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Probability measures and Milyutin maps between metric spaces

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Abstract

We prove that the functor $\hat{P}$ of Radon probability measures transforms any open map between completely metrizable spaces into a soft map. This result is applied to establish some properties of Milyutin maps between completely metrizable spaces.

Keywords: Probability measures; Support of measures; Milyutin maps; Metric spaces

1. Introduction

In this paper we deal with metrizable spaces and continuous maps. By a (complete) space we mean a (completely) metrizable space, and by a measure a probability Radon measure. Recall that a measure μ on X is said to be:

- *probability* if $\mu(X) = 1$;
- *Radon* if $\mu(B) = \sup\{\mu(K)\colon K \subset B \text{ and } K \text{ is compact}\}$ for any Borel set $B \subset X$.

The support $\operatorname{supp}\mu$ of a measure μ is the intersection of all closed subsets A of X with $\mu(A) = \mu(X)$. It is well known that the support of any measure is non-empty and separable.

Everywhere below $\hat{P}(X)$ stands for the space of all probability Radon measures on X equipped with the weak topology with respect to $C^*(X)$. Here, $C^*(X)$ is the space of bounded continuous functions on X with the uniform convergence topology. According to [2], $\hat{P}$ is a functor in the category of metrizable spaces and continuous maps. In particular, for any map $f : X \to Y$ there exists a map $\hat{P}(f) : \hat{P}(X) \to \hat{P}(Y)$. A systematic study of the functor $\hat{P}$ can be found in [2] and [3]. We also consider the subspace $P_\beta(X) \subset \hat{P}(X)$ consisting of all measures μ such that $\operatorname{supp}\mu$ is compact.

This paper is devoted to some properties of Milyutin maps between metrizable spaces. We say that $f : X \to Y$ is a *Milyutin map* if there exists a map $g : Y \to \hat{P}(X)$ such that $\operatorname{supp} g(y) \subset f^{-1}(y)$ for every $y \in Y$. Such g is called a choice map associated with f. According to [3, Theorem 3.15], for any metrizable X there exists a barycentric map $b_{\hat{P}(X)} : \hat{P}(\hat{P}(X)) \to \hat{P}(X)$ such that $b_{\hat{P}(X)}(\nu) = \nu$ for all $\nu \in \hat{P}(X)$. Hence, if g is a choice map associated with f,

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then the map $b_{\hat{P}(X)} \circ \hat{P}(g) : \hat{P}(Y) \to \hat{P}(X)$ is a right inverse of $\hat{P}(f)$. Consequently, f is a Milyutin map if and only if $\hat{P}(f)$ admits a right inverse.

Our first principal result concerns the question when $\hat{P}(f)$ is a soft map. Recall that a map $f : X \to Y$ is soft if for any space Z and its closed subset A and any maps $g : Z \to Y$, $h : A \to X$ with $(f \circ h)|A = g$ there exists a map $\bar{g} : Z \to X$ such that $\bar{g}$ extends h and $f \circ \bar{g} = g$. It is easily seen that every soft map is surjective and open.

Theorem 1.1. *Let $f : X \to Y$ be a surjective open map between complete spaces. Then $\hat{P}(f) : \hat{P}(X) \to \hat{P}(Y)$ is a soft map.*

The particular cases of Theorem 1.1 when both X and Y are either compact or separable were established in [8] and [4], respectively.

Since any soft map admits a right inverse, a map f satisfying the hypotheses of Theorem 1.1 is a Milyutin map. We apply Theorem 1.1 to obtain some results about atomless and exact Milyutin maps introduced in [14]. If $f : X \to Y$ is a Milyutin map and there exists a choice map g such that $\operatorname{supp} g(y) = f^{-1}(y)$ (resp., $g(y)$ is an atomless measure on $f^{-1}(y)$ for each $y \in Y$, i.e. $g(y)(\{x\}) = 0$ for all $x \in f^{-1}(y)$), then f is said to be an *exact* (resp., *atomless*) Milyutin map. It was established in [14] that, in the realm of Polish spaces X and Y, f is exact Milyutin if and only if it is open. The classes of atomless exact Milyutin maps and atomless Milyutin maps between Polish spaces were characterized in [1, Theorem 1.6]. The first class consists of all open maps possessing perfect fibers (i.e., without isolated points) [1, Theorem 1.6], and the second one of all maps $f : X \to Y$ such that for some Polish space $X_0 \subset X$ the restriction $f_0 = f|X_0 : X_0 \to Y$ is an open surjection whose fibers are perfect [1, Theorem 1.7].

Next theorem is a non-separable analogue of [1, Theorem 1.7].

Theorem 1.2. *A continuous surjection $f : X \to Y$ of complete spaces is an atomless Milyutin map if and only if there exists a complete subspace $X_0 \subset X$ such that $f_0 = f|X_0 : X_0 \to Y$ is an open surjection and all fibers of f_0 are perfect sets. Moreover, for any such f there exists a map $f^* : P_\beta(Y) \to \hat{P}(X)$ such that any $f^*(\mu)$ is atomless and $\hat{P}(f)(f^*(\mu)) = \mu$, $\mu \in P_\beta(Y)$.*

We do not know whether under the hypotheses in Theorem 1.2 there exists a map $f^* : \hat{P}(Y) \to \hat{P}(X)$ such that each $f^*(\mu)$ is atomless and $\hat{P}(f)(f^*(\mu)) = \mu$, $\mu \in \hat{P}(Y)$. But if we are interested in atomless maps defined on Y, we have the following:

Theorem 1.3. *Every open surjection $f : X \to Y$ with perfect fibers is a densely atomless Milyutin map provided X and Y are complete spaces.*

Here, a Milyutin map $f : X \to Y$ is *densely atomless* if

$$\{g \in Ch_f(Y, X):\ g(y) \text{ is atomless for all } y \in Y\}$$

is a dense G_δ-set in the space $Ch_f(Y, X)$ of all choice maps associated with f equipped with the source limitation topology. A few words about this topology. If (X, d) is a bounded (complete) metric space, then there exists a (complete) metric $\hat{d}$ on $\hat{P}(X)$ generating its topology and extending d, see [3]. Then $Ch_f(Y, X)$ is a subspace of the function space $C(Y, \hat{P}(X))$ with the source limitation topology whose local base at a given $h \in C(Y, \hat{P}(X))$ consists of all sets

$$B_{\hat{d}}(h, \alpha) = \{g \in C(Y, \hat{P}(X)):\ \hat{d}(g(y), h(y)) < \alpha(y) \text{ for all } y \in Y\},$$

where α is a continuous map from Y into $(0, \infty)$. It is well known that this topology does not depend on the metric $\hat{d}$ and it has the Baire property in case $\hat{P}(X)$ is complete. Similarly, f is said to be *densely exact* provided the set

$$\{g \in Ch_f(Y, X):\ \operatorname{supp} g(y) = f^{-1}(y) \text{ for every } y \in Y\}$$

is a dense and G_δ-set in $Ch_f(Y, X)$. When f is both densely atomless and densely exact, it is called densely exact atomless.

Theorem 1.4. *Let $f : X \to Y$ be an open surjection of complete spaces and $\pi : X \to M$ a map into a separable space M. Then all choice maps $h \in Ch_f(Y, X)$ such that $\pi(\operatorname{supp} h(y))$ is dense in $\pi(f^{-1}(y))$ for every $y \in Y$ form a dense G_δ-set in $Ch_f(Y, X)$.*

It is interesting whether in Theorem 1.4 one can substitute the phrase "$\pi(\operatorname{supp} h(y))$ is dense in $\pi(f^{-1}(y))$" by "$\pi(\operatorname{supp} h(y)) = \pi(f^{-1}(y))$."

Next corollary is a parametrization of the Parthasarathy [12] result that perfect Polish spaces admit atomless measures. It also provides a partial answer of the question [1] whether any open surjection f of complete spaces is an exact atomless Milyutin map provided all fibers of f are perfect Polish spaces.

Corollary 1.5. *Let $f : X \to Y$ be an open and closed surjection of complete spaces such that all fibers of f are separable (and perfect). Then f is densely exact (atomless) Milyutin map.*

Finally, we generalize [14, Corollary 1.4] and [1, Corollary 1.9] as follows (below a continuous set-valued map means a map which is both lower and upper semi-continuous):

Corollary 1.6. *Let X and Y be complete spaces and $\Phi : Y \to X$ a continuous set-valued map such that all values $\Phi(y)$ are closed separable subsets of X. Then there exists a map $h : Y \to \hat{P}(X)$ such that $\operatorname{supp} h(y) = \Phi(y)$ for every $y \in Y$. If, in addition, all $\Phi(y)$ are perfect sets, the map h can be chosen so that every $h(y)$ is atomless.*

2. Preliminaries

In this section we provide some preliminary results and establish the proof of Theorem 1.1.

Probability Radon measures on a complete space X can be described as positive linear functionals μ on $C^*(X)$ such that $\|\mu\| = 1$ and $\lim \mu(h_\alpha) = 0$ for any decreasing net $\{h_\alpha\} \subset C^*(X)$ which pointwisely converges to 0, see [15]. Under this interpretation, $\operatorname{supp}\mu$ coincides with the set of all $x \in X$ such that for every neighborhood U_x of x in X there exists $\varphi \in C^*(X)$ such that $\varphi(X \setminus U_x) = 0$ and $\mu(\varphi) \neq 0$. This representation of $\operatorname{supp}\mu$ easily implies that the set-valued map $\operatorname{supp} : \hat{P}(X) \to X$ (assigning to each μ its support) is lower semi-continuous, i.e., $\{\mu \in \hat{P}(X)\colon \operatorname{supp}\mu \cap U \neq \emptyset\}$ is open in $\hat{P}(X)$ for any open $U \subset X$. For every closed $F \subset X$, we have $\mu(F) = \inf\{\mu(\varphi)\colon \varphi \in C(F)\}$ (see for example [7] in case X is compact), where $C(F) = \{\varphi \in C^*(X)\colon 0 \leqslant \varphi \leqslant 1 \text{ and } \varphi(F) = 1\}$.

According to [4], any compatible (complete) metric d on X generates a compatible (complete) metric $\hat{d}$ on $\hat{P}(X)$ such that

$$\hat{d}\big(t\mu + (1-t)\mu', t\nu + (1-t)\nu'\big) \leqslant t\hat{d}(\mu, \mu') + (1-t)\hat{d}(\nu, \nu')$$

for all $t \in [0, 1]$ and $\mu, \mu', \nu, \nu' \in \hat{P}(X)$. It is easily seen that every ball (open or closed) with respect to $\hat{d}$ is convex.

Let $A_\varepsilon(X)$ denote the set of all $\mu \in \hat{P}(X)$ such that $\mu(\{x\}) \geqslant \varepsilon$ for some $x \in \operatorname{supp}\mu$. For any closed $K \subset X$ there exists a closed embedding $i : \hat{P}(K) \to \hat{P}(X)$ defined by $i(\nu)(h) = \nu(h|K)$ for all $\nu \in \hat{P}(K)$ and $h \in C^*(X)$. Everywhere below we identify $\hat{P}(K)$ with the set $i(\hat{P}(K)) = \{\mu \in \hat{P}(X)\colon \operatorname{supp}\mu \subset K\}$ which is closed in $\hat{P}(X)$.

Lemma 2.1. *Let X be a complete space, K a perfect closed subset of X and G a convex open subset of $\hat{P}(K)$. Then for every $\varepsilon > 0$ we have*:

(1) *$A_\varepsilon(X)$ is a closed subset of $\hat{P}(X)$;*
(2) *$A_\varepsilon(X) \cap \overline{G}$ is a nowhere dense set in the closure $\overline{G}$.*

Proof. (1) Since $\hat{P}(X)$ is metrizable, it suffices to check that $\mu_0 = \lim \mu_n \in A_\varepsilon(X)$ for every convergent sequence $\{\mu_n\}_{n\geqslant 1}$ in $\hat{P}(X)$ with $\{\mu_n\} \subset A_\varepsilon(X)$. To this end, let H be the closure in X of the set $\bigcup_{n\geqslant 0} \operatorname{supp}\mu_n$. Because every $\mu \in \hat{P}(X)$ has a separable support, H is a Polish subset of X. Considering all μ_n, $n \geqslant 0$, as elements of $\hat{P}(H)$, we have that the sequence $\{\mu_n\}_{n\geqslant 1}$ is contained in $A_\varepsilon(H)$ and converges to μ_0. Therefore, by [12, Theorem 8.1], $\mu_0 \in A_\varepsilon(H)$. Consequently, there exists $x_0 \in H$ with $\mu_0(\{x_0\}) \geqslant \varepsilon$. Therefore, $A_\varepsilon(X)$ is closed in $\hat{P}(X)$.

(2) Since $A_\varepsilon(K) = A_\varepsilon(X) \cap \hat{P}(K)$, it suffices to show that $A_\varepsilon(K)$ is nowhere dense in $\hat{P}(K)$. Suppose $A_\varepsilon(K)$ contains an open subset W of $\hat{P}(K)$ and let $P_\omega(K)$ be the set of all $\mu \in \hat{P}(K)$ having a finite support. Since

$P_\omega(K)$ is dense in $\hat{P}(K)$, there exists $\mu_0 = \sum_{i=1}^{i=k} \lambda_i \delta_{x_i} \in P_\omega(K) \cap W$. Here, δ_{x_i} denotes Dirac's measures at x_i and $\lambda_i = \mu_0(\{x_i\})$. Moreover, $\lambda_i \geqslant \varepsilon$ for at least one i. For each $i \leqslant k$ and $n \geqslant 1$ choose a neighborhood $V_i \subset K$ of x_i and n different points $x_{i(1)}, \dots, x_{i(n)} \in V_i$ such that the family $\{V_i\colon\ 1 \leqslant i \leqslant k\}$ is disjoint and $\operatorname{dist}(x_i, x_{i(j)}) \leqslant 1/n$ for all $1 \leqslant j \leqslant n$. This can be done because K is perfect, so every neighborhood of x_i contains infinitely many points. Consider now the measures $\mu_n = \sum_{i=1}^{i=k} \sum_{j=1}^{j=n} \frac{\lambda_i \delta_{x_{i(j)}}}{n}$. Since $\lim \mu_n = \mu_0$, there exists n_0 such that $\mu_n \in W$ for all $n \geqslant n_0$. Consequently, for every $n \geqslant n_0$ there exists $i \leqslant k$ with $\lambda_i / n \geqslant \varepsilon$, a contradiction. □

Lemma 2.2. *Let $f : X \to Y$ be an open surjection between complete spaces such that* $\dim Y = 0$. *Then $\hat{P}(f) : \hat{P}(X) \to \hat{P}(Y)$ is a soft map.*

Proof. According to Theorem 1.3 from [4], it suffices to show that f is everywhere locally invertible. The last notion is defined as follows: for any space Z, a point $a \in Z$, a map $g : Z \to Y$ and an open set $U \subset X$ with $g(a) \in f(U)$ there exist a neighborhood V of a in Z and a map $h : V \to U$ such that $f \circ h = g|V$. Obviously, f is everywhere locally invertible provided it satisfies the following condition:

(∗) For any open $U \subset X$ and $a \in f(U)$ there exists a map $g : V \to U$ with V being a neighborhood of a in Y such that $f(g(y)) = y$ for all $y \in V$.

To show f satisfies (∗), fix an open set $U \subset X$ and $a \in f(U)$. Since f is open, the set $V = f(U) \subset Y$ is also open and the set-valued map $\Phi : V \to U$, $\Phi(y) = f^{-1}(y) \cap U$, is lower semi-continuous with closed values. Moreover, U admits a complete metric because X is complete. Then, by the 0-dimensional selection theorem of Michael [11], Φ has a continuous selection g. Obviously, g is as required. □

Proof of Theorem 1.1. First, let us show that $\hat{f} = \hat{P}(f)|\hat{P}(f)^{-1}(Y)$ is everywhere locally invertible. It suffices to show that $\hat{f}$ satisfies condition (∗) from Lemma 2.2. Suppose that $U \subset \hat{P}(f)^{-1}(Y)$ is open and $y_0 \in \hat{f}(U)$. We need to find a map $\alpha : V \to U$, where V is a neighborhood of y_0 in Y, such that $\hat{f}(\alpha(y)) = y$ for every $y \in V$. To this end, choose a 0-dimensional complete space Z and a perfect Milyutin map $g : Z \to Y$, see [6] (recall that a map is perfect if it is closed and has compact fibers). Next, consider the pull-back $T = \{(z, x) \in Z \times X\colon\ g(z) = f(x)\}$ of Z and X with respect to the maps g and f, and let $p_f : T \to Z$, $p_g : T \to X$ be the corresponding projections. Since f is open, so is p_f. For any $y \in Y$ we have $p_f^{-1}(g^{-1}(y)) = p_g^{-1}(f^{-1}(y)) = g^{-1}(y) \times f^{-1}(y)$. Since g is Milyutin, there exists a map $g^* : Y \to \hat{P}(Z)$ such that $\operatorname{supp} g^*(y) \subset g^{-1}(y)$ for all $y \in Y$. Let $\hat{p_f} = \hat{P}(p_f) : \hat{P}(T) \to \hat{P}(Z)$ and $\hat{p_g} = \hat{P}(p_g) : \hat{P}(T) \to \hat{P}(X)$. Take an open set $G \subset \hat{P}(X)$ with $G \cap \hat{P}(f)^{-1}(Y) = U$ and let $W = \hat{p_g}^{-1}(G)$. Pick $\mu^* \in G \cap \hat{P}(f^{-1}(y_0))$ and let $\nu_0 = \mu_0 \times \mu^*$ be the product measure, where $\mu_0 = g^*(y_0)$. Obviously, $\nu_0 \in \hat{P}(g^{-1}(y_0) \times f^{-1}(y_0)) \subset \hat{P}(T)$. Moreover, $\hat{p_f}(\nu_0) = \mu_0$ and $\nu_0 \in W$ because $\hat{p_g}(\nu_0) = \mu_* \in G$.

Now we can complete the proof that $\hat{f}$ is everywhere locally invertible. Let $g_0 : \{y_0\} \to \hat{P}(T)$ be the constant map $g_0(y_0) = \nu_0$. Since $\hat{p_f}(\nu_0) = g^*(y_0)$ and, by Lemma 2.2, the map $\hat{p_f}$ is soft, there exists a map $\theta : Y \to \hat{P}(T)$ extending g_0 such that $\hat{p_f} \circ \theta = g^*$. Obviously, $V = \theta^{-1}(W)$ is a neighborhood of y_0, and define $\alpha = \hat{p_g} \circ \theta$. Since for any $y \in V$ we have $\hat{p_f}(\theta(y)) = g^*(y)$, $p_f(\operatorname{supp} \theta(y)) = \operatorname{supp} g^*(y) \subset g^{-1}(y)$ and $\operatorname{supp} \theta(y) \subset g^{-1}(y) \times f^{-1}(y)$. So, $\operatorname{supp} \alpha(y) = p_g(\operatorname{supp} \theta(y)) \subset f^{-1}(y)$. Consequently, $\hat{f}(\alpha(y)) = y$. Moreover, $\alpha(y) \in U$ for all $y \in V$.

Since $\hat{f}$ is everywhere locally invertible, by [4, Theorem 1.3], the map $\hat{P}(\hat{f}) : \hat{P}(\hat{Y}) \to \hat{P}(Y)$ is soft, where $\hat{Y} = \hat{f}^{-1}(Y)$. Moreover, $\hat{P}(X) \subset \hat{P}(\hat{Y}) \subset \hat{P}(\hat{P}(X))$ because $X \subset \hat{Y} \subset \hat{P}(X)$. Therefore the following diagram

$$\begin{array}{ccc} \hat{P}(\hat{Y}) & \xrightarrow{\ b_{\hat{P}}\ } & \hat{P}(X) \\ {\scriptstyle \hat{P}(\hat{f})}\big\downarrow & & \big\downarrow{\scriptstyle \hat{P}(f)} \\ \hat{P}(Y) & \xrightarrow{\ i_{\hat{P}(Y)}\ } & \hat{P}(Y) \end{array}$$

is commutative. Here, $b_{\hat{P}}$ denotes the restriction $b_{\hat{P}(X)}|\hat{P}(\hat{Y})$ of the barycentric map $b_{\hat{P}(X)} : \hat{P}(\hat{P}(X)) \to \hat{P}(X)$, see [3], and $i_{\hat{P}(Y)}$ is the identity on $\hat{P}(Y)$. Since $b_{\hat{P}}$ retracts each $\hat{P}(\hat{f})^{-1}(\mu)$ onto $\hat{P}(f)^{-1}(\mu)$, $\mu \in \hat{P}(Y)$, and $\hat{P}(\hat{f})$ is soft, we finally obtain that $\hat{P}(f)$ is also soft. The proof is completed. □

3. Atomless Milyutin maps

In this section we provide the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Suppose that $f: X \to Y$ is a surjective atomless Milyutin map with X and Y complete spaces. Then there exists a choice map $h: Y \to \hat{P}(X)$ associated with f such that $h(y)$ is an atomless measure for all $y \in Y$. Let $X_0 = \bigcup\{\operatorname{supp} h(y):\ y \in Y\}$ and $f_0 = f|X_0$. Since $f_0^{-1} = \operatorname{supp} \circ h$ is lower semi-continuous, f_0 is open. Hence, by [1, Theorem 3.6], X_0 is complete. Moreover, all $f_0^{-1}(y)$ are perfect sets because $h(y)$ are atomless measures.

For the other implication, assume that $f: X \to Y$ is a surjection between complete spaces and there exists a complete subspace $X_0 \subset X$ such that $f_0 = f|X_0$ is an open surjection possessing perfect fibers. Considering X_0 and $f_0|X_0$, we may suppose that f is open and all of its fibers $f^{-1}(y)$, $y \in Y$, are perfect sets. Then, by Theorem 1.1, f is Milyutin because $\hat{P}(f)$ has a right inverse as a soft map. To show f is atomless, as in the proof of Theorem 1.1 take a 0-dimensional complete space Z and a perfect Milyutin map $g: Z \to Y$. Since g is Milyutin, there exists a map $g^*: \hat{P}(Y) \to \hat{P}(Z)$ such that $\hat{P}(g)(g^*(\mu)) = \mu$ for all $\mu \in \hat{P}(Y)$. By Theorem 1.1, $\hat{P}(f)$ is open (as a soft map). Hence, $\hat{f}: \hat{P}(f)^{-1}(Y) \to Y$ is also open (as a restriction of an open map onto a preimage-set). So, the set-valued map $\Phi: Z \to \hat{P}(f)^{-1}(Y)$, $\Phi(z) = \hat{f}^{-1}(g(z))$, is lower semi-continuous. Actually, $\Phi(z) = \hat{P}(f^{-1}(g(z)))$ for every $z \in Z$. Let A_n, $n \geqslant 1$, be the set of all $\mu \in \hat{P}(X)$ such that $\mu(\{x\}) \geqslant 1/n$ for some point $x \in \operatorname{supp}\mu$. Since the fibers $f^{-1}(y)$ are perfect sets, by Lemma 2.1, A_n are closed in $\hat{P}(X)$ and all intersections $A_n \cap \hat{P}(f^{-1}(y))$ are nowhere dense in $\hat{P}(f^{-1}(y))$, $y \in Y$. Then, by [9, Theorem 1.2], Φ admits a selection $\theta: Z \to \hat{P}(f)^{-1}(Y)$ such that $\theta(z) \in \Phi(z) \setminus \bigcup_{n=1}^{\infty} A_n$, $z \in Z$. This means that each measure $\theta(z) \in \hat{P}(f^{-1}(g(z)))$ is atomless. The selection θ generates a regular operator $u: C^*(X) \to C^*(Z)$, $u(\phi)(z) = \theta(z)(\phi)$ for all $\phi \in C^*(X)$ and $z \in Z$. Finally, for every $\mu \in P_\beta(Y)$ let $f^*(\mu) \in \hat{P}(X)$ be the measure defined by $f^*(\mu)(\phi) = g^*(\mu)(u(\phi))$, $\phi \in C^*(X)$. It is easily seen that this definition is correct (i.e., $f^*(\mu) \in \hat{P}(X)$) and $f^*: P_\beta(Y) \to \hat{P}(X)$ is a continuous map.

Let us show that $\hat{P}(f)(f^*(\mu)) = \mu$ for every $\mu \in P_\beta(Y)$. It suffices to prove that $f^*(\mu)(\alpha \circ f) = \mu(\alpha)$ for any $\alpha \in C^*(Y)$. And this is really true because $\phi = \alpha \circ f$ is the constant $\alpha(y)$ on each set $f^{-1}(y)$, $y \in Y$. So, $u(\phi)(z) = \theta(z)(\phi) = \alpha(y)$ for any $z \in g^{-1}(y)$. Thus, $u(\phi) = \alpha \circ g$ and $f^*(\mu)(\alpha \circ f) = g^*(\mu)(\alpha \circ g)$. Finally, since $\hat{P}(g)(g^*(\mu)) = \mu$, we have $g^*(\mu)(\alpha \circ g) = \mu(\alpha)$.

So, it remains to prove only that every $f^*(\mu)$, $\mu \in P_\beta(Y)$, is an atomless measure. To this end, fix $\mu_0 \in P_\beta(Y)$, $x_0 \in \operatorname{supp} f^*(\mu_0)$ and $\eta > 0$. It suffices to find a function $\phi_0 \in C^*(X)$ with $0 \leqslant \phi_0 \leqslant 1$ such that $\phi_0(x_0) = 1$ and $f^*(\mu_0)(\phi_0) \leqslant \eta$. Since $\theta(z)(\{x_0\}) = 0$, for every $z \in Z$ there exists $\phi_z \in C^*(X)$ and a neighborhood U_z of z in Z such that $0 \leqslant \phi_z \leqslant 1$, $\phi_z(x_0) = 1$ and $\theta(z')(\phi_z) < \eta$ whenever $z' \in U_z$. Using the compactness of $g^{-1}(\operatorname{supp}\mu_0)$ (recall that μ_0 has a compact support and g is a perfect map), we find neighborhoods $U_{z(i)}$, $i = 1, \dots, k$, covering $g^{-1}(\operatorname{supp}\mu_0)$, and let $\phi_0 = \phi_{z(1)} \cdot \phi_{z(2)} \cdot \dots \cdot \phi_{z(k)}$. Then ϕ_0 is as required. Indeed, since $\hat{P}(g)(g^*(\mu_0)) = \mu_0$, $g^{-1}(\operatorname{supp}\mu_0)$ contains the support of $g^*(\mu_0)$. Consequently, $g^*(\mu_0)(u(\phi_0)) \leqslant \max\{u(\phi_0)(z):\ z \in g^{-1}(\operatorname{supp}\mu_0)\}$. So, there exists $z_0 \in g^{-1}(\operatorname{supp}\mu_0)$ such that $g^*(\mu_0)(u(\phi_0)) \leqslant u(\phi_0)(z_0)$. Next, choose j with $z_0 \in U_{z(j)}$ and observe that $\phi_0 \leqslant \phi_j$ implies $u(\phi_0)(z_0) \leqslant u(\phi_j)(z_0) = \theta(z_0)(\phi_j)$. Therefore, $f^*(\mu_0)(\phi_0) \leqslant \theta(z_0)(\phi_j) < \eta$ because $z_0 \in U_{z(j)}$. The proof is completed. □

Proof of Theorem 1.3. Take a 0-dimensional complete space Z, a perfect Milyutin map $g: Z \to Y$ and a map $g^*: \hat{P}(Y) \to \hat{P}(Z)$ which is right inverse of $\hat{P}(g)$. We equip $\hat{P}(X)$ with a convex metric $\hat{d}$, and let A_n, $n \geqslant 1$, be the closed subsets of $\hat{P}(X)$ considered in the proof of Theorem 1.2. We need to show that the set $\mathcal{A}$ of all atomless choice maps form a dense G_δ-subset of $Ch_f(Y, X)$. Since each A_n is closed in $\hat{P}(X)$, it is easily seen that the sets

$$\mathcal{U}_n = \{h \in Ch_f(Y, X):\ h(y) \notin A_n \text{ for all } y \in Y\}$$

are open in $Ch_f(Y, X)$ and $\mathcal{A} = \bigcap_{n \geqslant 1} \mathcal{U}_n$. To prove that $\mathcal{A}$ is dense in $Ch_f(Y, X)$, fix $h \in Ch_f(Y, X)$ and a function $\eta: Y \to (0, \infty)$. We are going to find a map $h' \in \mathcal{A}$ such that $\hat{d}(h(y), h'(y)) \leqslant \eta(y)$ for all $y \in Y$.

Denote by $B(h(g(z)), \eta(g(z)))$ the open ball in $\hat{P}(X)$ (with respect to $\hat{d}$) which is centered at $h(g(z))$ and has a radius $\eta(g(z))$. Define the set-valued map $\Phi: Z \to \hat{P}(X)$, $\Phi(z) = \overline{\hat{P}(f^{-1}(g(z))) \cap B(h(g(z)), \eta(g(z)))}$. This is a convex and closed-valued map because any ball in $\hat{P}(X)$ with respect to $\hat{d}$ is convex. Since $\hat{f} = \hat{P}(f)|(\hat{P}(f)^{-1}(Y))$ is open (as a soft map, see Theorem 1.1), the set-valued map $z \mapsto \hat{P}(f)^{-1}(g(z))$ is lower semi-continuous. Hence, by [10, Proposition 2.5], so is Φ. Moreover, each $\Phi(z)$ is the closure of the convex open set $\hat{P}(f^{-1}(g(z))) \cap$

$B(h(g(z)), \eta(g(z)))$ in $\hat{P}(f^{-1}(g(z)))$. Hence, according to Lemma 2.1, $A_n \cap \Phi(z)$, $n \geqslant 1$, are nowhere dense sets in $\Phi(z)$ for every $z \in Z$. Then, by [9, Theorem 1.2], Φ has a continuous selection $\theta : Z \to \hat{P}(X)$ avoiding the set $\bigcup_{n=1}^{\infty} A_n$, i.e., with $\theta(z) \in \Phi(z) \setminus \bigcup_{n=1}^{\infty} A_n$ for every $z \in Z$. Following the notations from the proof of Theorem 1.2, we extend θ to a map $\bar{\theta} : P_\beta(Z) \to \hat{P}(X)$ by $\bar{\theta}(\nu)(\phi) = \nu(u(\phi))$, $\phi \in C^*(X)$. Now let $h' : Y \to \hat{P}(X)$ be the composition $\bar{\theta} \circ g^*$. It follows from the proof of Theorem 1.2 that $h'(y)$ is atomless and $h'(y) \in \hat{P}(f^{-1}(y))$ for all $y \in Y$. So, $h' \in \mathcal{A}$.

It remains to show that $\hat{d}(h(y), h'(y)) \leqslant \eta(y)$, $y \in Y$. To this end, we fix $y \in Y$ and take a sequence $\{\nu_n\} \subset P_\beta(g^{-1}(y))$ converging to $g^*(y)$ such that each ν_n has a finite support. It is easily seen that if $\nu = \sum_{i=1}^{i=k} t_i \delta_{z(i)} \in P_\beta(g^{-1}(y))$ is a measure with a finite support, then $\bar{\theta}(\nu) = \sum_{i=1}^{i=k} t_i \theta(z(i))$. Since $\hat{d}(\theta(z(i)), h(y)) \leqslant \eta(y)$ for all i and the metric $\hat{d}$ is convex, we have $\hat{d}(\bar{\theta}(\nu), h(y)) \leqslant \eta(y)$. In particular, $\hat{d}(\bar{\theta}(\nu_n), h(y)) \leqslant \eta(y)$ for every n. This implies that $\hat{d}(h'(y), h(y)) \leqslant \eta(y)$ because $h'(y)$ is the limit of the sequence $\{\bar{\theta}(\nu_n)\}$. □

4. Exact Milyutin maps

In this section the proofs of Theorem 1.4 and Corollaries 1.5–1.6 are established.

Lemma 4.1. *Let $U \subset X$ be a non-empty open set in a space X. Then the set $\hat{U} = \{\nu \in \hat{P}(X)\colon \operatorname{supp}\nu \cap U \neq \emptyset\}$ is open convex and dense in $\hat{P}(X)$.*

Proof. Since the support map $\nu \to \operatorname{supp}\nu$ is a lower semi-continuous map, $\hat{U} \subset \hat{P}(X)$ is open. To show it is dense, suppose there exists an open set $W = \{\nu \in \hat{P}(X)\colon |\nu(\phi_i) - \nu_0(\phi_i)| < \varepsilon, 1 \leqslant i \leqslant k\}$ in $\hat{P}(X)$ with $W \subset \hat{P}(X) \setminus \hat{U}$, where $\phi_i \in C^*(X)$ and $\varepsilon > 0$. We can suppose that ν_0 has a finite support (recall that the measures with a finite support form a dense set in $\hat{P}(X)$). Let $\nu_0 = \sum_{j=1}^{j=m} \lambda_j \delta_{x(j)}$ such that $\lambda_j > 0$ and $\sum_{j=1}^{j=m} \lambda_j = 1$. Then $\operatorname{supp}\nu_0 = \{x(j)\colon 1 \leqslant j \leqslant m\} \subset X \setminus U$. Now, let $\nu' = \lambda_0 \delta_{x(0)} + (\lambda_1 - \lambda_0)\delta_{x(1)} + \sum_{j=2}^{j=m} \lambda_j \delta_{x(j)}$, where $x_0 \in U$ and $0 < \lambda_0 < \lambda_1$ such that $\lambda_0 |\phi_i(x_0) - \phi_i(x_1)| < \epsilon$ for every $i = 1, 2, \ldots, k$. The choice of λ_0 yields that $\nu' \in W$. Consequently, $\nu' \notin \hat{U}$ and $\operatorname{supp}\nu' \subset X \setminus U$. This contradicts $x_0 \in U \cap \operatorname{supp}\nu'$.

To show $\hat{U}$ is convex, it suffices to prove that $\operatorname{supp}(t\nu_1 + (1-t)\nu_2) = \operatorname{supp}\nu_1 \cup \operatorname{supp}\nu_2$ for any $\nu_1, \nu_2 \in \hat{P}(X)$ and any $t \in (0,1)$. Obviously, $\operatorname{supp}\nu_1 \cup \operatorname{supp}\nu_2 \supset \operatorname{supp}(t\nu_1 + (1-t)\nu_2)$. Assume $x \in \operatorname{supp}\nu_1$. Then for every neighborhood V_x of x there exists a function $\phi_x \in C^*(X)$ with $\phi_x(X \setminus V_x) = 0$ and $\nu_1(\phi_x) \neq 0$. Since $\nu_1(\phi_x) = \nu_1(\phi_x^+) - \nu_1(\phi_x^-)$, where ϕ_x^+ and ϕ_x^- are the positive and negative parts of ϕ_x, we can suppose ϕ_x is non-negative. Then, $\nu(\phi_x) \geqslant \nu_1(\phi_x) > 0$ with $\nu = \nu = t\nu_1 + (1-t)\nu_2$. Hence, $x \in \operatorname{supp}\nu$ which completes the proof. □

Proof of Theorem 1.4. Choose a countable base $\{V_n\colon n \geqslant 1\}$ for the topology of M, and let $B_n = \{\nu \in \hat{P}(X)\colon \operatorname{supp}\nu \cap \pi^{-1}(V_n) = \emptyset\}$. By Lemma 4.1, each B_n is closed in $\hat{P}(X)$. Let $\mathcal{B}$ be the set of all maps $h \in Ch_f(Y, X)$ such that $\pi(\operatorname{supp} h(y))$ is dense in $\pi(f^{-1}(y))$ for any $y \in Y$. Obviously, $\mathcal{B} = \bigcap_{n \geqslant 1} \mathcal{G}_n$, where $\mathcal{G}_n = \{h \in Ch_f(Y, X)\colon h(y) \notin B_n$ for all $y \in Y\}$. It suffices to show that each $\mathcal{G}_n$ is open and dense in $Ch_f(Y, X)$ with respect to the source limitation topology.

Claim 1. *Each $\mathcal{G}_n$ is open in $Ch_f(Y, X)$.*

We can suppose that each V_n is of the form $V_n = g_n^{-1}(0, \infty)$ for some non-negative function $g_n \in C^*(M)$. Then $\nu \in B_n$ if and only if $\nu(g_n \circ \pi) = 0$, $n \geqslant 1$. Obviously the equality $D_n(\mu, \mu') = \hat{d}(\mu, \mu') + |\mu(g_n \circ \pi) - \mu'(g_n \circ \pi)|$, where $\mu, \mu' \in \hat{P}(X)$ and $\hat{d}$ is a compatible metric on $\hat{P}(X)$, defines a compatible metric on $\hat{P}(X)$ for every $n \geqslant 1$. Given $h \in \mathcal{G}_n$ we consider the continuous function $\alpha : Y \to (0, \infty)$, $\alpha(y) = h(y)(g_n \circ \pi)/2$. We have $B_{D_n}(h, \alpha) \subset \mathcal{G}_n$. Indeed, if $h' \in B_{D_n}(h, \alpha)$, then $|h'(y)(g_n \circ \pi) - h(y)(g_n \circ \pi)| \leqslant D_n(h(y), h'(y)) < \alpha(y)$ for all $y \in Y$. The last inequality implies $h'(y)(g_n \circ \pi) > \alpha(y) > 0$, $y \in Y$. Hence, $h'(y) \notin B_n$ for all $y \in Y$. So, $h' \in \mathcal{G}_n$ which completes the proof of Claim 1.

To show that any $\mathcal{G}_n$ is dense in $Ch_f(Y, X)$, we fix $m \geqslant 1$, $h \in Ch_f(Y, X)$ and a function $\eta : Y \to (0, \infty)$. We are going to find a map $h' \in \mathcal{G}_m$ with $\hat{d}(h'(y), h(y)) \leqslant \eta(y)$ for all $y \in Y$. To this end, following the proof of Theorems 1.2 and 1.3, take a complete 0-dimensional space Z and a perfect Milyutin map $g : Z \to Y$ with a right inverse $g^*: Y \to P_\beta(Z)$. We also consider the lower semi-continuous convex and closed-valued map $\Phi: Z \to \hat{P}(X)$, $\Phi(z) = \overline{\hat{P}(f^{-1}(g(z))) \cap B(h(g(z)), \eta(g(z)))}$. According to Lemma 4.1, $B_m \cap \hat{P}(f^{-1}(g(z)))$ is a closed nowhere dense subsets of $\hat{P}(f^{-1}(g(z)))$ for every $z \in Z$. Hence, all $B_m \cap \Phi(z)$ are closed and nowhere dense in $\Phi(z)$. Then, by [9, Theorem 1.2], Φ has a continuous selection $\theta: Z \to \hat{P}(X)$ such that $\theta(z) \in \Phi(z) \backslash B_m$, $z \in Z$. As in the proof of Theorem 1.3, let $h': Y \to \hat{P}(X)$ be the composition $\bar{\theta} \circ g^*$, where $\bar{\theta}: P_\beta(Z) \to \hat{P}(X)$ is an extension of θ defined by $\bar{\theta}(\nu)(\phi) = \nu(u(\phi))$, $\phi \in C^*(X)$. Following the arguments from Theorem 1.3, we can show that $\hat{d}(h'(y), h(y)) \leqslant \eta(y)$ for all $y \in Y$. Next claim completes the proof of Theorem 1.4.

Claim 2. $h'(y) \notin B_m$ *for any* $y \in Y$.

The proof of this claim is reduced to find a function $\phi_y \in C^*(X)$ such that $\phi_y(X \backslash \pi^{-1}(V_m)) = 0$ and $h(y)(\phi_y) \neq 0$. Indeed, in such a case $\operatorname{supp} h(y) \cap \pi^{-1}(V_m) \neq \emptyset$. Since $\theta(z) \notin B_m$ for all $z \in g^{-1}(y)$, $\operatorname{supp} \theta(z) \cap \pi^{-1}(V_m) \neq \emptyset$. Consequently, for any $z \in g^{-1}(y)$ there exists a function $\phi_z \in C^*(X)$ with $\phi_z(X \backslash \pi^{-1}(V_m)) = 0$ and $\theta(z)(\phi_z) \neq 0$. Considering the positive or negative parts of ϕ_z, we may assume each $\phi_z \geqslant 0$. Next, use the continuity of θ and the compactness of $g^{-1}(y)$ to find finitely many points $z(i) \in g^{-1}(y)$, $i = 1, 2, \ldots, k$, and neighborhoods $U_{z(i)}$ such that $\theta(z)(\phi_{z(i)}) > 0$ provided $z \in U_{z(i)}$. Finally, let $\phi_y = \sum_{i=1}^{i=k} \phi_{z(i)}$. Then $\phi_y(X \backslash \pi^{-1}(V_m)) = 0$ and $u(\phi_y)(z) = \theta(z)(\phi_y) > 0$ for any $z \in g^{-1}(y)$. So, $h(y)(\phi_y) \geqslant \min\{u(\phi_y)(z)\colon z \in g^{-1}(y)\} > 0$ because $g^{-1}(y)$ is compact. This completes the proof of the claim. □

Proof of Corollary 1.5. Since f is closed with separable fibers, there exists a map $\pi: X \to Q$ such that all restrictions $\pi | f^{-1}(y)$, $y \in Y$, are embeddings, see [13]. Here, Q is the Hilbert cube. Then, by Theorem 1.4 (with M replaced by Q), f is densely exact. If, in addition, the fibers of f are perfect, both Theorems 1.3 and 1.4 imply that f is densely exact atomless. □

Proof of Corollary 1.6. Consider the graph $G(\Phi) = \bigcup\{\{y\} \times \Phi(y)\colon y \in Y\} \subset Y \times X$ of Φ and the projection $f: G(\Phi) \to Y$. Since Φ is continuous, $G(\Phi)$ is closed in $Y \times X$ and f is both open and closed. Then $G(\Phi)$ is a complete space. Now, by Corollary 1.5, there exists a map $h': Y \to \hat{P}(G(\Phi))$ with each $h'(y) \in \hat{P}(f^{-1}(y))$ being exact measure. Therefore, $\operatorname{supp} h'(y) = f^{-1}(y)$. Let $h = \hat{P}(\pi) \circ h'$, where $\pi: G(\Phi) \to X$ is the projection into X. Since π embeds each $f^{-1}(y)$ onto $\Phi(y)$, h is a map from Y into $\hat{P}(X)$ such that $\operatorname{supp} h(y) = \Phi(y)$ for every $y \in Y$. If $\Phi(y)$ are perfect sets, so are the fibers $f^{-1}(y)$, and h' can be chosen to be atomless and exact. In such a case h is also atomless. □

5. Note added in proof

Recently T. Banakh informed the author that V. Bogachev and A. Kolesnikov [5] proved the following result: The map $\hat{P}(f)$ from Theorem 1.1 is open. This, in combination with Michael's convex-valued selection theorem [10], provides another proof of Theorem 1.1.

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