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Finite-to-one maps into Euclidean manifolds and spaces with disjoint disks properties

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ABSTRACT

It is shown that every Euclidean manifold M has the following property for any $m \geqslant 1$: If $f : X \to Y$ is a perfect surjection between finite-dimensional metric spaces, then the mapping space $C(X, M)$ with the source limitation topology contains a dense G_δ-subset of maps g such that $\dim B_m(g) \leqslant m \dim f + \dim Y - (m-1)\dim M$. Here, $B_m(g) = \{(y, z) \in Y \times M \mid |f^{-1}(y) \cap g^{-1}(z)| \geqslant m\}$. The existence of residual sets of finite-to-one maps into product of manifolds and spaces having disjoint disks properties is also obtained.

Contents

1. Introduction

All spaces in the paper are assumed to be metrizable and all maps continuous. By $C(X, M)$ we denote all maps from X into M. Unless stated otherwise, the function spaces are endowed with the source limitation topology (see Section 2 for the definition).

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In [13] the last two authors established a parametric version of the Hurewicz theorem [7] on regularly branched maps into Euclidean spaces. In the present paper we extend this result by considering maps into more general spaces, in particular Euclidean manifolds. More precisely, if $m \geqslant 1$ is a fixed integer, we say that a finite-dimensional space M has the *parametric regularly m-branched maps property* (resp. *parametric regularly m-branched maps property with respect to compact spaces*) provided for every perfect surjection $f : X \to Y$ between finite-dimensional metric (resp. finite-dimensional compact metric) spaces the set of all f-regularly m-branched maps $g \in C(X, M)$ is dense in $C(X, M)$. Here, a map $g \in C(X, M)$ is said to be f-regularly m-branched if the set $B_m(g) = \{(y, z) \in Y \times M \mid |f^{-1}(y) \cap g^{-1}(z)| \geqslant m\}$ is of dimension $\leqslant q(m) = m \dim f + \dim Y - (m-1)\dim M$. If M has the above property for all $m \geqslant 1$, then we say that M has the *parametric regularly branched maps property*.

Our main results can be outlined as follows. First, we show that the parametric regularly m-branched maps property is a local property.

Theorem 1.1. *A complete finite-dimensional metrizable ANR-space M has the parametric regularly m-branched maps property provided it has it locally, i.e. every $z \in M$ has a neighborhood U_z with the same property and* $\dim U_z = \dim M$*. Moreover, all f-regularly m-branched maps $g \in C(X, M)$ form a dense G_δ-subset of $C(X, M)$ for every perfect surjection $f : X \to Y$ with X, Y finite-dimensional.*

According to [13, Theorem 1.1], every Euclidean space has the parametric regularly branched maps property. Therefore we obtain the following corollary.

Corollary 1.2. *Every Euclidean manifold has the parametric regularly branched maps property.*

Recall that a map $\varphi : A \to B$ is called m-to-1 if $|\varphi^{-1}(b)| \leqslant m$ for all $b \in B$. Theorem 1.1 implies another corollary:

Corollary 1.3. *Let the integers n_1, n_2, m, n satisfy the inequality $n_1 + n_2 + 1 \leqslant m(n - n_1)$ and M be an n-manifold. Then for every perfect map $f : X \to Y$ with* $\dim f \leqslant n_1$ *and* $\dim Y \leqslant n_2$ *the space $C(X, M)$ contains a dense G_δ-subset $\mathcal{G}$ of maps g such that g restricted to any fiber of f is an m-to-1 map.*

Indeed, under the hypotheses of the last corollary, $\dim B_{m+1}(g) \leqslant (m+1)n_1 + n_2 - mn \leqslant -1$ for every f-regularly m-branched map g. So, $B_{m+1}(g) = \varnothing$. In particular, if $\dim M \geqslant 2\dim f + \dim Y + 1$, then the set $\mathcal{G}$ from Corollary 1.3 consists of maps g such that g embeds any fiber of f into M.

For the reader's convenience, we briefly describe the steps that lead to the proof of Theorem 1.1. In Section 3 we show that for a perfect map $f : X \to Y$ between finite-dimensional metric spaces the set of all f-regularly m-branched maps $g : X \to M$ is a G_δ-subset in the space $C(X, M)$, endowed with the source limitation topology. In Section 4 we prove Theorem 1.1 in the case when X and Y are metric compacta. The following theorem, obtained in Section 5, finalizes the proof of Theorem 1.1.

Theorem 1.4. *Let M be a complete finite-dimensional metric ANR-space having the parametric regularly m-branched maps property with respect to compact spaces. Then M has the parametric regularly m-branched maps property.*

In Section 6 we consider finite-to-one maps into products of Euclidean manifolds and spaces having disjoint disks properties. The following result is established.

Theorem 1.5. *Let $f : X \to Y$ be a perfect map between metric spaces such that* $\dim f \leqslant n_1$ *and* $\dim Y \leqslant n_2$*, and M_1 a complete ANR-space having the $DD\{n_1 - 1, n_1 + n_2\}$-property. If $0 \leqslant m \leqslant n_2$ and M_2 is a complete finite-dimensional ANR-space possessing the parametric regularly branched maps property and such that* $\dim M_2 \geqslant n_2 + 1 - m$*, then there exists a dense G_δ-subset $\mathcal{G} \subset C(X, M_1 \times M_2)$ such that the diagonal product $f \triangle g$ is an $(m+1)$-to-1 map for every $g \in \mathcal{G}$.*

The general position properties $m - DD\{n, k\}$ were introduced in [1], see also [2]. We say that an *ANR*-space M has the $0 - DD\{n, k\}$-property (briefly, $DD\{n, k\}$) if any two maps $f_1 : \mathbb{I}^n \to M$, $f_2 : \mathbb{I}^k \to M$ can be approximated by maps $g_1 : \mathbb{I}^n \to M$ and $g_2 : \mathbb{I}^k \to M$, respectively, such that $g_1(\mathbb{I}^n) \cap g_2(\mathbb{I}^k) = \varnothing$. Here $\mathbb{I} = [0, 1]$ is the unit interval. Note that a product of m dendrites with dense sets of endpoints has the $DD\{m - 1, k\}$-property for any $k \geqslant 0$ (see Proposition 6.4). Theorem 1.5 was established by Kato and Matsuhashi [5] in case when X and Y are compact spaces, M_1 is a product of m dendrites with dense sets of end points, and $M_2 = \mathbb{I}^{n+1-k}$. Combining Corollary 1.3 and Theorem 1.5, we obtain the following corollary.

Corollary 1.6. *Let M_1 be a product of n_1 dendrites with dense sets of endpoints and M_2 an $(n_2 + 1 - m)$-manifold. Then for every perfect surjection $f : X \to Y$ between metric spaces such that* $\dim f \leqslant n_1$ *and* $\dim Y \leqslant n_2$ *there exists a dense G_δ-set $\mathcal{G} \subset C(X, M_1 \times M_2)$ such that $f \triangle g$ is $(m+1)$-to-1 for all $g \in \mathcal{G}$.*

2. Notations

In this section we introduce some notations that will be used in the rest of the paper. Everywhere below, by $f \triangle g$ we denote the diagonal product of maps $f : X \to Y$ and $g : X \to M$. It is easy to see that $(f \triangle g)^{-1}(y, z) = f^{-1}(y) \cap g^{-1}(z)$.

Let X be a metric space with a metric ρ. For all $x \in X$ and $\delta > 0$ we let $O(x, \delta) = \{x' \in X \mid \rho(x, x') < \delta\}$ and $B(x, \delta) = \{x' \in X \mid \rho(x, x') \leqslant \delta\}$. For a subset $A \subset X$ and an integer n we write $d_n(A) \leqslant \delta$ to denote that A admits an open in X cover ω of mesh $\leqslant \delta$ and of order $n + 1$. The latter means that every point of X is contained in at most $n + 1$ elements of ω. For a negative n, we assume in this definition that the cover ω, and hence the set A, is empty.

Recall that if (M, ρ_M) is a metric space and X is a topological space, then the neighborhood base for *the source limitation topology* at given $f \in C(X, M)$ consists of all sets

$$O_{\rho_M}(f, \varepsilon) = \{g \in C(X, M) \mid \rho_M(g, f) < \varepsilon\}$$

with $\varepsilon : X \to (0, 1]$ being a continuous positive function on X. The symbol $\rho_M(f, g) < \varepsilon$ means that $\rho_M(f(x), g(x)) < \varepsilon(x)$ for all $x \in X$. It is well known that for a paracompact space X the source limitation topology does not depend on the metric ρ_M and it has the Baire property provided M is completely metrizable. If X is compact, the source limitation topology coincides with the uniform convergence topology.

3. Parametric regularly (l, m, n_1, n_2)-branched maps

In this section M is a fixed metric space, Z is a closed subset of M, and $f : X \to Y$ is a perfect surjection such that both X and Y are finite-dimensional metric spaces.

Let $m \geqslant 1$ and $l, n_1, n_2 \geqslant 0$ be fixed integers. We say that a space M has the *parametric regularly* (l, m, n_1, n_2)*-branched maps property* (resp. *parametric regularly* (l, m, n_1, n_2)*-branched maps property with respect to compact spaces*) if for every perfect surjection $f : X \to Y$ between metric (resp. compact metric) spaces X and Y such that $\dim f \leqslant n_1$ and $\dim Y \leqslant n_2$ the set of all f-regularly (l, m)-branched maps $g \in C(X, M)$ is dense in $C(X, M)$. Here, a map $g \in C(X, M)$ is said to be f-regularly (l, m)-branched if the set $B_m(g) = \{(y, z) \in Y \times M \mid |(f \triangle g)^{-1}(y, z)| \geqslant m\}$ is of dimension $\leqslant q(l, m) = m \dim f + \dim Y - (m - 1)l$. If M has parametric regularly (l, m, n_1, n_2)-branched maps property for all n_1 and n_2 we say that M has the parametric regularly (l, m)-branched maps property. Note that a finite-dimensional space M has the parametric regularly m-branched maps property if and only if it has the parametric regularly (l, m)-branched maps property for $l = \dim M$. Note also that if $m \geqslant 2$, $\dim f \leqslant n_1$, $\dim Y \leqslant n_2$, and l is any integer such that $(m - 1)l > mn_1 + n_2$ then g is f-regularly (l, m)-branched if and only if $f \triangle g$ is $(m - 1)$-to-1. This last observation, as well as our definition of regularly (l, m, n_1, n_2)-branched maps property, will be useful in Section 6.

For a map $g \in C(X, M)$ we let

$$B_m(g, Z) = \{(y, z) \in Y \times Z \mid |(f \triangle g)^{-1}(y, z)| \geqslant m\}$$

and

$$\mathcal{R}_f(Z, l, m) = \{g \in C(X, M) \mid \dim B_m(g, Z) \leqslant q(l, m)\}.$$

Here, as before, $q(l, m) = m \dim f + \dim Y - (m - 1)l$. If X' is a closed subset of X and $g' \in C(X', M)$, the sets $B_m(g', Z)$ and $\mathcal{R}_{f|X'}(Z, l, m)$ are defined by replacing in the above formulas X, Y, f, and g by X', $f(X')$, $f|X'$, and g', respectively. Note that $B_m(g) = B_m(g, M)$. Note also that the set $R_f(M, l, m)$ is precisely the set of all $g \in C(X, M)$ such that g is f-regularly (l, m)-branched. We will show that the set $R_f(Z, l, m)$ is G_δ in $C(X, M)$ with respect to the source limitation topology for all l and m, and any closed subset Z of M. To this end we introduce some notations. For every $g \in C(X, M)$, $\delta > 0$ and $m \geqslant 2$ let $A_{m-1}(g, \delta)$ be the set of all $(y, z) \in Y \times M$ satisfying the following condition:

(1) there exists a neighborhood $V(y, z)$ of (y, z) in $Y \times M$ such that $(f \triangle g)^{-1}(V(y, z)) = S_1 \cup \cdots \cup S_{m-1}$ with $\operatorname{diam} S_i \leqslant \delta$ for any $i = 1, \ldots, m - 1$.

When $m = 1$ we let $A_0(g, \delta) = \varnothing$. Obviously, $A_{m-1}(g, \delta)$ is open in $Y \times M$ for any m and $\delta > 0$.

For a closed subset $Z \subset M$ we let

$$\mathcal{R}_f(Z, l, m, \delta) = \{g \in C(X, M) \mid d_{q(l,m)}\big(\big((f \triangle g)(X) \cap (Y \times Z)\big) \backslash A_{m-1}(g, \delta)\big) \leqslant \delta\},$$

where $q(l, m) = m \dim f + \dim Y - (m - 1)l$ and the function d_n was defined in Section 2.

Lemma 3.1. *The set $\mathcal{R}_f(Z, l, m, \delta)$ is open in $C(X, M)$ in source limitation topology for any $\delta > 0$.*

Proof. Let $g_0 \in \mathcal{R}_f(Z, l, m, \delta)$ and $h_0 = f \triangle g_0$. Since f is a perfect map, so is h_0. Thus, $h_0(X)$ is closed in $Y \times M$. By the definition of $R_f(Z, l, m, \delta)$ there exists an open in $Y \times Z$ family $\gamma(g_0, \delta)$ of mesh $\leqslant \delta$ and order $\leqslant q(l, m) + 1$, which covers $(h_0(X) \cap (Y \times Z)) \backslash A_{m-1}(g_0, \delta)$. Let $U = \bigcup \gamma(g_0, \delta)$ be the union of all elements of $\gamma(g_0, \delta)$. We fix a metric ρ_Y on Y and ρ_M on M, and consider the metric $\rho = \rho_Y + \rho_M$ on $Y \times M$.

Suppose first that $m \geqslant 2$. Then every point $w \in (Y \times Z)\backslash U$ has an open in $Y \times M$ neighborhood V_w satisfying condition (1) on page 781 and disjoint with $h_0(X)\backslash A_{m-1}(g_0, \delta)$. Put $\omega_1 = \{U, V_w \mid w \in (Y \times Z)\backslash U\}$ and let ω_2 be a locally finite open in $Y \times M$ cover of $Y \times Z$ such that the family $\{St(W, \omega_2) \mid W \in \omega_2\}$ refines the system ω_1. Here $St(W, \omega_2)$ denotes the star of the set W with respect to the family ω_2. Let also O be an open in $Y \times M$ set such that $\overline{O} \cap (Y \times Z) = \varnothing$ and $(Y \times M)\backslash(\bigcup \omega_2) \subset O$. Let $\omega = \omega_2 \cup \{O\}$ and define the function $\alpha_\omega : X \to (0, \infty)$, letting $\alpha_\omega(x) = \sup\{\rho(h_0(x), (Y \times M)\backslash W) \mid W \in \omega\}$. Since ω is locally finite, α_ω is continuous. Next claim completes the proof in the case $m \geqslant 2$.

Claim 1. *A map $g \in C(X, M)$ belongs to $\mathcal{R}_f(Z, l, m, \delta)$ provided g satisfies the inequality $\rho_M(g_0(x), g(x)) < \alpha_\omega(x)$ for all $x \in X$.*

Indeed, for any such g and all $x \in X$ we have $\rho(h_0(x), h(x)) < \alpha_\omega(x)$, where $h = f \triangle g$. This implies that for every $x \in h^{-1}((Y \times M)\backslash\overline{O})$ there exists $W_x \in \omega_2$ containing the points $h(x)$ and $h_0(x)$. It suffices to show that $(Y \times Z)\backslash U \subset A_{m-1}(g, \delta)$ since in this case $\gamma(g_0, \delta)$ would cover $(h(X) \cap (Y \times Z))\backslash A_{m-1}(g, \delta)$. Since $h(X)$ is closed in $Y \times M$, every $w \in (Y \times Z)\backslash(U \cup h(X))$ belongs to $A_{m-1}(g, \delta)$. Suppose $w^* \in (h(X) \cap (Y \times Z))\backslash U$. Then $w^* = h(x^*)$ for some $x^* \in X$ and there exists $W^* \in \omega_2$ containing both $h(x^*)$ and $h_0(x^*)$. We will show that

$$(2) \qquad h^{-1}(W^*\backslash\overline{O}) \subset h_0^{-1}(V_w) \quad \text{for some } w \in (Y \times Z)\backslash U.$$

Take $x \in h^{-1}(W^*\backslash\overline{O})$ and $W_x \in \omega_2$ with $h(x), h_0(x) \in W_x$. This implies $h_0(x) \in St(W^*, \omega_2)$. Since $St(W^*, \omega_2)$ is contained in an element of ω_1 and intersects the complement of U in $Y \times Z$, there exists $w \in (Y \times Z)\backslash U$ with $St(W^*, \omega_2) \subset V_w$. Consequently, $x \in h_0^{-1}(St(W^*, \omega_2)) \subset h_0^{-1}(V_w)$. Thus (2) holds. Since $h_0^{-1}(V_w)$ can be covered by $m - 1$ sets S_i, each of diameter $\leqslant \delta$, (2) implies that $h^{-1}(W^*\backslash\overline{O})$ also admits such a cover. Therefore, $w^* \in A_{m-1}(g, \delta)$ which completes the proof of the claim.

If $m = 1$, then $A_0(g_0, \delta) = \varnothing$, $h_0(X) \cap (Y \times Z) \subset U$, and $\mathcal{R}_f(Z, l, 1, \delta)$ consists of all $g \in C(X, M)$ such that $(f \triangle g)(X) \cap (Y \times Z)$ admits an open cover of mesh $\leqslant \delta$ and order $\leqslant q(l, 1)$. Since U admits such a cover, it suffices to find a positive function α on X such that $(f \triangle g)(X) \cap (Y \times Z) \subset U$ for any $g \in C(X, M)$ which is α-close to g_0. Let $\omega = \{U, (Y \times M)\backslash h_0(X), (Y \times M)\backslash(Y \times Z)\}$ and consider the function $\alpha_\omega : X \to (0, \infty)$ such that $\alpha_\omega(x) = \sup\{\rho(h_0(x), (Y \times M)\backslash W) \mid W \in \omega\}$. Then, for any $x \in X$, $\rho_M(g_0(x), g(x)) < \alpha_\omega(x)$ and $(f \triangle g)(x) \in Y \times Z$ implies $(f \triangle g)(x) \in U$ since there exists an element of ω that contains both points $h_0(x)$ and $(f \triangle g)(x)$. □

Corollary 3.2. *$\mathcal{R}_f(Z, l, m) = \bigcap_{k \geqslant 1} \mathcal{R}_f(Z, l, m, 1/k)$. In particular, $\mathcal{R}_f(Z, l, m)$ is a G_δ-subset of $C(X, M)$ in source limitation topology.*

Proof. Let $h : A \to B$ be a closed map between metric spaces. Suppose that $|h^{-1}(b)| = \{a_1, a_2, \dots, a_k\}$ for some point $b \in B$. Then $V = B\backslash f(A\backslash \bigcup_{i=1}^k O(a_i, \delta))$ is an open neighborhood of b and $f^{-1}(V) \subset \bigcup_{i=1}^k O(a_i, \delta)$. Conversely, if for each $\delta > 0$ a point $b \in B$ has an open neighborhood V such that $h^{-1}(V) = S_1 \cup \dots \cup S_k$ with $\operatorname{diam} S_i \leqslant \delta$ for any $i = 1, \dots, k$, then $|h^{-1}(b)| \leqslant k$. This observation implies that for a map $g : X \to M$ we have

$$\left\{(y, z) \in Y \times M \;\middle|\; \left|(f \triangle g)^{-1}(y, z)\right| < m\right\} = \bigcap_{k \geqslant 1} A_{m-1}(g, 1/k)$$

and therefore

$$(3) \qquad B_m(g, Z) = \left(\bigcup_{k \geqslant 1} (f \triangle g)(X)\backslash A_{m-1}(g, 1/k)\right) \cap (Y \times Z).$$

Thus the inclusion $R_f(Z, l, m) \subset \bigcap_{k \geqslant 1} \mathcal{R}_f(Z, l, m, 1/k)$ holds.

Consider now $g \in \bigcap_{k \geqslant 1} \mathcal{R}_f(Z, l, m, 1/k)$. Note that if $k > k'$ then $A_{m-1}(g, 1/k) \subset A_{m-1}(g, 1/k')$. Note also that each $A_{m-1}(g, 1/k)$ is open in $Y \times M$. This and (3) imply that the set $B_m(g, Z)$ is the union of closed in $Y \times M$ sets that admit open covers of order $\leqslant q(l, m)$ and of arbitrary small mesh. Therefore $\dim B_m(g, Z) \leqslant q(l, m)$ and hence $g \in R_f(Z, l, m)$. □

4. Local regularly branched maps property implies global: compact case

In this section we will prove Theorem 1.1 in the compact case. Everywhere in this section (M, ρ_M) is a finite-dimensional complete metric *ANR*-space. The following statement is Lemma 2.1 from [14].

Proposition 4.1. *Let M be a completely metrizable ANR-space. There exists a complete metric ρ on M satisfying the following extension property*:

if X is a paracompact space, A is a closed subset of X, and $\varphi : X \to M$ is a map, then for every continuous function $\alpha : X \to (0, 1]$ and every map $\psi : A \to M$ with $\rho(\varphi(x), \psi(x)) < \alpha(x)/8$ for all $x \in A$ there exists a map $\overline{\psi} : X \to M$ extending ψ such that $\rho(\varphi(x), \overline{\psi}(x)) < \alpha(x)$ for all $x \in X$.

In what follows we assume that the metric ρ_M has the above property.

Theorem 4.2. *A complete metric finite-dimensional ANR-space M possesses the parametric regularly m-branched maps property with respect to compact spaces provided every $z \in M$ has a neighborhood U_z with this property such that $\dim U_z = \dim M$.*

Proof. We use an idea from the proof of [8, Theorem 3.6]. For every $z \in M$ choose $\varepsilon_z > 0$ such that U_z contains closed ball $B(z, 2\varepsilon_z)$. Let $f : X \to Y$ be a map between finite-dimensional metric compacta. Following notations from Section 3, for every $g \in C(X, M)$ we let $\mathcal{R}(z) = \mathcal{R}_f(B(z, \varepsilon_z), \dim M, m)$.

Claim 2. *For every $z \in M$ the set $\mathcal{R}(z)$ is dense in $C(X, M)$.*

We fix $z_0 \in M$, $g_0 \in C(X, M)$, and $\varepsilon > 0$ with $\varepsilon < \varepsilon_z$. Let $A = g_0^{-1}(B(z_0, 2\varepsilon_{z_0}))$ and $f_A = f|A$. Since U_{z_0} has the parametric regularly m-branched maps property, there exists an f_A-regularly m-branched map $g_A : A \to U_{z_0}$ which is $\varepsilon/8$-close to $g_0|A$. For the set

$$B_m(g_A) = \bigl\{(y, z) \in f(A) \times U_{z_0} \bigm| \bigl|(f_A \triangle g_A)^{-1}(y, z)\bigr| \geqslant m\bigr\}$$

we have

$$\dim B_m(g_A) \leqslant m \dim f_A + \dim f(A) - (m-1) \dim U_{z_0}.$$

Since $\dim U_{z_0} = \dim M$, $\dim f_A \leqslant \dim f$, and $\dim f(A) \leqslant \dim Y$, we have $\dim B_m(g_A) \leqslant q(m) = m \dim f + \dim Y - (m-1) \dim M$. According to the extension property of (M, ρ_M) from Proposition 4.1, g_A can be extended to a map $g \in C(X, M)$ with g being ε-close to g_0. It is easily seen that $g^{-1}(z) \subset A$ for every $z \in B(z_0, \varepsilon_{z_0})$. Hence $B_m(g, B(z_0, \varepsilon_{z_0})) \subset B_m(g_A)$ and therefore $\dim B_m(g, B(z_0, \varepsilon_{z_0})) \leqslant q(m)$. Consequently, $g \in \mathcal{R}(z)$ which completes the proof of the claim.

Now we will show that the set $\mathcal{R}_f(M, m)$ of all f-regularly m-branched maps $g \in C(X, M)$ is dense in $C(X, M)$. To this end, fix $g_0 \in C(X, M)$ and $\eta > 0$, and choose finitely many points $z_i \in M$, $i = 1, \dots, n$, such that $g_0(X) \subset \bigcup_{i=1}^n B(z_i, \varepsilon_{z_i}/2)$. The latter is possible since X is a compact space. Let $\delta = \min\{\eta, \varepsilon_{z_i}/2 \mid i = 1, \dots, n\}$. By Corollary 3.2 and the above claim, each $\mathcal{R}(z_i)$ is a dense G_δ-subset of $C(X, M)$. Therefore so is $\mathcal{R} = \bigcap_{i=1}^n \mathcal{R}(z_i)$. Hence there exists $g \in \mathcal{R}$ which is δ-close to g_0. It is easily seen that $g(X) \subset \bigcup_{i=1}^n B(z_i, \varepsilon_{z_i})$ and $B_m(g, M) \subset \bigcup_{i \leqslant n} B_m(g, B(z_i, \varepsilon_{z_i}))$. Note that $\dim B_m(g, B(z_i, \varepsilon_{z_i})) \leqslant q(m)$ for every i since $g \in \mathcal{R}(z_i)$. Each set $B_m(g, B(z_i, \varepsilon_{z_i}))$ is F_σ in M and therefore we can apply the countable sum theorem for dim. This implies $\dim B_m(g, M) \leqslant q(m)$. Thus, $g \in \mathcal{R}_f(M, \dim M, m)$, which completes the proof. □

5. Regularly branched maps: general case

In this section we prove Theorem 1.1 in general case. It is done by showing in Theorem 5.4 that if M has the regularly m-branched maps property with respect to compact spaces, then it has the regularly m-branched maps property. The idea of the proof is to first show that the property holds with respect to perfect maps between CW complexes and then approximate maps between metric spaces by maps between complexes. In this section, as before, M is a completely metrizable *ANR*-space. We also assume that ρ_M is a complete metric on M such that (M, ρ_M) satisfies the extension property from Proposition 4.1.

We begin by establishing several technical results.

Lemma 5.1. *Let $f : N \to L$ be a perfect map between simplicial complexes endowed with CW topology and $g \in C(N, M)$. Then we have*:

- *A set $A \subset (f \triangle g)(N)$ is closed in $L \times M$ iff $A \cap (\sigma \times M)$ is closed in $(\sigma \times M)$ for any simplex σ of L*;
- *The set $B_m(g, M)$ is F_σ in $L \times M$.*

Proof. Since f is perfect, for every simplex $\tau \subset N$ there are finitely many simplexes $\sigma_i \subset L$, $i = 1, \dots, k$, with $\tau \subset \bigcup_{i=1}^k f^{-1}(\sigma_i)$. This implies that a set $F \subset N$ is closed in N if and only if $F \cap f^{-1}(\sigma)$ is closed in $f^{-1}(\sigma)$ for all $\sigma \subset L$. Suppose $A \cap (\sigma \times M)$ is closed in $(\sigma \times M)$ for any simplex $\sigma \subset L$. Let $h = f \triangle g$. Since $h^{-1}(\sigma \times M) = f^{-1}(\sigma)$, $h^{-1}(A) \cap f^{-1}(\sigma)$ is closed in $f^{-1}(\sigma)$ for all σ. Consequently, $h^{-1}(A)$ is closed in N. On the other hand, h is a closed map and $h(h^{-1}(A)) = A$ since $A \subset h(X)$. Hence, A is closed in $L \times M$.

To prove the second item, for every $k \geqslant 1$ let $B_m^k(g, M)$ be the set of all $(y, z) \in B_m(g, M)$ such that $(y, z) = h(x_1) = \cdots = h(x_m)$ for some points $x_1, \dots, x_m \in N$ with $d(x_i, x_j) \geqslant 1/k$, $i \neq j$. Here, d is any metric generating the metric topology of N. It is easy to see that $B_m(g, M) = \bigcup_{k \geqslant 1} B_m^k(g, M)$. Using the fact that h is a perfect map and $\sigma \times M$ is metrizable, and compactness of σ, one can show that each $B_m^k(g,M)\cap(\sigma\times M)$ is closed in $\sigma\times M$ for every simplex $\sigma\subset L$. Therefore, all sets $B_m^k(g,M)$, $k\geqslant 1$, are closed in $L\times M$, and hence $B_m(g,M)$ is F_σ. □

The following lemma is about extensions of regularly branched maps.

Lemma 5.2. *Let (M,ρ_M) be a completely metrizable ANR space that has the parametric regularly (l,m,n_1,n_2)-branched maps property with respect to compact spaces. Suppose that the metric ρ_M satisfies the extension property from Proposition* 4.1. *Let $f: X\to Y$ be a map between metric compacta, such that* $\dim f\leqslant n_1$ *and* $\dim Y\leqslant n_2$. *Let also Y_0 be a closed subspace of Y, $X_0=f^{-1}(Y_0)$, and $\varepsilon>0$ be a number. If $g: X\to M$ and $g_0: X_0\to M$ are maps such that g_0 is $f|X_0$-regularly (l,m)-branched and $\rho_M(g|X_0,g_0)<\varepsilon/16$ then there exists an extension $g': X\to M$ of g_0 which is f-regularly (l,m)-branched and such that $\rho_M(g,g')<\varepsilon$.*

Proof. For a space A, a function $\varphi: A\to M$, and a number $\delta>0$ by $O(\varphi,\delta)$ we denote open δ-neighborhood of φ in $C(A,M)$ with respect to ρ_M. Let $\{Y_i\}_{i\geqslant 1}$ be a sequence of closed subsets of Y_i such that $Y_i\subset Y_{i+1}$ and $\bigcup_{i\geqslant 1}Y_i=Y\backslash Y_0$. Let also $X_i=f^{-1}(Y_i)$. There exists a map $\bar g_0: X\to M$, extending g_0, such that $\rho_M(\bar g_0,g)<\varepsilon/2$. By induction we construct a sequence of maps $\bar g_i: X\to M$ and $g_i=\bar g_i|X_i$, and numbers $\delta_i>0$, $i\geqslant 0$, such that the following conditions are satisfied for all $i\geqslant 1$ (see page 781 for the definition of the sets $\mathcal{R}_{f|X_i}(M,l,m)$ and $\mathcal{R}_{f|X_i}(M,l,m,1/i)$):

(i) $\bar g_i|X_0=g_0$,

(ii) $\varepsilon/2=\delta_0>\delta_1>\cdots>\delta_i>\delta_{i+1}>\cdots$,

(iii) $g_i\in\mathcal{R}_{f|X_i}(M,l,m)$,

(iv) $O(g_i,\delta_i)\subset\mathcal{R}_{f|X_i}(M,l,m,1/i)$,

(v) $\rho_M(\bar g_{i-1},\bar g_i)<\delta_i/2^{i+1}$.

Let $\delta_0=\varepsilon/2$ and suppose that $\bar g_{i-1}$ has been constructed, where $i\geqslant 1$. We construct $\bar g_i$ as follows. Note first that $\dim f|X_i\leqslant\dim f\leqslant n_1$ and $\dim Y_i\leqslant\dim Y\leqslant n_2$. Therefore there exists a map $g_i: X_i\to M$ such that $g_i\in\mathcal{R}_{f|X_i}(M,l,m)$ and $\rho_M(g_i,\bar g_{i-1}|X_i)<\delta_{i-1}/(8\cdot 2^{i+1})$. By Lemma 3.1 and Corollary 3.2 there exists $\delta_i>0$ such that $O(g_i,\delta_i)\subset\mathcal{R}_{f|X_i}(M,l,m,1/i)$. Let $\bar g_i: X\to M$ be an extension of the map $g_0\cup g_i: X_0\cup X_i\to M$ such that $\rho_M(\bar g_{i-1},\bar g_i)<\delta_{i-1}/2^{i+1}$. It is easy to verify that the conditions (i)–(v) are satisfied.

We let $g'(x)=\lim_{i\to\infty}g_i(x)$ for all $x\in X$. It is easy to see that g' is a continuous extension of g_0. Note also that

$$\rho_M(g',g)\leqslant\rho_M(g,\bar g_0)+\sum_{i=1}^{\infty}\rho_M(\bar g_{i-1},\bar g_i)<\varepsilon/2+\sum_{i=1}^{\infty}\delta_{i-1}/2^{i+1}<\varepsilon/2+\sum_{i=1}^{\infty}\varepsilon/2^{i+1}=\varepsilon.$$

It remains to check that g' is f-regularly (l,m)-branched. Note first that for all $i\geqslant 1$,

$$\rho_M(g_i,g'|X_i)\leqslant\sum_{j=i}^{\infty}\rho_M(\bar g_j,\bar g_{j+1})<\sum_{j=i}^{\infty}\delta_j/2^{j+2}<\sum_{j=i}^{\infty}\delta_i/2^{j+2}<\delta_i.$$

Therefore due to condition (iv) we have $g'|X_i\in\mathcal{R}_{f|X_i}(M,l,m,1/i)$ for all $i\geqslant 1$. Due to the fact that $X_i\subset X_k$ for $k>i$, for a function $\varphi: X\to M$ the condition $\varphi|X_k\in\mathcal{R}_{f|X_k}(M,l,m,\eta)$ implies $\varphi|X_i\in\mathcal{R}_{f|X_i}(M,l,m,\eta)$ for any $\eta>0$. Thus $g'|X_i\in\mathcal{R}_{f|X_i}(M,l,m,1/k)$ for all $k\geqslant i$. On the other hand, it is not hard to see that for a closed subset $A\subset X$ and positive numbers $\eta_1<\eta_2$ we have $\mathcal{R}_{f|A}(M,l,m,\eta_1)\subset\mathcal{R}_{f|A}(M,l,m,\eta_2)$. Hence $g'|X_i\in\mathcal{R}_{f|X_i}(M,l,m,1/k)$ for all $k\geqslant 1$ and therefore $g'|X_i\in\bigcap_{k\geqslant 1}\mathcal{R}_{f|X_i}(M,l,m,1/k)=\mathcal{R}_{f|X_i}(M,l,m)$ by Corollary 3.2. This implies that the set $B_m(g'|X_i)=\{(y,z)\in Y\times M \mid |(f|X_i\triangle g'|X_i)^{-1}(z)|\geqslant m\}$ is of dimension $\leqslant m\dim f|X_i+\dim Y_i-(m-1)l\leqslant q(l,m)=m\dim f+\dim Y-(m-1)l$ for all $i\geqslant 1$. By the condition of the lemma and since g' extends g_0 the same is true about $B_m(g'|X_0)$, i.e. $\dim B_m(g'|X_0)\leqslant q(l,m)$. Due to the fact that $X_i=f^{-1}(Y_i)$ for all $i\geqslant 0$ and $\bigcup_{i\geqslant 0}Y_i=Y$ we have $B_m(g',M)=\bigcup_{i\geqslant 0}B_m(g'|X_i)$. Since each $B_m(g'|X_i)$ is F_σ in $Y\times M$, we can apply the countable sum theorem for dim. This implies $\dim B_m(g',M)\leqslant q(l,m)$ and hence $g\in\mathcal{R}_f(M,l,m)$, as required. □

Let $f: X\to Y$ be a map between finite-dimensional spaces. Note that our definition of an f-regularly (l,m)-branched map $g: X\to M$ is applicable even in the case when X and Y are general (not necessarily metrizable) topological spaces. In the following proposition, the roles of X and Y are played by simplicial complexes, endowed with CW topology.

Lemma 5.3. *Suppose that a complete metric ANR space M has the parametric regularly (l,m,n_1,n_2)-branched maps property with respect to compact spaces. Let N, L be simplicial complexes, endowed with CW topology, and $f: N\to L$ a finite-dimensional perfect map with* $\dim f\leqslant n_1$ *and* $\dim L\leqslant n_2$. *Then all f-regularly (l,m)-branched maps $g\in C(N,M)$ form a dense subset of $C(N,M)$.*

Proof. Fix a metric ρ_M on M that satisfies the extension property from Proposition 4.1. Consider an arbitrary map $g_{-1} \in C(N, M)$ and a function $\alpha \in C(N, (0, 1])$. Our goal is to find an f-regularly m-branched map $g \in C(N, M)$ which is α-close to g_{-1}.

Let $L^{(i)}$, $i \geqslant 0$, denote the i-dimensional skeleton of L and $K^i = f^{-1}(L^{(i)})$. We put $L^{(-1)} = \varnothing$. Construct inductively a sequence $g_i : N \to M$ of maps such that for all $i = 0, 1, 2, \ldots, d = \dim L$ the following conditions are satisfied:

$(4)_i$ $g_i|K^{i-1} = g_{i-1}|K^{i-1}$,

$(5)_i$ $g_i|K^i \in \mathcal{R}_{f|K^i}(M, l, m)$,

$(6)_i$ $\rho_M(g_{i-1}, g_i) < \alpha/2^{i+1}$.

Assume that the map $g_{i-1} : N \to M$ has been constructed and consider the complement $L^{(i)} \setminus L^{(i-1)} = \bigsqcup_{j\in\mathcal{J}} \overset{\circ}{\sigma}_j$, which is the discrete union of open i-dimensional simplexes $\overset{\circ}{\sigma}_j$. Fix a simplex $\sigma = \sigma_j$. Let $\sigma^{(i-1)}$ denote the $(i-1)$-dimensional skeleton (the boundary) of σ. We put $S = f^{-1}(\sigma)$ and $\widetilde{S} = f^{-1}(\sigma^{(i-1)})$. Note that, due to condition $(4)_{i-1}$ and $(5)_{i-1}$ above, $g_{i-1}|\widetilde{S} \in \mathcal{R}_{f|\widetilde{S}}(M, l, m)$. Note also that $\dim f|S \leqslant \dim f \leqslant n_1$ and $\dim \sigma \leqslant \dim L \leqslant n_2$. Thus, we can apply Lemma 5.2 to the maps $f|S$, $g_{i-1}|S$, and $g_{i-1}|\widetilde{S}$ (in place of f, g, and g_0 from this lemma), to the metric compact spaces S, σ, and $\sigma^{(i-1)}$ (in place of X, Y, and Y_0 from the lemma), and to the number $\varepsilon = (\min \alpha|S)/(8 \cdot 2^{i+1}) > 0$. Therefore there exists a map $g_\sigma : S \to M$ such that

(i)′ $g_\sigma|\widetilde{S} = g_{i-1}|\widetilde{S}$,

(ii)′ g_σ is $f|S$-regularly (l, m)-branched,

(iii)′ $\rho_M(g_{i-1}|S, g_\sigma) < \alpha/(8 \cdot 2^{i+1})$.

Define a map $g_i' : K^i \to M$ by the formula

$$g_i'(x) = \begin{cases} g_{i-1}(x) & \text{if } x \in f^{-1}(L^{(i-1)}), \\ g_{\sigma_j}(x) & \text{if } x \in f^{-1}(\sigma_j). \end{cases}$$

The extension property of the metric ρ_M and the conditions (i)′ and (iii)′ from above guarantee the existence of a map $g_i : N \to M$, extending g_i', such that $\rho_M(g_{i-1}, g_i) < \alpha/2^{i+1}$. It is easy to see that conditions $(4)_i$ and $(6)_i$ are satisfied. It remains to verify condition $(5)_i$. Consider the sets

$$B_m(g_i|K^i) = \{(y, z) \in L^{(i)} \times M \mid |(f \triangle g_i)^{-1}(y, z)| \geqslant m\}.$$

Condition $(5)_{i-1}$ implies that

$$\dim B_m(g_{i-1}|K^{i-1}) \leqslant m \dim f + \dim L^{(i-1)} - (m-1)l. \tag{7}$$

It is not hard to see that

$$B_m(g_i|K^i) = B_m(g_{i-1}|K^{i-1}) \cup \Big(\bigcup_{j\in\mathcal{J}} B_m(g_i|f^{-1}(\sigma_j))\Big). \tag{8}$$

Since $g_i|f^{-1}(\sigma_j) = g_{\sigma_j}$, condition (ii)′ implies

$$\dim B_m(g_i|f^{-1}(\sigma_j)) \leqslant m \dim f + \dim \sigma_j - (m-1)l. \tag{9}$$

Note that $\dim L^{(i-1)} = i - 1 < i = \dim L^{(i)} = \dim \sigma_j$. The first item of Lemma 5.1 implies that a set A is closed in $L^{(i)} \times M$ if and only if all intersections $A \cap (L^{(i-1)} \times M)$ and $A \cap (\sigma_j \times M)$, $j \in J$, are closed in $L^{(i-1)} \times M$ and $\sigma_j \times M$, respectively. Moreover, due to second item of Lemma 5.1, the set $B_m(g_i|K^i)$ is the union of countably many closed sets $F_k \subset L^{(i)} \times M$, $k \geqslant 1$. By (8) for every $k \geqslant 1$ and $j \in J$ we have

$$F_k \cap (L^{(i-1)} \times M) \subset B_m(g_{i-1}|K^{i-1}) \quad \text{and} \quad F_k \cap (\sigma_j \times M) \subset B_m(g_i|f^{-1}(\sigma_j)).$$

So, according to (7) and (9), $\dim F_k \cap (L^{(i-1)} \times M) \leqslant m \dim f + \dim L^{(i)} - (m-1)l$ and $\dim F_k \cap (\sigma_j \times M) \leqslant m \dim f + \dim L^{(i)} - (m-1)l$. This yields $\dim F_k \leqslant m \dim f + \dim L^{(i)} - (m-1)l$ for every k. Consequently, $\dim B_m(g_i|K^i) \leqslant m \dim f + \dim L^{(i)} - (m-1)l$. Thus $g_i|K^i \in \mathcal{R}_{f|K^i}(M, l, m)$.

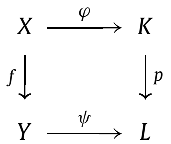
Let $g = g_d$, where $d = \dim L$. Conditions $(6)_i$ imply $\rho_M(g_0, g) < \alpha$, and due to conditions $(5)_i$ g is f-regularly (l, m)-branched. This completes the proof. □

Now we are ready to show that the parametric regularly branched maps property with respect to compact spaces implies the parametric regularly branched maps property. Our proof is based on some arguments from the proof of [2, Proposition 3.4].

Theorem 5.4. *Suppose a complete metric ANR-space M has the parametric regularly (l, m, n_1, n_2)-branched maps property with respect to compact spaces. Then M has the parametric regularly (l, m, n_1, n_2)-branched maps property.*

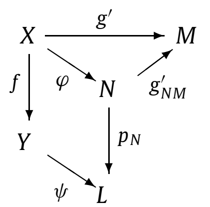
Proof. We fix a metric ρ_M on M. Let $f : X \to Y$ be a perfect map such that X and Y are finite-dimensional metric spaces, and $\dim f \leqslant n_1$, $\dim Y \leqslant n_2$. Since $C(X, M)$, endowed with the source limitation topology, has the Baire property, according to Proposition 3.1 it suffices to show that the set $\mathcal{R}_f(M, l, m, \delta)$ is dense in $C(X, M)$ for any fixed $\delta > 0$. Let $\varepsilon \in C(X, (0, 1])$ and $g_0 \in C(X, M)$. We will find a map $g \in \mathcal{R}_f(M, l, m, \delta)$ such that $\rho_M(g_0, g) < \varepsilon$. Since M is an *ANR*, g_0 can be approximated by simplicially factorizable maps (i.e. maps $\widetilde{g} \in C(X, M)$ such that $\widetilde{g} = g^D \circ g_D$, where g_D is a map from X into a simplicial complex D and $g^D : D \to M$). Therefore g_0 itself can be assumed to be simplicially factorizable. Choose a simplicial complex D and maps $g_D : X \to D$, $g^D : D \to M$ with $g_0 = g^D \circ g_D$. The metric ρ_M induces a continuous pseudometric ρ_D on D defined by $\rho_D(x, y) = \rho_M(g^D(x), g^D(y))$. By [3] and [10] D, being a stratifiable *ANR*, is a neighborhood retract of a locally convex space. Hence we can apply [1, Lemma 8.1] to find an open cover ω_0 of X satisfying the following condition: if $\varphi : X \to K$ is an ω_0-map into a paracompact space K (i.e. $\varphi^{-1}(\upsilon)$ refines ω_0 for some open cover υ of K), then there exists a map $\varphi_D : G \to D$, where G is an open neighborhood of the closure $\overline{\varphi(X)}$ in K, such that g_D and $\varphi_D \circ \varphi$ are $\varepsilon/2$-close with respect to the pseudometric ρ_D. Let ω be an open cover of X refining ω_0 with $\operatorname{mesh} \omega < \delta/2$ and $\inf\{\varepsilon(x) \mid x \in U\} > 0$ for all $U \in \omega$.

According to [1, Theorem 6], there exists an open cover ν of Y such that for any ν-map $\psi : Y \to L$ into a simplicial complex L there exists an ω-map $\varphi : X \to K$ into a simplicial complex K and a perfect PL-map $p : K \to L$ with $\psi \circ f = p \circ \varphi$ and $\dim p = \dim f$. We can assume that the cover ν is locally finite of mesh $\leqslant \delta/2$ and of order $\leqslant \dim Y + 1$. Take L to be the nerve of the cover ν and $\psi : Y \to L$ the corresponding natural map. Then there exist a simplicial complex K and maps p and φ satisfying the above conditions. Hence, the following diagram is commutative:



Note that we have $\dim p = \dim f \leqslant n_1$ and $\dim L \leqslant \dim Y \leqslant n_2$.

Since K is paracompact, the choice of the cover ω (and ω_0) guarantees the existence of a map $\varphi_D : G \to D$, where $G \subset K$ is an open neighborhood of $\overline{\phi(X)}$, such that g_D and $\varphi_D \circ \alpha$ are $\varepsilon/2$-close with respect to ρ_D. Then, according to the definition of ρ_D, $g' = g^D \circ \varphi_D \circ \varphi$ is $\varepsilon/2$-close to g_0 with respect to ρ_M. Replacing the triangulation of K by a suitable subdivision, we may additionally assume that no simplex of K meets both $\overline{\varphi(X)}$ and $K \backslash G$. Therefore the union N of all simplices $\sigma \in K$ with $\sigma \cap \overline{\varphi(X)} \neq \varnothing$ is a subcomplex of K and $N \subset G$. Moreover, since N is closed in K, $p_N = p|N : N \to L$ is a perfect map. Clearly, $\dim p_N \leqslant \dim p = \dim f \leqslant n_1$. We have the following commutative diagram, where $g'_{NM} = g^D \circ \varphi_D$:



Using that φ is an ω-map and $\inf\{\varepsilon(x) \mid x \in U\} > 0$ for all $U \in \omega$, we can construct a continuous function $\varepsilon' : N \to (0, 1]$ with $\varepsilon' \circ \varphi \leqslant \varepsilon$. By Lemma 5.3, there exists a p_N-regularly (l, m)-branched map $g_{NM} \in C(N, M)$ which is $\varepsilon'/2$-close to g'_{NM}. Let $g = g_{NM} \circ \varphi$. Then g and $g'_{NM} \circ \varphi$ are $\varepsilon/2$-close since $\varepsilon' \circ \varphi \leqslant \varepsilon$. On the other hand, $g'_{NM} \circ \varphi = g'$ is $\varepsilon/2$-close to g_0. Hence, g and g_0 are ε-close.

It remains to show that $g \in \mathcal{R}_f(M, l, m, \delta)$. Recall that this is equivalent to the property $d_{q(l,m)}((f \triangle g)(X) \backslash A_{m-1}(g, \delta)) \leqslant \delta$, where $q(l, m) = m \dim f + \dim Y - (m-1)l$. Recall also that the function d_q and the set $A_{m-1}(g, \delta)$ were defined in Section 2, see page 781. Consider the map $\Psi = \psi \times id_M : Y \times M \to L \times M$.

Claim 3. *The set $\Psi((f \triangle g)(X) \backslash A_{m-1}(g, \delta))$ is contained in the set $B_m(g_{NM}) = \{(w, z) \in L \times M \mid |(p_N \triangle g_{NM})^{-1}(w, z)| \geqslant m\}$.*

Obviously, the claim is true for $m = 1$. Suppose $m \geqslant 2$ and $\Psi(y, z) \notin B_m(g_{NM})$ for some $(y, z) \in (f \triangle g)(X) \backslash A_{m-1}(g, \delta)$. Hence, $|(p_N \triangle g_{NM})^{-1}(\psi(y), z)| \leqslant m - 1$. Then $|(f \triangle g)^{-1}(y, z)| \leqslant m - 1$ and since all fibers of φ are of diameter $\delta/2$ (recall that $\operatorname{mesh} \omega < \delta/2$ and φ is an ω-map), the set $(f \triangle g)^{-1}(y, z)$ is covered by $m - 1$ closed subsets S_i of X, $i = 1, 2, \ldots, m - 1$, each of diameter $\leqslant \delta/2$. Further, for each i choose an open set $W_i \subset X$ such that $S_i \subset W_i$ and

$\operatorname{diam} W_i < \delta$. Then $(f \triangle g)^{-1}(y,z) \subset \bigcup_{i=1}^{m-1} W_i$. Since $f \triangle g$ is a perfect map, there exists a neighborhood V of $(y,z) \in Y \times M$ with $(f \triangle g)^{-1}(V) \subset \bigcup_{i=1}^{m-1} W_i$. Consequently, $(y,z) \in A_{m-1}(h,\delta)$, a contradiction. The claim is proved.

Since ψ is a ν-map, there exists an open cover γ of L with $\psi^{-1}(\gamma)$ refining ν. Then $\gamma' = \{\Gamma \times O(z,\delta/2) \mid \Gamma \in \gamma,\ z \in M\}$ is an open cover of $L \times M$. The space $L \times M$, being a product of two stratifiable spaces, is hereditary paracompact, see [6]. Therefore $B_m(g_{NM})$ is also paracompact. On the other hand, since g_{NM} is p_N-regularly (l,m)-branched, $\dim B_m(g_{NM}) \leqslant q_{p_N}(l,m)$, where $q_{p_N}(l,m) = m \dim p_N + \dim L - (m-1)l$. Consequently, $B_m(g_{NM})$ admits a locally finite open cover η of order $\leqslant q_{p_N}(m)+1$ such that η refines γ'. According to Claim 3, $\Psi^{-1}(\eta)$ is a locally-finite open cover of $(f \triangle g)(X)\backslash A_{m-1}(g,\delta)$ of order $\leqslant q_{p_N}(m)+1$. Moreover, each element of $\Psi^{-1}(\eta)$ is contained in $\psi^{-1}(\Gamma) \times B(z,\delta/2)$ for some $\Gamma \in \gamma$ and $z \in M$. Note that each $\psi^{-1}(\Gamma)$, $\Gamma \in \gamma$, is of diameter $< \delta/2$. Hence $\operatorname{mesh} \Psi^{-1}(\eta) < \delta$. Finally, $q_{p_N}(l,m) \leqslant q(l,m)$ since $\dim p_N \leqslant \dim f$ and $\dim L \leqslant \dim Y$. Therefore $(f \triangle g)(X)\backslash A_{m-1}(g,\delta)$ admits an open cover of mesh $\leqslant \delta$ and order $\leqslant q(m)+1$. This implies $g \in \mathcal{R}_f(M,l,m,\delta)$, which completes the proof. □

Combining Theorems 4.2 and 5.4, and Corollary 3.2 we obtain the main result of this section.

Theorem 5.5. *A complete finite-dimensional metrizable ANR-space M has the parametric regularly m-branched maps property provided for every $z \in M$ there exists a neighborhood U_z with the same property and such that $\dim U_z = \dim M$. Moreover, all f-regularly m-branched maps $g \in C(X,M)$ form a dense G_δ-subset of $C(X,M)$ for every perfect surjection $f : X \to Y$ with X, Y finite-dimensional.*

6. Finite-to-one maps into products

In this section we prove Theorem 1.5. The following characterization of spaces with the $DD\{n,k\}$-property follows from the more general Proposition 6.4 in [2].

Lemma 6.1. *A complete ANR-space M has the $DD\{n,k\}$-property, where $n \leqslant k$, if and only if M satisfies the following condition:*

Let X be a metric compactum and $A \subset B \subset X$ two σ-compact subsets with $\dim A \leqslant n$ and $\dim B \leqslant k$. Then every $g \in C(X,M)$ can be approximated by a map $g' \in C(X,M)$ such that $(g')^{-1}(g'(x)) \cap B = x$ for all $x \in A$.

We will also need the following observation.

Proposition 6.2. *Let M be a metric space, $f : X \to Y$ a perfect map between finite-dimensional metric spaces and $m \geqslant 1$. Then $\mathcal{F}_m(X,M) = \{g \in C(X,M) \mid f \triangle g \text{ is } m\text{-to-}1\}$ is a G_δ-subset of $C(X,M)$.*

Proof. Chose integers l and n such that $\dim f \leqslant n$, $\dim Y \leqslant n$, and $ml > (m+1)(n+1)$. Then a map $g \in C(X,M)$ is f-regularly $(l,m+1)$-branched if and only if $f \triangle g$ is m-to-1. Now the proposition follows from Corollary 3.2. □

Now we obtain the central result of this section.

Theorem 6.3. *Let $f : X \to Y$ be a perfect map between metric spaces with $\dim f \leqslant n_1$ and $\dim Y \leqslant n_2$, and M_1 a complete ANR-space having the $DD\{n_1-1, n_1+n_2\}$-property. If $0 \leqslant m \leqslant n_1$ and M_2 is a complete finite-dimensional ANR-space possessing the parametric regularly branched maps property and such that $\dim M_2 \geqslant n_2+1-m$, then there exists a dense G_δ-subset $\mathcal{F}_{m+1}(X, M_1 \times M_2) \subset C(X, M_1 \times M_2)$ such that the diagonal product $f \triangle g$ is an $(m+1)$-to-1 map for every $g \in \mathcal{F}_{m+1}(X, M_1 \times M_2)$.*

Proof. First we establish the theorem in the case when X and Y are compacta. By Proposition 6.2, we need only to show that $\mathcal{F}_{m+1}(X,M)$ is a dense set in $C(X,M)$, where $M = M_1 \times M_2$. We fix a map $g \in C(X,M)$, $\varepsilon > 0$, and metrics ρ_1 and ρ_2 on M_1 and M_2, respectively. We define a metric ρ on $M_1 \times M_2$ by $\rho = \rho_1 + \rho_2$. Note that $g = g_1 \triangle g_2$ with $g_i \in C(X, M_i)$, $i = 1,2$. According to [12] or [9], there exists an F_σ-subset $F \subset X$ such that $\dim F \leqslant n_1 - 1$ and $\dim f|(X\backslash F) \leqslant 0$. Since f is a perfect map with $\dim f \leqslant n_1$ and $\dim Y \leqslant n_2$, by the Hurewicz theorem [4] $\dim X \leqslant n_1 + n_2$. Therefore by Lemma 6.1, there exists a map $g_1' \in C(X, M_1)$ which is $\varepsilon/2$-close to g_1 with respect to ρ_1 and $(g_1')^{-1}(g_1'(x)) = x$ for all $x \in F$. Since the set $G = \{x \in X \mid (g_1')^{-1}(g_1'(x)) = x\}$ is G_δ in X, $X\backslash G = \bigcup_{i \geqslant 1} F_i$, where $\{F_i\}_{i \geqslant 1}$ is an increasing sequence of compact sets. For each $i \geqslant 1$ consider the restriction map $f_i = f|F_i : F_i \to f(F_i)$. Since M_2 possesses the parametric regularly branched maps property, the set $\mathcal{R}_i$ of all f_i-regularly branched maps is a dense G_δ-subset of $C(F_i, M_2)$. Consequently, since the restriction maps $p_i : C(X, M_2) \to C(F_i, M_2)$, $p_i(g) = g|F_i$, are open, each $p_i^{-1}(\mathcal{R}_i)$ is also dense and G_δ in $C(X, M_2)$. Thus, there exists $g_2' \in \bigcap_{i \geqslant 1} p_i^{-1}(\mathcal{R}_i)$ which is $\varepsilon/2$-close to g_2 with respect to ρ_2. Since $\dim f_i = 0$, for the set

$$B_{m+2}\big(g_2'\big|F_i\big) = \big\{(y,z) \in f(F_i) \times M_2 \;\big|\; \big|\big(f_i \triangle g_2'\big)^{-1}(y,z)\big| \geqslant m+2\big\}$$

we have

$$\dim B_{m+2}\big(g_2'\big|F_i\big) \leqslant \dim f(F_i) - (m+1)\dim M_2 \leqslant n_2 - (m+1)(n_2+1-m) \leqslant -1.$$

Hence, each $(f \triangle g'_2)|F_i$ is $(m+1)$-to-1. Since $F_i \subset F_{i+1}$ for each i, the map $f \triangle g'_2$ is also $(m+1)$-to-1. It remains to notice that $g' = g'_1 \triangle g'_2$ is ε-close to g with respect to ρ and $f \triangle g'$ is $(m+1)$-to-1.

In the second part of the proof we obtain the theorem in general case. Let l be any integer such that $(m+1)l > (m+2)n_1 + n_2$. It is not hard to see that the following property is satisfied:

(10) for any perfect map $f : X \to Y$ between metric spaces X and Y with $\dim f \leqslant n_1$ and $\dim Y \leqslant n_2$ a map $g \in C(X, M_1 \times M_2)$ is f-regularly $(l, m+2)$-branched if and only if $f \triangle g$ is $(m+1)$-to-1.

Therefore, by the first part of the proof we conclude that the space $M_1 \times M_2$ possesses the parametric regularly $(l, m+2, n_1, n_2)$-property with respect to compact spaces. Hence, due to Theorem 5.4, the space $M_1 \times M_2$ has the parametric regularly $(l, m+2, n_1, n_2)$-property. Applying the property (10) again, we obtain the statement of the theorem. □

The following proposition provides a particular type of spaces M_1 for which Theorem 6.3 is applicable.

Proposition 6.4. *Any product of n dendrites with dense sets of end points has the $DD\{n-1, k\}$-property for all $k \geqslant 0$.*

Proof. Let $M = \prod_{i=1}^{n} D_i$ such that each D_i is a dendrite with a dense set of endpoints. Since $M \in DD\{n-1, k\}$ implies $M \in DD\{n-1, k'\}$ for all $k' \leqslant k$, we suppose that $k \geqslant n-1$. Let $g \in C(X, M)$ with X being a compact metric space and A a σ-compact subset of X such that $\dim A \leqslant n-1$. Then $g = (g_1, g_2, \ldots, g_n)$ with $g_i \in C(X, D_i)$. Take a 0-dimensional F_σ-subset A_1 of A such that $\dim A \backslash A_1 \leqslant n-2$. By [11, Theorem 1.1], g_1 can be approximated by a map $h_1 \in C(X, D_1)$ such that $h_1^{-1}(h_1(x)) = x$ for all $x \in A_1$. Since the set $S(h_1) = \{x \in X \mid h_1^{-1}(h_1(x)) = x\}$ is G_δ, $B_1 = A \backslash S(h_1)$ is F_σ and $\dim B_1 \leqslant m-2$. Take a 0-dimensional F_σ-subset A_2 of B_1 with $\dim B_1 \backslash A_2 \leqslant n-3$ and a map $h_2 \in C(X, D_2)$, that approximates g_2, such that $h_2^{-1}(h_2(x)) = x$ for all $x \in A_2$. Proceeding in this way, for all $i = 1, 2, \ldots, n$ we construct F_σ-subsets $B_i \subset A$, 0-dimensional F_σ-sets $A_i \subset B_{i-1}$, and maps $h_i \in C(X, D_i)$ such that:

- h_i approximates g_i;
- $B_{n-1} = A_n$ and $B_n = \varnothing$;
- $\dim B_{i-1} \backslash A_i \leqslant n-1-i$;
- $B_i = B_{i-1} \backslash S(h_i)$;
- $h_i^{-1}(h_i(x)) = x$ for all $x \in A_i$;
- $A \subset \bigcup_{i=1}^{n} S(h_i)$.

Then the map $h \in C(X, M)$, $h = \triangle_{i=1}^{n} h_i$, approximates g and for all $x \in A$ we have $h^{-1}(h(x)) = x$. Hence, by Lemma 6.1, M has the $DD\{n-1, k\}$-property for all $k \geqslant n-1$. □

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