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# A non-separable Christensen's theorem and set tri-quotient maps ☆

Stoyan Nedev [a], Jan Pelant [1], Vesko Valov [b,*]

[a] Institute of Mathematics, Bulgarian Academy of Sciences, Acad. G. Bonchev str., bl. 8, Sofia 1113, Bulgaria
[b] Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, PO Box 5002, North Bay, ON, P1B 8L7, Canada

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ABSTRACT

For every space $X$ let $\mathcal{K}(X)$ be the set of all compact subsets of $X$. Christensen [J.P.R. Christensen, Necessary and sufficient conditions for measurability of certain sets of closed subsets, Math. Ann. 200 (1973) 189–193] proved that if $X, Y$ are separable metrizable spaces and $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ is a monotone map such that any $L \in \mathcal{K}(Y)$ is covered by $F(K)$ for some $K \in \mathcal{K}(X)$, then $Y$ is complete provided $X$ is complete. It is well known [J. Baars, J. de Groot, J. Pelant, Function spaces of completely metrizable space, Trans. Amer. Math. Soc. 340 (1993) 871–879] that this result is not true for non-separable spaces. In this paper we discuss some additional properties of $F$ which guarantee the validity of Christensen's result for more general spaces.



## 1. Introduction

All spaces in this paper are assumed to be completely regular.

The following characterization of Polish spaces established by J.P. Christensen [6] (see also [18] for another proof) is well-known.

**Theorem 1.1.** *([6]) A separable metric space $Y$ is complete iff there exists a Polish space $X$ and a map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ such that*:

(1) *$F$ is monotone (i.e. if $K, L \in \mathcal{K}(X)$ with $K \subset L$, then $F(K) \subset F(L)$);*
(2) *$F(\mathcal{K}(X))$ is cofinal in $\mathcal{K}(Y)$ (i.e. for each $L \in \mathcal{K}(Y)$ there is $K \in \mathcal{K}(X)$ with $L \subset F(K)$).*

According to Proposition 2.2(b) and Theorem 1.4 below, Theorem 1.1 remains valid if condition (2) is replaced by the weaker one:

$(2)_c$ For any countable $L \in \mathcal{K}(Y)$ there exists $K \in \mathcal{K}(X)$ with $L \subset F(K)$.

Theorem 1.1 is not valid for non-separable $X$. Indeed, let $\mathbb{Q}$ be rational numbers and $X$ the discrete sum of all compact subsets of $\mathbb{Q}$. Then there exists a map $F : \mathcal{K}(X) \to \mathcal{K}(\mathbb{Q})$ satisfying conditions (1) and (2), see [3]. Our first principal result shows that Theorem 1.1 remains valid for arbitrary metrizable $X$ and $Y$ if $F$ satisfies an extra condition:

☆ The last author was partially supported by NSERC Grant 261914.
* Corresponding author.
*E-mail addresses:* nedev@math.bas.bg (S. Nedev), veskov@nipissingu.ca (V. Valov).
[1] Deceased.

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**Theorem 1.2.** *Let $X$ and $Y$ be metrizable spaces and $F:\mathcal{K}(X)\to\mathcal{K}(Y)$ be a map satisfying conditions* (1), $(2)_c$ *and condition* $(3)_c$ *below*:

$(3)_c$ *If $U\subset X$ and $V\subset Y$ are non-empty open sets such that for each countable compact set $L\subset V$ there is a compact $K\subset U$ with $L\subset F(K)$, then for any open cover $\mathcal{W}$ of $U$ and any point $y\in V$ there exist a finite subfamily $\mathcal{E}\subset\mathcal{W}$ and a neighborhood $V_y$ of $y$ such that each countable compact $K\subset V_y$ is covered by $F(K)$ for some compact $K\subset\bigcup\mathcal{E}$.*

*Then $Y$ is completely metrizable and* $\operatorname{dens} Y\leqslant\operatorname{dens} X$ *provided $X$ is completely metrizable.*

Any map $F:\mathcal{K}(X)\to\mathcal{K}(Y)$ satisfies $(3)_c$ if $X$ and $Y$ are metrizable with $X$ being separable (see Proposition 2.2(b)). So, Theorem 1.2 is a generalization of Christensen's result.

A non-metrizable analog of Theorem 1.1 was established in [8] (see [4] for related results).

**Theorem 1.3.** *([8]) Let $X$ be a Lindelöf Čech-complete space and $F:\mathcal{K}(X)\to\mathcal{K}(Y)$ be a map satisfying conditions* (1), (2). *If $Y$ is a $\mu$-complete $q$-space, then $Y$ is also Lindelöf and Čech-complete.*

Recall that $X$ is said to be a *$\mu$-space* or *$\mu$-complete* if every closed and bounded set in $X$ is compact. Here, a set $A\subset X$ is *bounded* in $X$ if each continuous real-valued function on $X$ is bounded on $A$. All paracompact, in particular, Lindelöf spaces, are $\mu$-complete. The notion of a $q$-space was introduced in [11]: $X$ is a $q$-space if every $x\in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n\in U_n$ for each $n\in\mathbb{N}$, then $\{x_n\}$ has a cluster point in $X$. Obviously, every first countable, in particular, every metric space is a $q$-space.

In order to obtain a general version of Theorem 1.2 which implies Theorem 1.3, we introduce a special type of set-valued maps called *set tri-quotient maps* (see Section 2). Recall that tri-quotient maps (single-valued) introduced by Michael [12] are extensively investigated, see [9,10,13–15,17,20].

Every map $F:\mathcal{K}(X)\to\mathcal{K}(Y)$ satisfying conditions (1), $(2)_c$ and $(3)_c$ is a monotone set tri-quotient map (see Proposition 2.4). This allows us to derive Theorems 1.2 and 1.3 from the following one which in turn follows from Theorem 3.3 (recall that sieve-completeness, see [7] and [12], is a more general property than Čech-completeness and both they are equivalent in the class of paracompact spaces).

**Theorem 1.4.** *Let $X$ be a sieve-complete space and $F:\mathcal{K}(X)\to\mathcal{K}(Y)$ be a monotone set tri-quotient map. If $Y$ is a $\mu$-space, then $Y$ is also sieve-complete and the Lindelöf number $l(Y)$ of $Y$ is $\leqslant l(X)$.*

In the last section we apply Theorem 3.3 to show that sieve completeness is preserved under linear continuous surjections between function spaces, see Theorem 4.3. We also establish a locally compact version of Theorem 1.2.

## 2. Set tri-quotient maps

The topology of a space $X$ is denoted by $\mathcal{T}(X)$.

Let $\mathcal{S}(X)\subset 2^X$. A map $F:\mathcal{S}(X)\to 2^Y$ is called *set tri-quotient* if there exists a map $s:\mathcal{T}(X)\to\mathcal{T}(Y)$ such that:

(str1) $s(U)\subset\bigcup\{F(K)\colon K\in\mathcal{S}(X)\text{ and }K\subset U\}$;
(str2) $s(X)=Y$;
(str3) $U\subset V$ implies $s(U)\subset s(V)$;
(str4) if $y\in s(U)$ and if $\mathcal{W}$ is a cover of $\bigcup\{K\in F^{-1}(y)\colon K\subset U\}$ by open subsets of $X$, then $y\in s(\bigcup\mathcal{E})$ for some finite $\mathcal{E}\subset\mathcal{W}$.

In the above definition $F^{-1}(y)$ stands for the family $\{K\in\mathcal{S}(X)\colon y\in F(K)\}$. Let us also observe that conditions (str1) and (str4) imply that $F$ is surjective, i.e. $Y=\bigcup\{F(K)\colon K\in\mathcal{S}(X)\}$.

There is a similarity between set tri-quotient maps and Michael's tri-quotient maps [12]. To clarify this similarity, let us consider another class of maps introduced in [8].

A map $F:X\to 2^Y$ is said to be *generalized tri-quotient* if one can assign to each open $U\subset X$ an open $t(U)\subset Y$ such that:

(gtr1) $t(U)\subset F(U)=\cup\{F(x)\colon x\in U\}$;
(gtr1) $t(X)=Y$;
(gtr1) $U\subset V$ implies $t(U)\subset t(V)$;
(gtr1) if $y\in t(U)$ and if $\mathcal{W}$ is a cover of $F^{-1}(y)\cap U$ by open subsets of $X$, then $y\in t(\bigcup\mathcal{E})$ for some finite $\mathcal{E}\subset\mathcal{W}$.

We call the function $t:\mathcal{T}(X)\to\mathcal{T}(Y)$ an *assignment* for $F$. By (gtr1), every generalized tri-quotient map is surjective, i.e. $Y=F(X)$. When $F:X\to Y$ is single-valued and continuous, the above definition coincides with the definition of a tri-quotient map [12]. It was shown [8, Proposition 2.1] that $F:X\to 2^Y$ is generalized tri-quotient if and only if the projection

$\pi_Y : G(F) \to Y$ is tri-quotient, where $G(F)$ is the graph of $F$. This result, compared with [16, Theorem 2.4], shows that generalized tri-quotient maps (as well as, set tri-quotient maps) are different from the class of set-valued tri-quotient maps introduced by Ostrovsky [16].

Next lemma describes the connection between generalized tri-quotient and set tri-quotient maps.

**Lemma 2.1.** *Let $F : X \to 2^Y$ be a generalized tri-quotient map. Then $\Phi : 2^X \to 2^Y$, $\Phi(A) = \mathrm{cl}_Y(F(A))$, is monotone set tri-quotient.*

**Proof.** It follows from the definition that $\Phi$ is monotone. Let $t : \mathcal{T}(X) \to \mathcal{T}(Y)$ be an assignment for $F$. We define $s(U) = t(U)$ for every open $U \subset X$. Obviously, $s$ satisfies the first three conditions (str1)–(str3). Since $F^{-1}(y) \cap U \subset \bigcup\{K \in \Phi^{-1}(y)\colon K \subset U\}$ for all $y \in Y$ and $U \in \mathcal{T}(X)$, condition (str4) also holds. □

Similarly, every tri-quotient map $f : X \to Y$ generates a monotone set tri-quotient map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by $F(K) = f(K)$, $K \in \mathcal{K}(X)$.

Now, let us show that the map $F$ from Theorems 1.1 and 1.3 is monotone set tri-quotient.

**Proposition 2.2.** *Suppose $F : \mathcal{K}(X) \to \mathcal{K}(Y)$. Then we have*:

(a) *$F$ is monotone set tri-quotient provided $F$ satisfies conditions* (1) *and* (2)*, $X$ is Lindelöf and $Y$ a $\mu$-complete $q$-space*;
(b) *$F$ satisfies condition $(3)_c$ provided $X$ is separable metric and $Y$ is first countable. Moreover, $F$ is monotone set tri-quotient if $F$ satisfies conditions* (1) *and $(2)_c$.*

**Proof.** To prove (a), suppose $X$ is Lindelöf, $Y$ is a $\mu$-complete $q$-space and $F$ satisfies conditions (1) and (2). We say that a set $A \subset Y$ is $F$-covered by a set $B \subset X$ if for any compact $L \subset A$ there exists a compact $K \subset B$ with $L \subset F(K)$.

**Claim 2.3.** *Let $U \subset X$ be functionally open and $V \subset Y$ open such that $V$ is $F$-covered by $U$. If $\mathcal{W}$ is an open cover of $U$ and $y \in V$, then there exist a neighborhood $V_y$ of $y$ and a finite subfamily $\mathcal{E} \subset \mathcal{W}$ such that $V_y$ is $F$-covered by $\bigcup \mathcal{E}$.*

Since $U$ is functionally open, it is Lindelöf. So, we can suppose that $\mathcal{W} = \{W_n\colon n \geqslant 1\}$ is countable. Let $\{V_n\}$ be a sequence of neighborhoods of $y$ witnessing that $y$ is a $q$-point and such that $\mathrm{cl}(V_{n+1}) \subset V_n \subset V$ for all $n$. Assume the claim is false and for each $n$ choose a compact set $L_n \subset V_n$ which is not covered by any $F(K)$, $K \in \mathcal{K}(\bigcup_{i=1}^{i=n} W_i)$. Then the set

$$L = \left(\bigcup_{n=1}^{\infty} L_n\right) \cup \left(\bigcap_{n=1}^{\infty} V_n\right)$$

is closed. It is bounded in $Y$ because every infinite subset of $L$ has a cluster point. Hence $L$ is compact (recall that $Y$ is a $\mu$-space). Since $L \subset V$ and $V$ is $F$-covered by $U$, there is a compact set $K \subset U$ with $L \subset F(K)$. Then $K \subset \bigcup_{i=1}^{i=m} W_i$ for some $m$. Consequently, $L_m$ is covered by $F(K)$, which contradicts the choice of $L_m$. The claim is proved.

Now, for every open $U \subset X$ let $s(U)$ be the set of all $y \in Y$ having a neighborhood in $Y$ which is $F$-covered by a functionally open subset $W$ of $X$ with $W \subset U$. Obviously, $s(U)$ is open in $Y$ (possibly empty) and $s$ satisfies first three conditions from the definition of a set tri-quotient map. To check the last one, let $z \in s(U)$ and $\mathcal{W}$ be a cover of $\bigcup\{K \in F^{-1}(z)\colon K \subset U\}$ consisting of open in $X$ sets. Then there is a functionally open subset $W_0$ of $X$ with $W_0 \subset U$ and a neighborhood $V_0$ of $z$ such that $V_0$ is $F$-covered by $W_0$. Since $F$ is monotone, $U = \bigcup\{K \in F^{-1}(z)\colon K \subset U\}$, so $\mathcal{W}$ is an open cover of $U$. Taking a refinement of $\mathcal{W}$, if necessary, we can assume that each element of $\mathcal{W}$ is functionally open in $X$. Then $\mathcal{W}_0 = \{G \cap W_0\colon G \in \mathcal{W}\}$ is a functionally open cover of $W_0$. According to Claim 2.3, there exist a neighborhood $V_z$ of $z$ and finite $\mathcal{E}_0 \subset \mathcal{W}_0$ such that $V_z$ is $F$-covered by $\bigcup \mathcal{E}_0$.

To finish the proof of (a), let $\mathcal{E} = \{G \in \mathcal{W}\colon G \cap W_0 \in \mathcal{E}_0\}$. Because $V_z$ is $F$-covered by $\bigcup \mathcal{E}_0$ which is functionally open in $X$ (as a finite union of functionally open sets) and $\bigcup \mathcal{E}_0 \subset \bigcup \mathcal{E}$, we have that $z \in s(\bigcup \mathcal{E})$. Therefore, $F$ is set tri-quotient and monotone.

To prove (b), assume $F$ does not satisfy $(3)_c$. Then there are open sets $U \subset X$ and $V \subset Y$, an open cover $\mathcal{W}$ of $U$ and a point $y \in V$ such that every countable compact set $L \subset V$ is covered by $F(K)$ for some compact set $K \subset U$, but $y$ does not have a neighborhood which is contained in any $\bigcup\{F(K)\colon K \in \mathcal{K}(\bigcup \mathcal{E})\}$ with $\mathcal{E} \subset \mathcal{W}$ being finite. Since $X$ is separable, we can suppose $\mathcal{W} = \{W_n\}_{n \geqslant 1}$ is countable. Next, choose neighborhoods $V_n \subset V$ of $Y$ and countable compact sets $L_n \subset V_n$ such that $\{V_n\}_{n \geqslant 1}$ is a local base at $y$ and $L_n$ is not covered by any $F(K)$, $K \subset \mathcal{K}(\bigcup_{i=1}^{i=n} W_i)$. Since $L = (\bigcup_{n=1}^{\infty} L_n) \cup \{y\}$ is countable and compact, there exists a compact set $K \subset U$ with $L \subset F(K)$. As in the proof of Claim 2.3, this contradicts the choice of the sets $L_n$. Hence, $F$ satisfies condition $(3)_c$.

It follows from Proposition 2.4 below that $F$ is monotone set tri-quotient provided it satisfies conditions (1) and $(2)_c$. □

**Proposition 2.4.** *Let $X$ and $Y$ be arbitrary spaces. Then any map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfying conditions* (1)*, $(2)_c$ and $(3)_c$ is monotone set tri-quotient.*

**Proof.** Because $F$ satisfies (1), it is monotone. For every open $U \subset X$ we define $s(U)$ to be the set of all $y \in Y$ having a neighborhood $V_y$ in $Y$ such that any countable compact $L \subset V_y$ is covered by $F(K)$ for some compact set $K \subset U$. Obviously, $s(U)$ is open in $Y$. Since $F$ satisfies conditions (1), $(2)_c$ and $(3)_c$, it is easily seen that $s$ satisfies conditions (str1)–(str4). So, $F$ is set tri-quotient. □

## 3. Sieve-complete spaces

### 3.1. Proof of Theorem 1.4

First, let us recall the definition of a sieve and a sieve-complete space (see [7] and [12]). A *sieve* on a space $X$ is a sequence of open covers $\{U_\alpha\colon \alpha \in A_n\}_{n\in\mathbb{N}}$ of $X$, together with maps $\pi_n : A_{n+1} \to A_n$ such that $U_\alpha = \bigcup\{U_\beta\colon \beta \in \pi_n^{-1}(\alpha)\}$ for all $n$ and $\alpha \in A_n$. A *$\pi$-chain* for such a sieve is a sequence $(\alpha_n)$ such that $\alpha_n \in A_n$ and $\pi(\alpha_{n+1}) = \alpha_n$ for all $n$. The sieve is *complete* if for every $\pi$-chain $(\alpha_n)$, every filter base $\mathcal{F}$ on $X$ which meshes with $\{U_{\alpha_n}\colon n \in \mathbb{N}\}$ (i.e. every $B \in \mathcal{F}$ meets every $U_{\alpha_n}$) has a cluster point in $X$, or equivalently, every filter base $\mathcal{F}$ on $X$ such that each $U_{\alpha_n}$ contains some $P \in \mathcal{F}$ clusters in $X$. A space $X$ with a complete sieve is called *sieve-complete*. A sieve $(\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ is said to be *finitely additive* [12] if every cover $\{U_\alpha\colon \alpha \in A_n\}$, as well as every collection of the form $\{U_\beta\colon \beta \in \pi_n^{-1}(\alpha)\}$ with $\alpha \in A_n$, is closed under finite unions. When $\mathrm{cl}_X(U_\beta) \subset U_\alpha$ for all $\alpha \in A_n$ and $\beta \in \pi_n^{-1}(\alpha)$, the sieve is called a *strong sieve* [7]. Every sieve-complete space has a finitely additive complete sieve [12, Lemma 2.3], as well as a strong complete sieve [12, Lemma 2.4]. Moreover, the proof of [12, Lemma 2.3] shows that the complete finitely additive sieve which is obtained from a strong complete sieve is also strong. Therefore, every sieve complete space has a strong complete finitely additive sieve.

Let $\mathcal{S}(X) \subset 2^X$. We will use $\tau_V^+$ to denote the *upper Vietoris topology* on $\mathcal{S}(X)$ generated by all collections of the form $\hat{U} = \{H \in \mathcal{S}(X)\colon H \subset U\}$, where $U$ runs over the open subsets of $X$.

**Lemma 3.1.** *If $(\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ is finitely additive and a strong complete sieve on $X$, then $(\{\hat{U}_\alpha\colon \alpha \in A_n\}, \pi_n)$ is a complete sieve on $(\mathcal{K}(X), \tau_V^+)$.*

**Proof.** Because $\gamma = (\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ is a finitely additive sieve on $X$, $\hat{\gamma} = (\{\hat{U}_\alpha\colon \alpha \in A_n\}, \pi_n)$ is a sieve on $(\mathcal{K}(X), \tau_V^+)$. Let us show that $\hat{\gamma}$ is complete. Suppose $(\alpha_n)$ is a $\pi$-chain and $\mathcal{F}$ a filter base on $\mathcal{K}(X)$ which meshes with $\{\hat{U}_{\alpha_n}\}$. By [12, Lemma 2.5], $K = \bigcap U_{\alpha_n}$ is a non-empty compact subset of $X$ such that every open $W \supset K$ contains some $U_{\alpha_n}$. Then every neighborhood $\hat{W}$ of $K$ in $(\mathcal{K}(X), \tau_V^+)$ contains some $\hat{U}_{\alpha_n}$, hence $\hat{W}$ meets every $H \in \mathcal{F}$. Therefore $K$ belongs to the closure (in $(\mathcal{K}(X), \tau_V^+)$) of each $H \in \mathcal{F}$, i.e. $K$ is a cluster point of $\mathcal{F}$ in $(\mathcal{K}(X), \tau_V^+)$. □

The following analogue of $q$-spaces was introduced in [19]: call $X$ a *wq-space* if every $x \in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n \in U_n$ for each $n$, then $\{x_n\}$ is bounded in $X$. The $wq$-space property is weaker than $q$-space property and they are equivalent for $\mu$-spaces.

We say that a set-valued map $F : X \to 2^Y$ is a *wq-map* if every $x \in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n \in U_n$ for each $n$, then $\bigcup\{F(x_n)\colon n \in \mathbb{N}\}$ has a compact closure in $Y$. A version of next lemma was established first in [8, Lemma 2.3]. In the present form it appears in [19, Proposition 3.14], and later on in [5, Theorem 2.2].

**Lemma 3.2.** *([19]) Let $F : X \to 2^Y$ be a wq-map with $Y$ being a $\mu$-space. Then there exists an usco map $\Phi : X \to Y$ such that $F(x) \subset \Phi(x)$ for every $x \in X$.*

Next theorem provides the proof of Theorem 1.4.

**Theorem 3.3.** *Let $X$ be a sieve-complete space and $Y$ a $\mu$-space. If there exists a monotone set tri-quotient map $F : \mathcal{K}(X) \to 2^Y$ such that each $F(K)$, $K \in \mathcal{K}(X)$ has a compact closure in $Y$, then $Y$ is sieve-complete and $l(Y) \leqslant l(X)$.*

**Proof.** As we already mentioned, there exists a strong complete sieve $\gamma = (\{U_\alpha\colon \alpha \in A_n\}, \pi_n)$ on $X$ which is finitely additive. Then, according to Lemma 3.1, $\hat{\gamma}$ is a complete sieve on $(\mathcal{K}(X), \tau_V^+)$.

First, let us show that $F$, considered as a set-valued map from $(\mathcal{K}(X), \tau_V^+)$ into $Y$, is a $wq$-map. Since $\gamma$ is a finitely additive and strong sieve on $X$, for every $K \in \mathcal{K}(X)$ there is a chain $(\alpha_n)$ such that $K \subset U_{\alpha_n}$ for all $n$. This yields (see [12, Lemma 2.5]) that $C = \bigcap U_{\alpha_n}$ is compact and $\{U_{\alpha_n}\}$ is a base for $C$. We assign to $K$ the sequence $\{\hat{U}_{\alpha_n}\}$. If $K_n \in \hat{U}_{\alpha_n}$ for all $n$, then $H = (\bigcup K_n) \cup C$ is a compact subset of $X$ and, since $F$ is monotone, $\bigcup F(K_n) \subset F(H)$. So, $\bigcup F(K_n)$ has a compact closure in $Y$. Therefore $F$ is a $wq$-map and, by Lemma 3.2, there exists an usco map $\Phi : (\mathcal{K}(X), \tau_V^+) \to Y$ with $F(K) \subset \Phi(K)$ for every $K \in \mathcal{K}(X)$. Let us observe that $\Phi$ is onto, i.e. $Y = \bigcup\{\Phi(K)\colon K \in \mathcal{K}(X)\}$. Since the Lindelöf number of $(\mathcal{K}(X), \tau_V^+)$ is $\leqslant l(X)$, the last equality yields $l(Y) \leqslant l(X)$.

Because $F$ is set tri-quotient, there is a map $s : \mathcal{T}(X) \to \mathcal{T}(Y)$ satisfying conditions (str1)–(str4). Let $W_\alpha = s(U_\alpha)$ for every $n$ and $\alpha \in A_n$. We are going to show that $\lambda = (\{W_\alpha\colon \alpha \in A_n\}, \pi_n)$ is a complete sieve on $Y$. Since all $\gamma_n = \{U_\alpha\colon \alpha \in A_n\}$ are open covers of $X$, it follows from conditions (str2) and (str4) that each $y \in Y$ is contained in $s(\bigcup \omega_n)$ for some finite

$\omega_n \subset \gamma_n$. But each $\gamma_n$ is finitely additive, so all systems $\{W_\alpha\colon \alpha \in A_n\}$, $n \geqslant 1$, are covers of $Y$. Similarly, we can show that $W_\alpha \subset \bigcup\{W_\beta\colon \beta \in \pi_n^{-1}(\alpha)\}$ for every $n$ and $\alpha \in A_n$. The inclusions $\bigcup\{W_\beta\colon \beta \in \pi_n^{-1}(\alpha)\} \subset W_\alpha$ follow from (str3) and $U_\alpha = \bigcup\{U_\beta\colon \beta \in \pi_n^{-1}(\alpha)\}$. Therefore, $\lambda$ is a sieve on $Y$. To show that $\lambda$ is a complete sieve, suppose $(\alpha_n)$ is a $\pi$-chain and $\mathcal{F}$ is a filter base on $Y$ which meshes with $\{W_{\alpha_n}\colon n \in \mathbb{N}\}$. Then $\Phi^{-1}(\mathcal{F}) = \{\Phi^{-1}(P)\colon P \in \mathcal{F}\}$ is a filter base on $(\mathcal{K}(X), \tau_V^+)$.

**Claim 3.4.** *$\Phi^{-1}(\mathcal{F})$ meshes with $\{\hat{U}_{\alpha_n}\colon n \in \mathbb{N}\}$.*

If $y \in P \cap W_{\alpha_n}$ for some $P \in \mathcal{F}$ and $n \in \mathbb{N}$, then, by (str1), there is $K \in \mathcal{K}(X)$ with $K \subset U_{\alpha_n}$ and $y \in F(K) \subset \Phi(K)$. Therefore, $K \in \Phi^{-1}(P) \cap \hat{U}_{\alpha_n}$ which completes the proof of the claim.

Since $\hat{\gamma}$ is a complete sieve, $\Phi^{-1}(\mathcal{F})$ has a cluster point, say $K_0$, in $(\mathcal{K}(X), \tau_V^+)$.

**Claim 3.5.** *$\Phi(K_0) \cap \mathrm{cl}_Y(P) \neq \emptyset$ for each $P \in \mathcal{F}$.*

Suppose $\Phi(K_0) \cap \mathrm{cl}_Y(P) = \emptyset$ for some $P \in \mathcal{F}$. Let $V \subset Y$ be open, disjoint with $P$ and containing $\Phi(K_0)$. Because $\Phi$ is usc, there is a neighborhood $\hat{U}$ of $K_0$ in $(\mathcal{K}(X), \tau_V^+)$ such that $\Phi(K) \subset V$ for every $K \in \hat{U}$. Since $\hat{U}$ meets $\Phi^{-1}(P)$, $\Phi(K) \subset V$ for some $K \in \Phi^{-1}(P)$ which is a contradiction.

By Claim 3.5, $\mathcal{F}_0 = \{\Phi(K_0) \cap \mathrm{cl}_Y(P)\colon P \in \mathcal{F}\}$ is a filter base on $\Phi(K_0)$. Because $\Phi(K_0)$ is compact, $\mathcal{F}_0$ has a cluster point. So, $\mathcal{F}$ has a cluster point in $Y$ and $\lambda$ is a complete sieve on $Y$. □

Let us observe that the restriction in Theorem 3.3 $Y$ to be $\mu$-complete and $F$ to be monotone were used only to apply Lemma 3.2 in order to find an usco map $\Phi : (\mathcal{K}(X), \tau_V^+) \to \mathcal{K}(Y)$ with $F(K) \subset \Phi(K)$, $K \in \mathcal{K}(X)$. Therefore, the following statement holds:

**Corollary 3.6.** *Let $F : (\mathcal{K}(X), \tau_V^+) \to \mathcal{K}(Y)$ be usc and set tri-quotient with $X$ a sieve-complete space. Then $Y$ is also sieve-complete.*

**Corollary 3.7.** *For a $\mu$-space $Y$ the following are equivalent*:

(a) *$Y$ is sieve-complete.*
(b) *There exist a paracompact Čech-complete space $X$ and an open (not necessary continuous) surjection $f : X \to Y$ such that $f(K)$ has a compact closure in $Y$ for every $K \in \mathcal{K}(X)$.*
(c) *There exist a paracompact Čech-complete space $X$ and a monotone set tri-quotient map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$.*

**Proof.** (a) ⇒ (b). This implication follows from [7, Theorem 3.7] stating that every sieve-complete space is an open and continuous image of a paracompact Čech-complete space.

(b) ⇒ (c). If $f$ satisfies (b), we simply define $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ by $F(K) = \mathrm{cl}_Y f(K)$. Since $f$ is open, $F$ is set tri-quotient.

(c) ⇒ (a). This implication follows from Theorem 3.3. □

*3.2. Proof of Theorem 1.2*

According to Proposition 2.4, Theorem 3.3 and the fact that sieve and Čech-completeness are equivalent in the realm of paracompact spaces, it follows that $Y$ is complete. Moreover, Theorem 3.3 also implies that $\mathrm{dens}\, Y \leqslant \mathrm{dens}\, X$.

## 4. Remarks and some applications

Let us consider the following analogs of condition $(3)_c$ in Theorem 1.2:

(3) If $U \subset X$ and $V \subset Y$ are non-empty open sets such that for each compact $L \subset V$ there is a compact $K \subset U$ with $L \subset F(K)$, then for any open cover $\mathcal{W}$ of $U$ and any point $y \in V$ there exist a finite subfamily $\mathcal{E} \subset \mathcal{W}$ and a neighborhood $V_y$ of $y$ such that for each compact $L \subset V_y$ there is a compact $K \subset \bigcup \mathcal{E}$ with $L \subset F(K)$.
(3′) For each open cover $\mathcal{W}$ of $X$ and for each point $y \in Y$ there exist a finite subfamily $\mathcal{E} \subset \mathcal{W}$ and a neighborhood $V_y$ such that every compact $L \subset V_y$ is covered by $F(K)$ for some compact $K \subset \bigcup \mathcal{E}$.

Obviously, conditions (3) and $(3)_c$ are not comparable, while conditions (2) and (3) imply (3′). As in Lemma 2.2(b), one can show that any map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfies condition (3) if $X$ is second countable and $Y$ first countable. Moreover, we have the following lemma whose proof is similar to that one of Proposition 2.4.

**Lemma 4.1.** *If $X$ and $Y$ are arbitrary spaces and $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfies conditions* (1), (2) *and* (3), *then $F$ is monotone set tri-quotient.*

We do not know whether Theorem 1.2 is valid when $F$ satisfies conditions (1), (2) and (3′). It seems now that the related claim in [3, Theorem 5.2] was overoptimistic.

It is interesting that a locally compact version of Theorem 1.2 is true if $F$ satisfies conditions (1) and (3′).

**Proposition 4.2.** *Let $X$ be a locally compact space and $F:\mathcal{K}(X)\to\mathcal{K}(Y)$ satisfy conditions* (1) *and* (3′)*. Then $Y$ is also locally compact.*

**Proof.** Let $\mathcal{U}=\{U_\alpha\colon\ \alpha\in A\}$ be an open cover of $X$ such that each $U_\alpha$ has a compact closure in $X$. Since $F$ satisfies condition (3′), for every $y\in Y$ there exist a neighborhood $V_y$ and a finite $\mathcal{E}_y\subset\mathcal{U}$ such that every compact set $L\subset V_y$ is covered by $F(K)$ for some compact $K\subset\bigcup\mathcal{E}_y$. So, $V_y\subset\bigcup\{F(K)\colon\ K\in\mathcal{K}(U_y)\}$, where $U_y=\bigcup\{U\colon\ U\in\mathcal{E}_y\}$. Because the closure $\overline{U}_y$ is compact and $F$ is monotone, $\bigcup\{F(K)\colon\ K\in\mathcal{K}(U_y)\}\subset F(\overline{U}_y)$. Hence, each $V_y$ has a compact closure in $Y$. □

As we already observed, if $X$ is second countable and $Y$ first countable, then condition (2) implies condition (3′). In this case, Proposition 4.2 is valid whenever $F$ satisfies conditions (1) and (2). The example provided in the introduction shows that conditions (1) and (2) are not enough for the validity of Proposition 4.2 if $X$ is not separable.

We are going now to apply Theorem 3.3 for obtaining alternative proofs and improvements of some results from [3] and [19] concerning preservation of Čech-completeness under linear surjections between function spaces. Everywhere below $C(X,E)$ denotes the set of all continuous maps from $X$ into $E$ (we write $C_p(X)$ when consider real-valued functions). The set $C(X,E)$ endowed with the compact-open or the pointwise convergence topology is denoted by $C_k(X,E)$ or $C_p(X,E)$, respectively. If $u:C_k(X,E)\to C_p(Y,F)$ is a linear map, where $E$ and $F$ are normed spaces, then for every $y\in Y$ there exists a continuous linear map $\mu_y:C_k(X,E)\to F$ defined by $\mu_y(f)=u(f)(y)$, $f\in C_k(X,E)$. Following Arhangel'skii [1], we define the support $\mathrm{supp}(\mu_y)$ of $\mu_y$ to be the set of all $x\in X$ such that for every neighborhood $U$ of $x$ in $X$ there is $f\in C(X,E)$ with $f(X\backslash U)=0$ and $\mu_y(f)\neq 0$, see [19]. So, we can consider the set-valued map $\varphi:Y\to 2^X$, $\varphi(y)=\mathrm{supp}(\mu_y)$. This map has the following properties (see [2,19]):

(a) $\varphi$ is lower semi-continuous;
(b) if $L$ is a bounded set in $Y$, then $\varphi(L)$ is bounded in $X$;
(c) if $K$ is a bounded set in $X$, then the set $\varphi^*(K)=\{y\in Y\colon\ \varphi(y)\subset K\}$ is bounded in $Y$;
(d) if $u$ is surjective, then $\varphi(y)\neq\emptyset$ for all $y\in Y$.

It is shown in [3, Theorem 3.3] that if $u:C_p(X)\to C_p(Y)$ is a continuous linear surjection with $X$ and $Y$ metrizable, then $Y$ is Čech-complete provided so is $X$. This result was generalized in [19, Corollary 3.15] to the case of non-metrizable $X$ and $Y$ and function spaces of maps into normed spaces (see the hypotheses of Theorem 4.3 below). Under the same hypotheses, we can establish a sieve completeness version of this result. Of course, if $X$ and $Y$ are paracompact spaces, then Theorem 4.3 and [19, Corollary 3.15] are equivalent. In such a situation, Theorem 4.3 provides an alternative proof of [19, Corollary 3.15].

**Theorem 4.3.** *Let $u:C_k(X,E)\to C_p(Y,F)$ be a continuous linear surjection, where both $X$ and $Y$ are $\mu$-spaces and $Y$ a $wq$-space. If $X$ is sieve-complete, then so is $Y$.*

**Proof.** Since $X$ is a $\mu$-space, $Y$ is a $wq$-space and $\varphi$ satisfies condition (b), $\varphi$ is a $wq$-map. So, by Lemma 3.2, there exists an usco map $\phi:Y\to 2^X$ such that $\varphi(y)\subset\phi(y)$ for every $y\in Y$. Now, define the map $F:\mathcal{K}(X)\to 2^Y$ by $F(K)=\phi^*(K)$. Let us note that $F(K)$ may not be a compact subset of $Y$, but it has a compact closure in $Y$. Indeed, $F(K)\subset\varphi^*(K)$ and the $\mu$-completeness of $Y$ implies that the set $\varphi^*(K)$ is compact as a closed and bounded subset of $Y$ (it is closed because $\varphi$ is lower semi-continuous, and it is bounded because of (c)). For every open $U\subset$ let $s(U)=\phi^*(U)$. Since $\phi$ is upper semi-continuous, every $s(U)$ is open in $Y$. We are going to show that $s$ satisfies conditions (str1)–(str4). Because $\varphi(y)\neq\emptyset$ for all $y\in Y$, the sets $\phi(y)$, $y\in Y$, are non-empty and compact. This yields that $s$ satisfies conditions (str1) and (str2). Obviously, condition (str3) also holds. Finally, if $y\in s(U)$ and $\mathcal{W}$ is an open cover of $U$, then $\phi(y)\subset U$ and choose a finite family $\mathcal{E}\subset\mathcal{W}$ covering $\phi(y)$. So, $y\in s(\bigcup\mathcal{E})$. Therefore, $F$ is set tri-quotient and we can apply Theorem 3.3 to conclude that $Y$ is sieve-complete. □

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