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Parametric bing and Krasinkiewicz maps

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Abstract

It is shown that if $f : X \to Y$ is a perfect map between metrizable spaces and Y is a C-space, then the function space $C(X, \mathbb{I})$ with the source limitation topology contains a dense G_δ-subset of maps g such that every restriction map $g_y = g|f^{-1}(y)$, $y \in Y$, satisfies the following condition: all fibers of g_y are hereditarily indecomposable and any continuum in $f^{-1}(y)$ either contains a component of a fiber of g_y or is contained in a fiber of g_y.

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1. Introduction

All spaces in the paper are assumed to be metrizable and all maps continuous. Unless stated otherwise, any function space $C(X, M)$ is endowed with the source limitation topology whose neighborhood base at a given function $f \in C(X, M)$ consists of the sets

$$B_\varrho(f, \varepsilon) = \{g \in C(X, M): \varrho(g, f) < \varepsilon\},$$

where ϱ is a fixed compatible metric on M and $\varepsilon : X \to (0, 1]$ runs over continuous functions into $(0, 1]$. The symbol $\varrho(f, g) < \varepsilon$ means that $\varrho(f(x), g(x)) < \varepsilon(x)$ for all $x \in X$. The source limitation topology does not depend on the metric ϱ [15] and has the Baire property provided M is completely metrizable [20]. Obviously, this topology coincides with the uniform convergence topology when X is compact.

A compactum is called a *Bing space* if each of its subcontinua is hereditarily indecomposable. A map g is said to be a *Bing map* [16] if all fibers of g are Bing spaces. Following Krasinkiewicz [13], we say that space M is a *free space* if for any compactum X the function space $C(X, M)$ contains a dense subset consisting of Bing maps. The class of free spaces is quite large, it contains all n-manifolds [13], in particular, the unit interval [16], all locally finite polyhedrons [24], as well as all manifolds modeled on Menger cubes M^n_{2n+1} or Nöbeling spaces N^n_{2n+1} [24] ($n \geqslant 1$). This class also contains 1-dimensional locally connected continua [24]. Surjective Bing maps were considered in [12].

In the present paper we provide a parametric versions of the above results concerning free spaces.

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Theorem 1.1. *Let $f : X \to Y$ be a perfect map with Y being a strongly countable-dimensional space. Then, for every complete ANR free space M, the function space $C(X, M)$ contains a dense G_δ-set of maps g such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are Bing maps. Moreover, if in addition M is a closed convex subset of a Banach space, then the same conclusion remains true provided Y is a C-space.*

Recall that X is a C-space if for any sequence $\{\nu_n\}_{n=1}^{\infty}$ of open covers of X there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of disjoint open families in X such that each γ_n refines ν_n and $\bigcup_{n=1}^{\infty} \gamma_n$ is a cover of X. Every strongly countable-dimensional space (i.e. a space which is a union of countably many closed finite-dimensional subsets), as well as every countable-dimensional space (a countable union of 0-dimensional subsets) is a C-space [7] and there exists a compact C-space which is not countable-dimensional [22].

We also consider the so-called Krasinkiewicz maps [17]. A map $g : X \to M$ between compacta is called a Krasinkiewicz map if every continuum in X is either contained in a fiber of g or contains a component of a fiber of g. Krasinkiewicz [14] proved that, for any compactum X, the Krasinkiewicz maps from space $C(X, \mathbb{I})$ form a dense subset. Levin and Lewis [17] established that this set is also a G_δ-subset of $C(X, \mathbb{I})$. Our second theorem in the present paper provides a parametric version of the Levin–Lewis result.

Theorem 1.2. *Let $f : X \to Y$ be a perfect map with Y being a C-space. Then $C(X, \mathbb{I})$ contains a dense G_δ-subset of maps g such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are Bing and Krasinkiewicz maps.*

The proof of Theorem 1.1 is given in Section 2. It is based on few selection theorems: the convex-valued selection theorem of Michael [19], a selection theorem for finite-dimensional spaces due to Gutev [9, Theorem 3.1], and a selection theorem for C-spaces established by Uspenskij [25, Theorem 1.3]. Section 3 is devoted to the proof of Theorem 1.2. In the final Section 4 we discuss some possible generalizations of the Uspenskij selection theorem for C-spaces which would imply the validity of Theorem 1.1 with Y being a complete ANR free space having the C-space property. In this section we also present a version of Theorem 1.1, where M is a free completely metrizable LC^n-space. As an application of Theorem 1.1, we establish a result concerning the Bula property of perfect open maps between metrizable spaces.

2. Proof of Theorem 1.1

Following [16], we consider the family $\mathcal{D}$ of all 4-tuples (F_0, F_1, V_0, V_1) such that F_0 and F_1 are disjoint closed subsets of the Hilbert cube Q and V_0, V_1 disjoint neighborhoods of F_0 and F_1, respectively, in Q. A set $P \subset Q$ is *D-crooked* (see [3] and [16]) for $D = ((F_0, F_1, V_0, V_1) \in \mathcal{D}$ if there exists a neighborhood G of P in Q such that for every map $h : \mathbb{I} = [0, 1] \to G$ with $h(0) \in F_0$ and $h(1) \in F_1$ there are $t_0, t_1 \in \mathbb{I}$ such that $0 < t_0 < t_1 < 1$, $h(t_0) \in V_1$ and $h(t_1) \in V_0$. According to [3] and [16], $\mathcal{D}$ has the following properties:

- a compactum $P \subset Q$ is a Bing space if and only if P is D-crooked for every $D \in \mathcal{D}$;
- there is a sequence $\{D_i\}_{i \geqslant 1} \subset \mathcal{D}$ such that for every compactum $P \subset Q$, P is a Bing space iff P is D_i-crooked for every i.

A map $g : P \subset Q \to M$ is called D-crooked, where $D \in \mathcal{D}$, if all fibers of g are D-crooked. We fix a sequence $\{D_i\}_{i \geqslant 1} \subset \mathcal{D}$ with the above property and a map $h : X \to Q$ such that h embeds every fiber $f^{-1}(y)$, $y \in Y$ (such a map exists by [21, Proposition 9.1]). For given $y \in Y$ we denote by h_y the restriction map $h|f^{-1}(y) : f^{-1}(y) \to h(f^{-1}(y))$. Let $\mathcal{H}_i(y)$, where $y \in Y$ and $i \geqslant 1$, be the set of all $g \in C(X, M)$ such that the map $g \circ h_y^{-1} : h(f^{-1}(y)) \to M$ is D_i-crooked. If $F \subset Y$, then $\mathcal{H}_i(F)$ is the intersection of all $\mathcal{H}_i(y)$, $y \in F$. Obviously, if $Y = \bigcup_{m=1}^{\infty} Y_m$ with each Y_m being closed in Y, the set $\mathcal{H} = \bigcap_{i,m=1}^{\infty} \mathcal{H}_i(Y_m)$ consists of maps $g \in C(X, M)$ such that all restriction maps $g|f^{-1}(y)$, $y \in Y$, are Bing maps. Therefore, it suffices to show that $\mathcal{H}_i(F)$ is open and dense in $C(X, M)$ with respect to the source limitation topology for any closed $F \subset Y$ in any of the following cases: (i) F is finite-dimensional provided Y is strongly countable-dimensional and M is a complete free *ANR*-space; (ii) F is arbitrary if Y is a C-space and M a closed convex free subspace of a Banach space.

2.1. Every $\mathcal{H}_i(F)$ is open in $C(X, M)$

In this subsection we suppose that M is a fix completely metrizable *ANR* and Y a metrizable space.

Lemma 2.1. *The space M admits a complete bounded metric ϱ generating its topology and satisfying the following condition: If Z is a paracompact space, $A \subset Z$ a closed set and $\varphi : Z \to M$ a map, then for every function $\alpha : Z \to (0, 1]$ and every map $g : A \to M$ with $\varrho(g(z), \varphi(z)) < \alpha(z)/8$ for all $z \in A$, there exists a map $\bar{g} : Z \to M$ extending g such that $\varrho(\bar{g}(z), \varphi(z)) < \alpha(z)$ for all $z \in Z$.*

Proof. We embed M as a closed subset of a Banach space E. Since $M \in ANR$, there exist a neighborhood W of M in E and a retraction $r : W \to M$. For every open $U \subset M$ let $T(U) = W \backslash r^{-1}(M \backslash U)$. Obviously, $T(U) \subset W$ is open, $T(U) \cap M = U$ and $r(T(U)) = U$. Let $\mathcal{T}$ be the collection of all pairs (U, V) of open sets in M such that $\overline{conv(V)} \subset T(U)$, where $\overline{conv(V)}$ is the closed convex hull of V in E. The family $\mathcal{T}$ has the following properties: (i) for any $z \in M$ and its neighborhood U in M there is a neighborhood $V \subset U$ of z with $(U, V) \in \mathcal{T}$; (ii) for any $(U, V) \in \mathcal{T}$ and open sets $U', V' \subset M$ we have $(U', V') \in \mathcal{T}$ provided $U \subset U'$ and $V' \subset V$. By [6, Proposition 2.3] (see also [1, Lemma 6.7]), there exists a complete bounded metric ϱ on M such that for every $z \in M$ and $\delta \in (0, 1)$ the pair of open balls $(B_\varrho(z, \delta), (B_\varrho(z, \delta/8))$ belongs to $\mathcal{T}$.

Suppose now that $A \subset Z$ is closed, $\varphi : Z \to M$ and $g : A \to M$, where Z is paracompact and $\varrho(g(z), \varphi(z)) < \alpha(z)/8$ for all $z \in A$. Consider the set-valued map $\Phi : Z \to E$, $\Phi(z) = g(z)$ if $z \in A$ and $\Phi(z) = \overline{conv(B_\varrho(\varphi(z), \alpha(z)/8))}$ if $z \notin A$. Then Φ is lower semi-continuous and has closed and convex values in E. So, by the Michael convex-valued selection theorem [19], Φ has a continuous selection g_1. According to the definition of $\mathcal{T}$, every $\overline{conv(B_\varrho(\varphi(z), \alpha(z)/8))}$ is contained in $r^{-1}(B_\varrho(\varphi(z), \alpha(z)))$. Hence, $\bar{g} = r \circ g_1$ is the required extension of g. □

Everywhere below we equip M with a complete metric ϱ satisfying the hypotheses of Lemma 2.1.

Lemma 2.2. *Let $g \in \mathcal{H}_i(y)$ for some $y \in Y$ and $i \geqslant 1$. Then there exists a neighborhood V_y of y in Y and $\delta_y > 0$ such that $y' \in V_y$ and $\varrho(g_1(x), g(x)) < \delta_y$ for all $x \in f^{-1}(y')$ yields $g_1 \in \mathcal{H}_i(y')$.*

Proof. Since $g \in \mathcal{H}_i(y)$, $g \circ h_y^{-1} : h(f^{-1}(y)) \to M$ is D_i-crooked. Hence, $h(f^{-1}(y) \cap g^{-1}(t))$ is D_i-crooked for every $t \in g(f^{-1}(y))$. Consequently, $h(f^{-1}(y) \cap g^{-1}(t))$ has a neighborhood $W(t)$ in Q which is also D_i-crooked. Then $U(t) = h^{-1}(W(t))$ is a neighborhood of $f^{-1}(y) \cap g^{-1}(t)$ in X. Hence, we can find neighborhoods $V_y(t)$ and $V(t) = B(t, 3\delta(t))$ of y and t in Y and M, respectively, such that $G(t, 3\delta(t)) = f^{-1}(V_y(t)) \cap g^{-1}(V(t)) \subset U(t)$ (this can be done because f is perfect, so is the map $f \triangle g : X \to Y \times M$). Here, $B(t, 3\delta(t))$ denotes the open ball in (M, ϱ) with center t and radius $3\delta(t)$. Next, choose finitely many points $\{t_j : j = 1, 2, \ldots, k\}$ and a neighborhood V_y of y with $V_y \subset \bigcap_{j=1}^{j=k} V_y(t_j)$ and $f^{-1}(y) \subset f^{-1}(V_y) \subset \bigcup_{j=1}^{j=k} G(t_j, \delta(t_j))$, and let $\delta_y = \min\{\delta(t_j): j = 1, \ldots, k\}$. Let us show that V_y and δ_y satisfy the requirement of the lemma. Suppose $y' \in V_y$ and $g_1 \in C(X, M)$ with $\varrho(g_1(x), g(x)) < \delta_y$ for all $x \in f^{-1}(y')$. Then any $x \in f^{-1}(y')$ is contained in some $G(t_j, \delta(t_j))$. So, $g(x) \in B(t_j, \delta(t_j))$. It easily follows that $g(g_1^{-1}(g_1(x))) \subset V(t_j)$. Hence, $g_1^{-1}(g_1(x)) \cap f^{-1}(y') \subset g^{-1}(V(t_j)) \cap f^{-1}(V_y(t_j)) \subset U(t_j)$. Consequently, $h(g_1^{-1}(g_1(x)) \cap f^{-1}(y')) \subset W(t_j)$ which implies that $h(g_1^{-1}(g_1(x)) \cap f^{-1}(y'))$ is D_i-crooked because so is $W(t_j)$. Therefore, $g_1 \circ h_{y'}^{-1}$ is a D_i-crooked map, i.e., $g_1 \in \mathcal{H}_i(y')$. This completes the proof. □

Now, we are in a position to show that the sets $\mathcal{H}_i(Y)$ are open in $C(X, M)$.

Proposition 2.3. *For any set $F \subset Y$ and any $i \geqslant 1$, the set $\mathcal{H}_i(F)$ is open in $C(X, M)$ with respect to the source limitation topology.*

Proof. Let $F \subset Y$ and $g_0 \in \mathcal{H}_i(F)$. Then, by Lemma 2.2, for every $y \in F$ there exist a neighborhood V_y and a positive $\delta_y \leqslant 1$ such that $g \in \mathcal{H}_i(V_y)$ provided $g|f^{-1}(V_y)$ is δ_y-close to $g_0|f^{-1}(V_y)$. The family $\{V_y \cap Y: y \in F\}$ can be supposed to be locally finite in F. Consider the set-valued lower semi-continuous map $\varphi : F \to (0, 1]$, $\varphi(y) = \bigcup\{(0, \delta_z]: y \in V_z\}$. By [23, Theorem 6.2, p. 116], φ admits a continuous selection $\beta : F \to (0, 1]$. Let $\overline{\beta} : Y \to (0, 1]$ be a continuous extension of β and $\alpha = \overline{\beta} \circ f$. It remains only to show that if $g \in C(X, M)$ with $\varrho(g_0(x), g(x)) <$

$\alpha(x)/8$ for all $x \in X$, then $g \in \mathcal{H}_i(F)$. So, we take such a g and fix $y \in F$. Then there exists $z \in F$ with $y \in V_z$ and $\alpha(x) \leqslant \delta_z$ for all $x \in f^{-1}(y)$. By Lemma 2.1, we can select a map $g' \in C(X, M)$ coinciding with g on $f^{-1}(y)$ and satisfying the inequality $\varrho(g'(x), g_0(x)) \leqslant \delta_z$ for each $x \in X$. According to the choice of V_z, $g' \in \mathcal{H}_i(y)$. Hence, $g \in \mathcal{H}_i(y)$ because $g|f^{-1}(y) = g'|f^{-1}(y)$. Therefore, $\mathcal{H}_i(F)$ is open in $C(X, M)$. □

2.2. *Every $\mathcal{H}_i(F)$ is dense in $C(X, M)$*

In this subsection we suppose M is a free *ANR*-space equipped with a complete metric ϱ satisfying the hypotheses of Lemma 2.1. We are going to show if $F \subset Y$ is closed, then the set $\mathcal{H}_i(F)$ is dense in $C(X, M)$ with respect to the source limitation topology in any of the following cases: (i) F is finite-dimensional; (ii) M is a convex free subset of a Banach space. In any of these two cases we need to show that $B_\varrho(g, \varepsilon) = \{g' \in C(X, M)\colon\ \varrho(g, g') < \varepsilon\}$ meets $\mathcal{H}_i(F)$ for every $g \in C(X, M)$ and every continuous function $\varepsilon : X \to (0, 1]$. To this end, fix $g_0 \in C(X, M)$ and $\varepsilon \in C(X, (0, 1])$. Consider the set-valued map $\Phi_\varepsilon : Y \to C(X, M)$, $\Phi_\varepsilon(y) = \mathcal{H}_i(y) \cap B_\varrho(g_0, \varepsilon)$, where $C(X, M)$ carries the compact open topology.

Lemma 2.4. *The map Φ_ε has the following property: If $y_0 \in Y$ and a compactum K is contained in $\Phi_\varepsilon(y_0)$, then there exists a neighborhood $V(y_0)$ of y_0 such that $K \subset \Phi_\varepsilon(y)$ for every $y \in V(y_0)$.*

Proof. Suppose there exists a sequence $\{y_j\}_{j \geqslant 1}$ converging to y_0 in Y such that $K \setminus \Phi_\varepsilon(y_j) \neq \emptyset$. Let $g_j \in K \setminus \Phi_\varepsilon(y_j)$, $j \geqslant 1$, and $P = f^{-1}(Y_0)$, where $Y_0 = \{y_j\}_{j \geqslant 1} \cup \{y_0\}$. Since the restriction map $\pi_P : C(X, M) \to C(P, M)$ is continuous when both $C(X, M)$ and $C(P, M)$ are equipped with the compact open topology, there exists a subsequence $\{g_{j_k}\}$ of $\{g_j\}$ such that $\pi_P(g_{j_k})$ converges to $\pi_P(g)$ in $C(P, M)$ for some $g \in K$. Obviously $g \in \Phi_\varepsilon(y_0) \subset \mathcal{H}_i(y_0)$. So, we can apply Lemma 2.2 to find a neighborhood V of y_0 in Y and a positive $\delta > 0$ such that $y' \in V$ and $\varrho(g(x), g'(x)) < \delta$ for all $x \in f^{-1}(y')$ implies $g' \in \mathcal{H}_i(y')$. Since $\pi_P(g_{j_k})$ converges to $\pi_P(g)$ in $C(P, M)$ and the compact open topology on $C(P, M)$ coincides with the uniform convergence (recall that P is compact), there exists j_k with $y_{j_k} \in V$ and $\varrho(g(x), g_{j_k}(x)) < \delta$ for any $x \in f^{-1}(y_{j_k})$. Hence, $g_{j_k} \in \mathcal{H}_i(y_{j_k})$. Consequently, $g_{j_k} \in \Phi_\varepsilon(y_{j_k})$ which contradicts the choice of the functions g_j. □

Lemma 2.5. *Every $\Phi_\varepsilon(y)$ has the following property: If $\hat{v} : \mathbb{S}^n \to \Phi_\varepsilon(y)$ is continuous, where $n \geqslant 0$, then $\hat{v}$ can be extended to a continuous map $\hat{u} : \mathbb{B}^{n+1} \to \Phi_{64\varepsilon}(y)$.*

Proof. Let us mention the following property of the function space $C(X, M)$ equipped with the compact open topology: For any metrizable space Z a map $\hat{w} : Z \to C(X, M)$ is continuous if and only if the map $w : Z \times X \to M$, $w(z, x) = \hat{w}(z)(x)$, is continuous. Hence, every map $\hat{v} : \mathbb{S}^n \to \Phi_\varepsilon(y)$ generates a continuous map $v : \mathbb{S}^n \times X \to M$ defined by $v(z, x) = \hat{v}(z)(x)$ such that $\varrho(v(z, x), g_0(x)) < \varepsilon(x)$ for all $(z, x) \in \mathbb{S}^n \times X$.

Claim. *Let $F \subset X$ be closed and $\pi_F : C(X, M) \to C(F, M)$ be the restriction map. Then π_F is continuous and open when both $C(X, M)$ and $C(F, M)$ are equipped with the source limitation topology.*

Suppose $G \subset C(F, M)$ is open and $\pi_F(g_0) \in G$ for some $g_0 \in C(X, M)$. Then there exists $\eta \in C(F, (0, 1])$ such that $\pi_F(g) \in G$ provided $g \in C(X, M)$ and $\varrho(g_0(x), g(x)) < \eta(x)$ for all $x \in F$. Obviously, $B_\varrho(g_0, \bar{\eta}) \subset \pi_F^{-1}(G)$, where $\bar{\eta} : X \to (0, 1]$ is a continuous extension of η. So, π_F is continuous. To show that π_F is open, let $W \subset C(X, M)$ be open and $\pi_F(g) \in \pi_F(W)$, $g \in W$. Hence, $B_\varrho(g, \alpha) \subset W$ for some continuous function $\alpha : X \to (0, 1]$. Then

$$O(g) = \bigl\{q \in C(F, M)\colon\ \varrho\bigl(q(x), g(x)\bigr) < \alpha(x)/8 \text{ for all } x \in F\bigr\}$$

is a neighborhood of $\pi_F(g)$ in $C(F, M)$. According to Lemma 2.1, $O(g) \subset \pi_F(\mathcal{H}_i(y))$ which completes the proof of the claim.

Now, let π_y be the restriction map $\pi_{f^{-1}(y)}$. Then, by the above claim, $\pi_y(\mathcal{H}_i(y))$ is open in $C(f^{-1}(y), M)$ with respect to the source limitation topology. Since $f^{-1}(y)$ is compact, the source limitation, the compact open and the uniform convergence topologies on $C(f^{-1}(y), M)$ coincide. On the other hand, π_y is continuous when both $C(X, M)$ and $C(f^{-1}(y), M)$ carry the compact open topology. Hence, $\pi_y(\hat{v}(\mathbb{S}^n))$ is a compact subset of $\pi_y(\mathcal{H}_i(y))$ and $\pi_y(\mathcal{H}_i(y))$ is open in $C(f^{-1}(y), M)$, where $C(f^{-1}(y), M)$ carries the uniform convergence topology generated by the metric ϱ. Consequently, there is $\delta_1 > 0$ such that if $\hat{\beta}: \mathbb{S}^n \to C(f^{-1}(y), M)$ and $\varrho(\beta(z,x), v(z,x)) < \delta_1$ for all $(z,x) \in \mathbb{S}^n \times f^{-1}(y)$, then $\hat{\beta}(\mathbb{S}^n) \subset \pi_y(\mathcal{H}_i(y))$.

Define $\delta_2 = \inf\{\varepsilon(x) - \varrho(v(z,x), g_0(x))\colon (z,x) \in \mathbb{S}^n \times f^{-1}(y)\}$. Obviously, $\delta_2 > 0$. According to Lemma 2.1, there exists a continuous extension $v_1: \mathbb{B}^{n+1} \times f^{-1}(y) \to M$ of the map $v|(\mathbb{S}^n \times f^{-1}(y))$ with $\varrho(v_1(z,x), g_0(x)) < 8\varepsilon(x)$ for all $(z,x) \in \mathbb{B}^{n+1} \times f^{-1}(y)$. Since $\mathbb{B}^{n+1} \times f^{-1}(y)$ is compact and M is a free space, we can find a Bing map $v_2: \mathbb{B}^{n+1} \times f^{-1}(y) \to M$ with $\varrho(v_2(z,x), v_1(z,x)) < \delta/8$ for all $(z,x) \in \mathbb{B}^{n+1} \times f^{-1}(y)$, where $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and $\delta_3 = \inf\{8\varepsilon(x) - \varrho(v_1(z,x), g_0(x))\colon (z,x) \in \mathbb{B}^{n+1} \times f^{-1}(y)\}$. Therefore, we have a map $\hat{v}_2: \mathbb{B}^{n+1} \to C(f^{-1}(y), M)$. The choice of δ_3 implies

$$\varrho\big(v_2(z,x), g_0(x)\big) < 8\varepsilon(x) \tag{1}$$

for all $(z,x) \in \mathbb{B}^{n+1} \times f^{-1}(y)$. Moreover, since v_2 is a Bing map, so are the maps $\hat{v}_2(z): f^{-1}(y) \to M$, $z \in \mathbb{B}^{n+1}$. On the other hand, by Lemma 2.1 and (1), every map $\hat{v}_2(z)$ can be extended to a map from X into M. Therefore,

$$\hat{v}_2(\mathbb{B}^{n+1}) \subset \pi_y\big(\mathcal{H}_i(y)\big). \tag{2}$$

Representing the ball $\mathbb{B}^{n+1}$ as a cone with a base $\mathbb{S}^n$ and a vertex z_0, we can consider v_2 as a homotopy from $\mathbb{S}^n \times f^{-1}(y) \times [0,1]$ into M between the maps $v_2|(\mathbb{S}^n \times f^{-1}(y) \times \{0\})$ and $v_2|(\{z_0\} \times f^{-1}(y))$. Observe also that $\varrho(v_2(z,x,0), v(z,x)) < \delta/8$ for any $(z,x) \in \mathbb{S}^n \times f^{-1}(y)$. Let E be the Banach space from the proof of Lemma 2.1 and $\varphi: \mathbb{S}^n \times f^{-1}(y) \times [0,1] \to E$ be the set valued map defined by

$$\varphi(z,x,t) = \begin{cases} v(z,x) & \text{if } t = 0; \\ v_2(z,x,0) & \text{if } t = 1; \\ \overline{conv(B_\varrho(v(z,x), \delta/8))} & \text{if } t \in (0,1). \end{cases}$$

By the convex-valued selection theorem of Michael, φ admits a continuous selection s. It follows from the proof of Lemma 2.1 that $v_3 = r \circ s$ is a map from $\mathbb{S}^n \times f^{-1}(y) \times [0,1]$ into M such that $v_3(z,x,0) = v(z,x)$, $v_3(z,x,1) = v_2(z,x,0)$ and $\varrho(v_3(z,x,t), v(z,x)) < \delta$ for every $(z,x,t) \in \mathbb{S}^n \times f^{-1}(y) \times [0,1]$. Since $\delta < \min\{\delta_1, \delta_2\}$, for any $(z,x,t) \in \mathbb{S}^n \times f^{-1}(y) \times [0,1]$ we have

$$\varrho\big(v_3(z,x,t), v(z,x)\big) < \delta_1, \tag{3}$$

and

$$\varrho\big(v_3(z,x,t), g_0(x)\big) < \varepsilon(x). \tag{4}$$

Therefore, v_3 is a homotopy connecting the maps v and $v_2|(\mathbb{S}^n \times f^{-1}(y) \times \{0\})$, while v_2 is a homotopy connecting the maps $v_2|(\mathbb{S}^n \times f^{-1}(y) \times \{0\})$ and $v_2|(\{z_0\} \times f^{-1}(y))$. Combining these two homotopies, we obtain a map $u_1: \mathbb{S}^n \times f^{-1}(y) \times [0,1] \to M$ such that $u_1(z,x,0) = v(z,x)$, $u_1(z,x,1) = v_2(z_0,x)$ and $\varrho(u_1(z,x,t), g_0(x)) < 8\varepsilon(x)$ for all $(z,x,t) \in \mathbb{S}^n \times f^{-1}(y) \times [0,1]$. Obviously, u_1 can also be considered as a map from $\mathbb{B}^{n+1} \times f^{-1}(y)$ into M such that $u_1|(\mathbb{S}^n \times f^{-1}(y)) = v$ and $\varrho(u_1(z,x), g_0(x)) < 8\varepsilon(x)$, $(z,x) \in \mathbb{B}^{n+1} \times f^{-1}(y)$. Now consider the map $u_2: (\mathbb{B}^{n+1} \times f^{-1}(y)) \cup (\mathbb{S}^n \times X) \to M$ with $u_2|(\mathbb{B}^{n+1} \times f^{-1}(y)) = u_1$ and $u_2|(\mathbb{S}^n \times X) = v$. Finally, using Lemma 2.1, we extend u_2 to a map $u: \mathbb{B}^{n+1} \times X \to M$ such that

$$\varrho\big(u(z,x), g_0(x)\big) < 64\varepsilon(x) \tag{5}$$

for any $(z,x) \in \mathbb{B}^{n+1} \times X$. Then $\hat{u}: \mathbb{B}^{n+1} \to C(X,M)$ extends the map $\hat{v}$. Moreover, (2), (3) and the choice of δ_1 implies that $\hat{u}(\mathbb{B}^{n+1}) \subset \mathcal{H}_i(y)$. On the other hand, (5) yields $\hat{u}(\mathbb{B}^{n+1}) \subset B_\varrho(g_0, 64\varepsilon)$. Hence, $\hat{u}(\mathbb{B}^{n+1}) \subset \Phi_{64\varepsilon}(y)$. □

Next proposition completes the proof of Theorem 1.1 in the case Y is strongly countable-dimensional.

Proposition 2.6. *The sets $\mathcal{H}_i(F)$, $i \geqslant 1$, are dense in $C(X,M)$ with respect to the source limitation topology for any closed finite-dimensional set $F \subset Y$.*

Proof. If $F \subset Y$ is closed and $\dim F \leqslant n$, we consider the sequence of the set-valued maps $\Phi_j: F \to C(X,M)$, $j = 0, \dots, n$, defined by $\Phi_j(y) = \Phi_{\varepsilon/8^{2(n-j)+1}}(y)$. Obviously, $\Phi_0(y) \subset \Phi_1(y) \subset \cdots \subset \Phi_n(y) = \Phi_{\varepsilon/8}(y)$. According to Lemma 2.5, every map from $\mathbb{S}^n$ into $\Phi_j(y)$ can be extended to a map from $\mathbb{B}^{n+1}$ into $\Phi_{j+1}(y)$, where

$j = 0, 1, \ldots, n-1$ and $y \in F$. Moreover, by Lemma 2.4, any $\Phi_j(y)$ has the following property: if $K \subset \Phi_j(y)$ is compact, then there exists a neighborhood V_y of y in Y such that $K \subset \Phi_j(z)$ for all $z \in V_y$. So, we may apply [9, Theorem 3.1] to find a continuous selection $\theta: F \to C(X, M)$ of Φ_n. Hence, $\theta(y) \in \Phi_{\varepsilon/8}(y)$ for all $y \in F$. Now, define a map $g: f^{-1}(F) \to M$, $g(x) = \theta(f(x))(x)$. To show that g is continuous, fix a sequence $\{x_k\} \subset f^{-1}(F)$ converging to some $x_0 \in f^{-1}(F)$ and let $y_k = f(x_k)$, $k \geqslant 0$. Since $C(X, M)$ carries the compact open topology, the sequence $\{\theta(y_k)\}$ restricted to any compact set $P \subset f^{-1}(F)$ converges uniformly to $\theta(y_0)|P$. Letting $P = f^{-1}(\{y_k\}_{k\geqslant 0})$, one can easily show that $\{g(x_k\}$ converges to $g(x_0)$. Hence g is continuous and $\varrho(g(x), g_0(x)) < \varepsilon(x)/8$ for all $x \in f^{-1}(F)$. Then, by Lemma 2.1, g can be extended to a continuous map $\bar{g}: X \to M$ with $\varrho(\bar{g}(x), g_0(x)) < \varepsilon(x)$, $x \in X$. It follows from the definition of g that $g|f^{-1}(y) = \theta(y)|f^{-1}(y)$ for every $y \in F$. Since $\theta(y) \in \mathcal{H}_i(y)$, we have $\bar{g} \in \mathcal{H}_i(F)$. Hence, $B_\varrho(g_0, \varepsilon) \cap \mathcal{H}_i(F) \neq \emptyset$. □

We now turn to the proof of Theorem 1.1 in the case Y is a C-space and M a closed convex free subset of a Banach space E. Let ϱ be the metric on M inherited from the norm of E and $\Psi_\varepsilon: Y \to C(X, M)$ be the set-valued map $\Psi_\varepsilon(y) = \overline{B}_\varrho(g_0, \varepsilon) \cap \mathcal{H}_i(y)$, where $C(X, M)$ is equipped with the compact open topology and

$$\overline{B}_\varrho(g_0, \varepsilon) = \bigl\{g \in C(X, M)\colon\ \varrho\bigl(g_0(x), g(x)\bigr) \leqslant \varepsilon(x) \text{ for all } x \in X\bigr\}.$$

Lemma 2.7. *Ψ_ε has the following property: Every map $\hat{v}: \mathbb{S}^n \to \Psi_\varepsilon(y)$, $n \geqslant 0$, can be extended to a map $\hat{u}: \mathbb{B}^{n+1} \to \Psi_\varepsilon(y)$.*

Proof. All function spaces in this proof are equipped with the compact open topology. Let $\pi_y: C(X, M) \to C(f^{-1}(y), M)$ be the restriction map and $P(y) = \overline{B}_\varrho(g_0, \varepsilon, y) \backslash \pi_y(\mathcal{H}_i(y))$, where $\overline{B}_\varrho(g_0, \varepsilon, y)$ is the set

$$\bigl\{g \in C\bigl(f^{-1}(y), M\bigr)\colon\ \varrho\bigl(g_0(x), g(x)\bigr) \leqslant \varepsilon(x) \text{ for all } x \in f^{-1}(y)\bigr\}.$$

According to the proof of Lemma 2.5, $P(y)$ is closed in $\overline{B}_\varrho(g_0, \varepsilon, y)$.

We are going to show that $P(y)$ is a Z-set in $\overline{B}_\varrho(g_0, \varepsilon, y)$, i.e., every map $\hat{w}: K \to \overline{B}_\varrho(g_0, \varepsilon, y)$, K is any compact, can be approximated by a map $\hat{w}_1: K \to \overline{B}_\varrho(g_0, \varepsilon, y) \backslash P(y) = \overline{B}_\varrho(g_0, \varepsilon, y) \cap \pi_y(\mathcal{H}_i(y))$. To this end, fix $\delta > 0$ and let $w: K \times f^{-1}(y) \to M$ be the map generated by $\hat{w}$. So, $\varrho(w(z, x), g_0(x)) \leqslant \varepsilon(x)$ for all $(z, x) \in K \times f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there exists $\lambda \in (0, 1)$ such that $\lambda \max\{\varepsilon(x)\colon x \in f^{-1}(y)\} < \delta/2$. Define the map $w_1: K \times f^{-1}(y) \to M$ by $w_1(z, x) = (1-\lambda)w(z, x) + \lambda g_0(x)$. Then, for all $(z, x) \in K \times f^{-1}(y)$ we have

$$\varrho\bigl(w_1(z, x), w(z, x)\bigr) \leqslant \lambda\varepsilon(x) < \delta/2$$

and

$$\varrho\bigl(w_1(z, x), g_0(x)\bigr) \leqslant (1-\lambda)\varepsilon(x) < \varepsilon(x).$$

Since M is a free space, there exists a Bing map $w_2: K \times f^{-1}(y) \to M$ which is δ_1-close to w_1, where $\delta_1 = \min\{\lambda\varepsilon(x)\colon x \in f^{-1}(y)\}$. Hence, $\varrho(w_2(z, x), g_0(x)) \leqslant \varepsilon(x)$ and $\varrho(w_2(z, x), w(z, x)) < \delta$, $(z, x) \in K \times f^{-1}(y)$. The last two inequalities imply that the map $\hat{w}_2: K \to C(f^{-1}(y), M)$ is δ-close to $\hat{w}$ and $\hat{w}_2(K) \subset \overline{B}_\varrho(g_0, \varepsilon, y)$. Moreover, every map $\hat{w}_2(z): f^{-1}(y) \to M$, $z \in K$, can be extended to a map from X to M because M is a closed convex subset of E. Since w_2 is a Bing map, so are the maps $\hat{w}_2(z)$, $z \in K$. Hence, $\hat{w}_2(K) \subset \pi_y(\mathcal{H}_i(y))$. So, $P(y)$ is a Z-set in $\overline{B}_\varrho(g_0, \varepsilon, y)$.

Now we can complete the proof of the lemma. For every map $\hat{v}: \mathbb{S}^n \to \Psi_\varepsilon(y)$ the composition $\pi_y \circ \hat{v}$ is a map from $\mathbb{S}^n$ into $\overline{B}_\varrho(g_0, \varepsilon, y) \cap \pi_y(\mathcal{H}_i(y))$. Since $P(y)$ is a Z-set in the convex set $\overline{B}_\varrho(g_0, \varepsilon, y)$, by [25, Proposition 6.3], there exists a map $\hat{v}_1: \mathbb{B}^{n+1} \to \overline{B}_\varrho(g_0, \varepsilon, y) \cap \pi_y(\mathcal{H}_i(y))$ extending $\pi_y \circ \hat{v}$. Consider the map $v_2: A \to M$, where $A = (\mathbb{B}^{n+1} \times f^{-1}(y)) \cup (\mathbb{S}^n \times X)$, defined by $v_2|(\mathbb{B}^{n+1} \times f^{-1}(y)) = v_1$ and $v_2|(\mathbb{S}^n \times X) = v$. Next, take a selection $u: \mathbb{B}^{n+1} \times X \to M$ for the set-valued map $\phi: \mathbb{B}^{n+1} \times X \to M$, $\phi(z, x) = v_2(z, x)$ if $(z, x) \in A$ and $\phi(z, x) = \overline{B_\varrho(g_0(x), \varepsilon(x))}$ if $(z, x) \notin A$. Such u exists by Michael's convex-valued selection theorem. Obviously u extends v_2 and $\varrho(u(z, x), g_0(x)) \leqslant \varepsilon(x)$ for every $(z, x) \in \mathbb{B}^{n+1} \times X$. Finally, observe that $\hat{u}$ is the required extension of $\hat{v}$. □

We can finally finish the proof of Theorem 1.1.

Proposition 2.8. *Suppose Y is a C-space and M a closed convex free subset of a Banach space E. Then the sets $\mathcal{H}_i(F)$, $i \geqslant 1$, are dense in $C(X,M)$ with respect to the source limitation topology for any closed set $F \subset Y$.*

Proof. For every ε, i and $g_0 \in C(X,M)$ consider the set-valued map $\Psi_\varepsilon : F \to C(X,M)$. It follows from the proof of Lemma 2.4 that if $K \subset \Psi_\varepsilon(y_0)$ for some compactum K and $y_0 \in F$, then y_0 admits a neighborhood V with $K \subset \Psi_\varepsilon(y)$ for all $y \in V$. Moreover, according to Lemma 2.7, every image $\Psi_\varepsilon(y)$ is aspherical, i.e., any map from $\mathbb{S}^n$ into $\Psi_\varepsilon(y)$, $n \geqslant 0$, can be extended to a map from $\mathbb{B}^{n+1}$ to $\Psi_\varepsilon(y)$. Then, by the Uspenskij selection theorem [25, Theorem 1.3], Ψ_ε admits a continuous selection $\theta : F \to C(X,M)$. Repeating the arguments from the proof of Proposition 2.6, we obtain a map $g : f^{-1}(F) \to M$ such that $\varrho(g(x), g_0(x)) \leqslant \varepsilon(x)$ for every $x \in f^{-1}(F)$ and $g|f^{-1}(y) = \theta(y)|f^{-1}(y)$, $y \in F$. Applying once more the Michael convex-valued selection theorem for the set-valued map $\vartheta : X \to M$, $\vartheta(x) = g(x)$ if $x \in F$ and $\vartheta(x) = \overline{B}_\varrho(g_0(x), \varepsilon(x))$ if $x \notin F$, we obtain a selection $\bar{g}$ for ϑ. Obviously, $\bar{g}$ extends g and $\bar{g} \in \overline{B}_\varrho(g_0, \varepsilon)$. Since $\theta(y) \in \mathcal{H}_i(y)$ for all $y \in F$, we have $\bar{g} \in \overline{B}_\varrho(g_0, \varepsilon) \cap \mathcal{H}_i(F)$. Hence, $\mathcal{H}_i(F)$ is dense in $C(X,M)$. □

3. Proof of Theorem 1.2

Let $f : X \to Y$ be a perfect map between with X, Y metrizable and Y is a C-space. We say that $g \in C(X,\mathbb{I})$ is an f-Krasinkiewicz map if the restrictions $g|f^{-1}(y)$ are Krasinkiewicz maps for every $y \in Y$. We are going to show that all f-Krasinkiewicz maps contains a dense G_δ-subset of $C(X,\mathbb{I})$ with respect to the source limitation topology. To this end, we use an idea of Levin–Lewis from the proof of [17, Proposition 3.3]. We fix a metric d on X and for every $y \in Y$, natural number i and rational numbers p, q with $0 \leqslant p < q \leqslant 1$ let $\mathcal{K}_i(p,q,y) = C(X,\mathbb{I}) \setminus \mathcal{L}_i(p,q,y)$, where $\mathcal{L}_i(p,q,y)$ is the set of all $g \in C(X,\mathbb{I})$ satisfying the following condition: there exists a continuum $F \subset f^{-1}(y)$ such that $[p,q] \subset g(F)$ and for every $t \in [p,q]$ there exists a component C of $g^{-1}(t) \cap f^{-1}(y)$ and a point $x \in C$ with $C \cap F \neq \emptyset$ and $d(x,F) \geqslant 1/i$. For a set $H \subset Y$ we denote by $\mathcal{K}_i(p,q,H)$ the intersection of all $\mathcal{K}_i(p,q,y)$, $y \in H$. It is easily seen that

$$\mathcal{K} = \bigcap \{ \mathcal{K}_i(p,q,Y) \colon i \geqslant 1 \text{ and } p,q \text{ rationals} \}$$

consists of f-Krasinkiewicz maps. Moreover every $g \in \mathcal{K}$ has the following property:

(∗) For every $y \in Y$ and every continuum $F \subset f^{-1}(y)$ with $g(F)$ being not a singleton there exists a dense subset $D \subset g(F)$ such that $g^{-1}(t) \cap F$ is the union of components of $g^{-1}(t) \cap f^{-1}(y)$ for every $t \in D$.

Therefore, it suffices to show that each set $\mathcal{K}_i(p,q,Y)$ is open and dense in $C(X,\mathbb{I})$ with respect to the source limitation topology.

Proposition 3.1. *For every closed $H \subset Y$, the set $\mathcal{K}_i(p,q,H)$ is open in $C(X,\mathbb{I})$.*

Proof. We are going first to prove the following

Claim. *If $g \in \mathcal{K}_i(p,q,y)$ for some $y \in Y$, then there exists a neighborhood V_y of y in Y and $\delta_y > 0$ such that $y' \in V_y$ and $|g'(x) - g(x)| < \delta_y$ for all $x \in f^{-1}(y')$ yields $g' \in \mathcal{K}_i(p,q,y')$.*

Indeed, otherwise we can find a local base of sequences $\{V_n\}$ of neighborhoods of y in Y, points $y_n \in V_n$ and functions $g_n \in C(X,\mathbb{I})$ such that $|g_n(x) - g(x)| < 1/n$ for all $x \in f^{-1}(y_n)$ but $g_n \notin \mathcal{K}_i(p,q,y_n)$, $n \geqslant 1$. Hence, $g_n \in \mathcal{L}_i(p,q,y_n)$ for all n. Consequently, for every n there is a continuum $F_n \subset f^{-1}(y_n)$ such that $[p,q] \subset g_n(F_n)$ and for any $t \in [p,q]$ there exists a component $C_n(t)$ of $g_n^{-1}(t) \cap f^{-1}(y_n)$ and a point $x_n(t) \in C_n(t)$ with $C_n(t) \cap F_n \neq \emptyset$ and $d(x_n(t), F_n) \geqslant 1/i$. Then all F_n are contained in the compact set $P = f^{-1}(\{y_n\} \cup \{y\})$. Passing to subsequences, we may suppose that $\{F_n\}$, considered as a sequence in the space $\exp_C(P)$ of all subcontinua of P equipped with the Hausdorff metric generated by d, converges to a continuum F. It follows that $F \subset f^{-1}(y)$ and $[p,q] \subset g(F)$. Since $g \in \mathcal{K}_i(p,q,y)$, there exists $t_0 \in [p,q]$ such that $d(C,F) < 1/i$ for every continuum $C \subset g^{-1}(t_0) \cap f^{-1}(y)$ with $C \cap F \neq \emptyset$. Next, consider the sequences $\{C_n(t_0)\}$ and $\{x_n(t_0)\}$. Passing again to subsequences, we may assume that $\{C_n(t_0)\}$ converges in $\exp_C(P)$ to a continuum C_0 and $\{x_n(t_0)\}$ converges in P to a point x_0. It is easily seen that

$C_0 \subset g^{-1}(t_0) \cap f^{-1}(y)$, $C_0 \cap F \neq \emptyset$, $x_0 \in C_0$ and $d(x_0, F) \geqslant 1/i$. On the other hand, according to the choice of t_0 we have $d(C_0, F) < 1/i$. This contradiction completes the proof of the claim.

Now, repeating the arguments from the proof of Proposition 2.3 and applying the above claim instead of Lemma 2.2, we can show that every $\mathcal{K}_i(p, q, H)$, $H \subset Y$ is closed, is open in $C(X, \mathbb{I})$. □

Proposition 3.2. *Every $\mathcal{K}_i(p, q, Y)$ is dense in $C(X, \mathbb{I})$ with respect to the source limitation topology.*

Proof. All function spaces in this proof are considered with the compact open topology. For every $\varepsilon \in C(X, (0, 1])$ and $g_0 \in C(X, \mathbb{I})$ consider the set-valued map $\Omega_\varepsilon : Y \to \mathbb{I}$ defined by $\Omega_\varepsilon(y) = \overline{B}(g_0, \varepsilon) \cap \mathcal{K}_i(p, q, y)$, where

$$\overline{B}(g_0, \varepsilon) = \{g \in C(X, \mathbb{I}): |g_0(x) - g(x)| \leqslant \varepsilon(x) \text{ for all } x \in X\}.$$

Claim 1. *Every map $\hat{v} : \mathbb{S}^n \to \Omega_\varepsilon(y)$, $n \geqslant 0$, can be extended to a map $\hat{u} : \mathbb{B}^{n+1} \to \Omega_\varepsilon(y)$.*

Following the notations from the proof of Lemma 2.7, let $P(y) = \overline{B}(g_0, \varepsilon, y) \backslash \pi_y(\mathcal{K}_i(p, q, y))$, where $\overline{B}(g_0, \varepsilon, y)$ is the set

$$\{g \in C(f^{-1}(y), \mathbb{I}): |g_0(x) - g(x)| \leqslant \varepsilon(x) \text{ for all } x \in f^{-1}(y)\}.$$

One can easily show that $P(y)$ is closed in $\overline{B}(g_0, \varepsilon, y)$. As in Lemma 2.7, we show first that $P(y)$ is a Z-set in $\overline{B}(g_0, \varepsilon, y)$. The only difference is that we take now $w_2 : K \times f^{-1}(y) \to \mathbb{I}$ to be a Krasinkiewicz map on $K \times f^{-1}(y)$ satisfying the following condition: For every continuum $F : K \times f^{-1}(y)$ with $w_2(F)$ being not a singleton there is a dense subset $D \subset w_2(F)$ such that $w_2^{-1}(t) \cap F$ is the union of components of $w_2^{-1}(t)$ for every $t \in D$. This can be done by the Levin–Lewis result [17, Proposition 3.3] stating that all maps with the above property form a dense subset of $C(K \times f^{-1}(y), \mathbb{I})$. Moreover, every map $\hat{w_2}(z) : f^{-1}(y) \to \mathbb{I}$, $z \in K$, belongs to $\pi_y(\mathcal{K}_i(p, q, y))$. Then, following the arguments from the proof of Lemma 2.7, we complete the proof of the claim.

Claim 2. *The map Ω_ε has the following property: If $y_0 \in Y$ and a compactum K is contained in $\Omega_\varepsilon(y_0)$, then there exists a neighborhood V of y_0 in Y such that $K \subset \Omega_\varepsilon(y)$ for every $y \in V$.*

The proof of this claim is the same as that one of Lemma 2.4. The only difference is that we need now to apply the claim from Proposition 3.1 instead of Lemma 2.2.

Finally, because of Claims 1 and 2, we can apply the Uspenskij selection theorem [25, Theorem 3.1] to find a continuous selection $\theta : Y \to C(X, \mathbb{I})$ for the map Ω_ε. Then the map $g : X \to \mathbb{I}$, $g(x) = \theta(f(x))(x)$ is continuous and $g \in \overline{B}(g_0, \varepsilon)$. Moreover, since $g|f^{-1}(y) = \theta(y)|f^{-1}(y)$ for every $y \in Y$, we have $g \in \mathcal{K}_i(p, q, Y)$. Hence, $\mathcal{K}_i(p, q, Y)$ is dense in $C(X, \mathbb{I})$. □

4. Some remarks and questions

As we mention in the introduction, there are free spaces which are not necessarily *ANR*. Here is a version of Theorem 1.1 for such spaces.

Theorem 4.1. *Let M be a completely metrizable* LC^n *free space. Then, for every perfect map $f : X \to Y$ with* $\dim Y + \dim X \leqslant n + 1$*, the function space $C(X, M)$ contains a dense G_δ-set of maps g such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are Bing maps.*

Proof. We need the following version of Lemma 2.1, see [2].

Lemma 4.2. *Every completely metrizable* LC^n*-space M admits a complete bounded metric ϱ generating its topology and satisfying the following condition: Let Z be a paracompact space with* $\dim Z \leqslant n + 1$*, $A \subset Z$ a closed set and $\varphi : Z \to M$ a map. Then, for every function $\alpha : Z \to (0, 1]$ and every map $g : A \to M$ with $\varrho(g(z), \varphi(z)) < \alpha(z)/8$ for all $z \in A$, there exists a map $\bar{g} : Z \to M$ extending g such that $\varrho(\bar{g}(z), \varphi(z)) < \alpha(z)$ for all $z \in Z$.*

Following the proof of Theorem 1.1 (the case M is a free *ANR*-space and Y strongly countable-dimensional) and using Lemma 4.2 instead of Lemma 2.1, we can show that the sets $\mathcal{H}_i(Y)$ are open in $C(X,M)$ with respect to the source limitation topology.

To show the density of the sets $\mathcal{H}_i(Y)$ in $C(X,M)$, let $\dim Y = k$. Then the arguments from Lemma 2.5 imply that the map Φ_ε has the following property: every map $\hat{v}:\mathbb{S}^{k-1} \to \Phi_\varepsilon(y)$ admits a continuous extension $\hat{u}:\mathbb{B}^k \to \Phi_{64\varepsilon}(y)$, $y \in Y$. Next, we consider the maps $\Phi_j : Y \to C(X,M)$, $\Phi_j(y) = \Phi_{\varepsilon/8^{2(k-j)}}(y)$, $j = 0,1,\dots,k$, and use [9, Theorem 3.1] to find a continuous selection $\theta : Y \to C(X,M)$ for Φ_k. Finally, the map $g : X \to M$ defined by $g(x) = \theta(f(x))(x)$ satisfies the conditions $\varrho(g_0(x), g(x)) < \varepsilon(x)$, $x \in X$, and $g \in \mathcal{H}_i(y)$ for every $y \in Y$ (see Proposition 2.6). □

As an application of Theorem 1.1, we establish that some maps have Bula's property. Following Kato and Levin [10], a map $f : X \to Y$ is said to have the Bula property if there exist two closed disjoint subsets F_0 and F_1 of X such that $f(F_0) = f(F_1) = Y$. Bula [4] has shown that every open map f from a compact Hausdorff space onto a finite-dimensional metric space has this property provided all fibers of f are dense in themselves. Gutev [8] generalized Bula's result to the case Y is countable-dimensional. Recently, Levin and Rogers [18] obtained a further generalization with Y being a C-space. The question whether the Levin–Rogers result remains true for perfect maps between metrizable spaces is still open [11] (if Y is strongly infinite-dimensional, this is not true, see [5] and [18]). Here, we provide a partial answer to this question:

Theorem 4.3. *Let $f : X \to Y$ be a perfect open map between metrizable spaces such that Y is a C-space and no fiber of f is a Bing space. Then f has the Bula property.*

Proof. We follow the arguments from the Kato–Levin proof of [10, Theorem 2.2]. According to Theorem 1.1, there exists $g : X \to \mathbb{I}$ such that every restriction $g|f^{-1}(y)$ ia a Bing map. Since no fiber $f^{-1}(y)$ is a Bing space, each $g(f^{-1}(y))$ contains at least two different points. Let $a_y = \min\{g(x):\ x \in f^{-1}(y)\}$, $b_y = \max\{g(x):\ x \in f^{-1}(y)\}$ and $h_y : [a_y, b_y] \to [0,1]$ be the linear transformation with $h_y(a_y) = 0$ and $h_y(b_y) = 1$. Then the map $h : X \to [0,1]$, $h(x) = h_y(g(x))$ for $x \in f^{-1}(y)$, is continuous because f is both closed and open. Moreover, $0, 1 \in h(f^{-1}(y))$, $y \in Y$. Obviously, $F_0 = h^{-1}(0)$ and $F_1 = h^{-1}(1)$ are closed disjoint subsets of X with $f(F_0) = f(F_1) = Y$. □

Most probably Theorem 1.1 remains true when M is a completely metrizable *ANR*-free space and Y is a C-space. The validity of this more general version of Theorem 1.1 is reduced to the existence of an appropriate selection theorem for C-spaces. The Uspenskij theorem does not work in this case because we do not have a single set-valued map $\Phi : Y \to C(X,M)$ with aspherical images such that $\Phi(y) \subset \mathcal{H}_i(y)$ for every $y \in Y$. But we can construct a decreasing sequence of maps $\Phi_n : Y \to C(X,M)$, $n = 0,1,\dots,$ such that each pair $(\Phi_{n+1}(y), \Phi_n(y))$, $n \geqslant 0$ and $y \in Y$, is UV^n (i.e., every map $v : \mathbb{S}^n \to \Phi_{n+1}(y)$ can be extended to a map $u : \mathbb{B}^{n+1} \to \Phi_n(y)$) and each Φ_n is strongly lower semi-continuous (i.e., if $K \subset \Phi_n(y_0)$ for some compact K, then there exists a neighborhood V of y_0 such that $K \subset \Phi_n(y)$ for any $y \in V$). Having in mind this observation, we can ask the following question.

Question. Let X be a C-space, Y a Tychonoff space and $\Phi_n : X \to Y$, $n \geqslant 0$, a decreasing sequence of strongly lower semi-continuous maps such that, for any n and $x \in X$, the pair $(\Phi_{n+1}(x), \Phi_n(x))$ is UV^n. Does Φ_0 admit a continuous selection? What about if Y is a completely metrizable *ANR* or each $\Phi_{n+1}(x)$ is contractible in $\Phi_n(x)$?

Note added in proof

In a recent paper with Gutev we provided a positive answer to the question about Bula's property discussed in this section. We proved that Theorem 4.3 is true if Y is a paracompact C-space, X a Tychonoff space and $f : X \to$ an open continuous surjection such that all fibers of f are infinite and C^*-embedded in X. Moreover, Gutev established a negative answer to the question formulated at the end of the paper.

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