) = \rho_m(\pi_{B(m)}(x), \pi_{B(m)}(y))$ . Denoting by  $\pi$  the natural projection  $\pi: Z_0 \rightarrow M_0$ , define a continuous pseudometric  $d_n$  on  $Z_0$  in the following manner: if  $x_i \in Z_0$ ,  $i = 1, 2$  with  $x_i = (m_i, x_i^*)$  where  $m_i \in M_0$  and  $x_i^* \in \prod \{X_\alpha : \alpha \in A\}$ , then  $d_n(x_1, x_2) = 1$  in the case  $m_1 \notin St_{\omega_n}(m_2)$  and  $d_n(x_1, x_2) = d^{m(U)}((m(U), x_1^*), (m(U), x_2^*))$  in the case  $m_1, m_2 \in U \in \omega_n$ . Finally, let

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x, y)$$
 for  $x, y \in Z_0$ . Obviously  $d$  is a continuous pseudometric

on  $Z_0$ . For  $x, y \in Y$  the equality  $d(x, y) = 0$  implies  $\pi(x) = \pi(y)$  (if not, then  $d_n(x, y) = 1$  for large  $n$ ). Moreover, on the same assumption,  $g(x) = r(x) = r(y) = g(y)$ . In fact, suppose  $d(x, y) = 0$  where  $x = (m_0, x^*)$ ,  $y = (m_0, y^*) \in Y$ . Define  $x_n = (m(U_n^{m_0}), x_n^*)$ ,  $y_n = (m(U_n^{m_0}), y_n^*)$ . Obviously  $x = \lim_{n \rightarrow \infty} x_n$ ,  $y = \lim_{n \rightarrow \infty} y_n$ . We have

$0 = d_n(x, y) = d^{m(U_n^{m_0})}(x_n, y_n)$  and hence  $\pi_{B(m(U_n^{m_0}))}(x_n) = \pi_{B(m(U_n^{m_0}))}(y_n)$ , thus  $r(x_n) \cap r(y_n) \neq \emptyset$  (note that  $x_n, y_n \in P(m(U_n^{m_0}))$ ). Suppose now that  $g(x) \neq g(y)$  and take disjoint open neighbourhoods  $U$  and  $V$  of  $g(x)$  and  $g(y)$ , respectively. For some  $n$  — natural number,  $r(x_n) \subset U$  and  $r(y_n) \subset V$  (because of u. s. continuity of  $r$ ), which contradicts  $r(x_n) \cap r(y_n) \neq \emptyset$ . Thus, if  $Y/d$  denotes natural quotient set and  $p: Y \rightarrow Y/d$  is the corresponding quotient mapping then there is a mapping  $h: Y/d \rightarrow X$  such that  $g = h \circ p$ . Let  $Y^*$  be the set  $Y/d$  endowed with the quotient topology and  $Y'$  be the set  $Y/d$  endowed with the metric topology induced by  $d$ . Then, obviously  $h: Y^* \rightarrow X$  is perfect, hence  $Y^*$  is a  $p$ -paracompact space and, on the other hand, the identity mapping  $\text{id}: Y^* \rightarrow Y'$  is a continuous bijection. Thus,  $Y^*$  is metrizable by a theorem of Arhangel'skii [1]. Finally,  $X$  is metrizable as a perfect image of a metric space (see [3])

*Mathematical Institute  
Bulgarian Academy of Sciences  
Sofia, Bulgaria*

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