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Spaces with fibered approximation property in dimension n

Research Article

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Abstract: A metric space M is said to have the fibered approximation property in dimension n (briefly, $M \in \mathrm{FAP}(n)$) if for any $\epsilon > 0$, $m \geq 0$ and any map $g\colon \mathbb{I}^m \times \mathbb{I}^n \to M$ there exists a map $g'\colon \mathbb{I}^m \times \mathbb{I}^n \to M$ such that g' is ϵ-homotopic to g and $\dim g'\big(\{z\} \times \mathbb{I}^n\big) \leq n$ for all $z \in \mathbb{I}^m$. The class of spaces having the FAP(n)-property is investigated in this paper. The main theorems are applied to obtain generalizations of some results due to Uspenskij [11] and Tuncali-Valov [10].

MSC: 54F45, 55M10

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1. Introduction

All spaces in the paper are assumed to be Tychonoff and all maps continuous. By $C(X, M)$ we denote the space of all maps from X into M.

We say that a metric space M has the *fibered approximation property in dimension* n (briefly, $M \in \mathrm{FAP}(n)$), where $n \geq 0$, if for any $\epsilon > 0$, any $m \geq 0$ and any map $g\colon \mathbb{I}^m \times \mathbb{I}^n \to M$ there exists a map $g'\colon \mathbb{I}^m \times \mathbb{I}^n \to M$ such that g' is ϵ-homotopic to g and $\dim g'(\{z\} \times \mathbb{I}^n) \leq n$ for all $z \in \mathbb{I}^m$.

In the paper we investigate the class of spaces having the FAP(n)-property, where $n \geq 0$. According to [10], this class contains all Euclidean spaces. It is shown in Theorem 2.1 below that a complete metric space has the FAP(n)-property if and only if it has locally the same property. So, any Euclidean manifold also has the FAP(n)-property, $n \geq 0$. Another FAP(n)-spaces are described in the last section. For example, if M is a manifold modeled on the n-dimensional Menger cube, or $M = \mathbb{I}^n$, then $M \times Z$ has the FAP(n)-property for any completely-metrizable space Z.

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We also introduce a subclass of FAP(n)-spaces, the *strong* FAP(n)*-spaces*, see Section 4. For example, any product of finitely many 1-dimensional completely metrizable LC(0)-spaces without isolated points is a strong FAP(n)-space for all $n \geq 0$ (Corollary 4.1).

Next theorem is the main result in this paper.

Theorem 1.1.

Let $f\colon X \to Y$ be a perfect map with $\dim_\triangle(f) \leq n$, where X and Y are paracompact spaces. If $M \in \mathrm{FAP}(n)$ is completely-metrizable, then $\mathcal{R}^f_n(Y, M) = \{g \in C(X, M) : \dim g(f^{-1}(y)) \leq n \ \text{ for all } y \in Y\}$ is a G_δ-subset of $C(X, M)$ and every simplicially factorizable map in $C(X, M)$ is homotopically approximated by maps from $\mathcal{R}^f_n(Y, M)$.

Corollary 1.1.

Let $f\colon X \to Y$ be a perfect 0-dimensional surjection between paracompact spaces and M be a completely metrizable ANR. *Then the maps $g \in C(X, M)$ with $\dim g(f^{-1}(y)) = 0$ for all $y \in Y$ form a dense G_δ-subset of $C(X, M)$.*

Corollary 1.1 was obtained in [10] in the particular case when Y is a C-space and M an Euclidean space (see also [11] for the case X compact, Y a C-space and $M = \mathbb{I}$).

Corollary 1.2.

Let $M \in$ FAP(n) be a completely metrizable ANR *and $f\colon X \to Y$ a perfect n-dimensional surjection between paracompact spaces with Y being a C-space. Then the maps $g \in C(X, M)$ such that $\dim g(f^{-1}(y)) \leq n$ for all $y \in Y$ form a dense G_δ-subset of $C(X, M)$.*

The version of Corollary 1.2 with M being an Euclidean space was established in [10].

Let us explain the notions in Theorem 1.1. A map $g \in C(X, M)$ is *homotopically approximated* by maps from a set $\mathcal{H} \subset C(X, M)$ means that for every function $\varepsilon \in C(X, (0, 1])$ there exists $g' \in \mathcal{H}$ which is ε-homotopic to g. Here, the maps g and g' are said to be *ε-homotopic*, if there is a homotopy $h\colon X \times \mathbb{I} \to M$ connecting g and g' such that each set $h(\{x\} \times \mathbb{I})$ has a diameter $< \varepsilon(x)$, $x \in X$.

The function space $C(X, M)$ appearing in this theorem is endowed with the source limitation topology whose neighborhood base at a given function $f \in C(X, M)$ consists of the sets

$$B_\rho(f, \varepsilon) = \{g \in C(X, M) : \forall\, x \in X \ \ \rho(g(x), f(x)) < \varepsilon(x)\},$$

where ρ is a fixed compatible metric on M and $\varepsilon : X \to (0, 1]$ runs over continuous positive functions on X. If X is paracompact, the source limitation topology does not depend on the metric ρ and it has the Baire property provided M is completely-metrizable.

We say that a map $g\colon X \to M$ is *simplicially factorizable* [1] if there exists a simplicial complex L and two maps $g_1\colon X \to L$ and $g_2\colon L \to M$ such that $g = g_2 \circ g_1$. In each of the following cases the set of simplicially factorizable maps is dense in $C(X, M)$ (see [1, Proposition 4]): (i) M is an ANR; (ii) $\dim X \leq k$ and M is LC^{k-1}; (iii) X is a C-space and M is locally contractible.

The dimension $\dim_\triangle(f)$ was defined in [1]: $\dim_\triangle(f)$ of a map $f : X \to Y$ is equal to the smallest cardinal number τ with the following property: for every open cover $\mathcal{U}$ of X there is a map $g : X \to \mathbb{I}^\tau$ such that the diagonal product $f\Delta g : X \to Y \times \mathbb{I}^\tau$ is a $\mathcal{U}$-disjoint map. The last one means that every $z \in (f\Delta g)(X)$ has a neighborhood V such that $(f\Delta g)^{-1}(V)$ is the union of a disjoint open in X family refining $\mathcal{U}$. According to results from [5], [6] and [9], for any perfect map $f : X \to Y$ between paracompact spaces we have: (i) $\dim f \leq \dim_\triangle(f)$; (ii) $\dim_\triangle(f) = 0$ iff $\dim f = 0$; (iii) $\dim_\triangle(f) = \dim f$ if Y a C-space; (iv) $\dim_\triangle(f) \leq \dim f + 1$ if the spaces X, Y are compact.

2. Some properties of $\mathrm{FAP}(n)$-spaces

Suppose that (M,ρ) is a complete metric space and $Z\subset M$ a closed set. For a perfect surjective map $f\colon X\to Y$ between paracompact spaces with $\dim f\le n$ and a subset $H\subset Y$, let

$$\mathcal{R}_n^f(H,Z)=\{g\in C(X,M):\forall\, y\in H\ \ \dim(g(f^{-1}(y))\cap Z)\le n\}.$$

Let also $\mathcal{R}_n^f(H,Z,k)$, where $H\subset Y$ and $k\ge 1$, denote the set of all maps $g\in C(X,M)$ satisfying the following condition:

- Each set $\Gamma(g,y)=g(f^{-1}(y))\cap Z$, $y\in H$, can be covered by an open family $\gamma(g,y)$ in M of mesh $\le 1/k$ and order $\le n$.

Recall that the order of $\gamma(g,y)$ is $\le n$ provided any point of M is contained in at most $n+1$ elements of $\gamma(g,y)$.

Lemma 2.1.
$\mathcal{R}_n^f(H,Z)$ is the intersection of all $\mathcal{R}_n^f(H,Z,k)$, $k\ge 1$, for any $H\subset Y$.

Proof. Let $g\in\mathcal{R}_n^f(H,Z)$. Then $\dim\Gamma(g,y)\le n$ for all $y\in H$. Hence, $\Gamma(g,y)$ admits an open in M cover of mesh $\le 1/k$ and order $\le n$ for any $k\ge 1$ and $y\in H$. Therefore, $\mathcal{R}_n^f(H,Z)$ is contained in the intersection of all $\mathcal{R}_n^f(H,Z,k)$, $k\ge 1$. On the other hand, if $g\in C(X,M)$ belongs to this intersection and $y\in Y$ is fixed, then each $\Gamma(g,y)$ admits open covers of arbitrary small mesh and order $\le n$. So, $\dim\Gamma(g,y)\le n$ and $g\in\mathcal{R}_n^f(H,Z)$. ☐

Lemma 2.2.
Suppose X and Y are metric spaces and $g\in\mathcal{R}_n^f(y,Z,k)$ for some $y\in Y$ and $k\ge 1$. Then there exists a neighborhood V_y of y in Y and $\delta_y>0$ such that $g'\in\mathcal{R}_n^f(y',Z,k)$ for any $y'\in V_y$ and $g'\in C(X,M)$ with $\rho(g'(x),g(x))<\delta_y$ for all $x\in f^{-1}(y')$. The same conclusion remains true if $Z=M$ and X,Y paracompact.

Proof. Assume first that X and Y are metric spaces. In case $\Gamma(g,y)\ne\varnothing$, it can be covered by an open in M family $\gamma(g,y)$ of mesh $\le 1/k$ and order $\le n$. Let $G=\cup\gamma(g,y)$ and $\Pi=M\backslash G$. If $\Gamma(g,y)=\varnothing$, let $\Pi=Z$. Hence, in both cases we have

$$Z\cap\Pi\cap g(f^{-1}(y))=\varnothing. \tag{1}$$

It suffices to show there exists a neighborhood V_y of y in Y such that

$$\delta_y=\rho\big(g(f^{-1}(V_y)),Z\cap\Pi\big)>0.$$

Indeed, otherwise there would be a sequence $\{x_i\}_{i\ge1}\subset X$ such that $\{f(x_i)\}_{i\ge1}$ converges to y and $\rho\big(g(x_i),Z\cap\Pi\big)\le 1/i$, $i\ge 1$. Passing to a subsequence, we may assume that $\{x_i\}_{i\ge1}$ also converges to a point $x\in f^{-1}(y)$. So, $g(x)\in Z\cap\Pi\cap g(f^{-1}(y))$, which contradicts (1).
If $Z=M$, we let $G=\cup\gamma(g,y)$ and $\delta_y=\dfrac{1}{2}\rho\big(g(f^{-1}(y)),M\backslash G\big)$, where $\gamma(g,y)$ is as above. Using that f is perfect, we can find a neighborhood V_y of y in Y such that $\rho\big(g(f^{-1}(V_y)),M\backslash G\big)\ge\delta_y$. Then V_y and δ_y are as required. ☐

Lemma 2.3.
Let $H\subset Y$ be closed. Then every $\mathcal{R}_n^f(H,Z,k)$ is open in $C(X,M)$ in each of the following two cases: (i) $Z\subset M$ is closed and both X and Y are metric spaces; (ii) $Z=M$ and X,Y are paracompact.

Proof. The lemma follows from the proof of [2, Proposition 3.3]. For completeness, we provide the arguments. We consider only the first case, the second one is similar. Suppose $g_0\in\mathcal{R}_n^f(H,Z,k)$. Then, by Lemma 2.2, for every $y\in H$ there exist a neighborhood V_y and a positive $\delta_y\le 1$ such that $g\in\mathcal{R}_n^f(y',Z,k)$ for any $y'\in V_y$ provided $g|f^{-1}(y')$ is δ_y-close to $g_0|f^{-1}(y')$. The family $\{V_y\cap H:y\in H\}$ can be supposed to be locally finite in H. Then the set-valued map

$\varphi\colon H \to (0,1]$, $\varphi(y) = \cup\{(0,\delta_z] : y \in V_z\}$ is lower semi-continuous. By [7, Theorem 6.2, p.116], φ admits a continuous selection $\beta\colon H \to (0,1]$. Let $\overline{\beta} : Y \to (0,1]$ be a continuous extension of β and $\alpha = \overline{\beta} \circ f$. It suffices to show that if $g \in C(X,M)$ with $\rho\big(g_0(x), g(x)\big) < \alpha(x)$ for all $x \in X$, then $g \in \mathcal{R}^f_n(y,Z,k)$ for every $y \in H$. So, we take such a g and fix $y \in H$. Then there exists $z \in H$ with $y \in V_z$ and $\alpha(x) \leq \delta_z$ for all $x \in f^{-1}(y)$. Hence, $\rho\big(g_0(x), g(x)\big) < \delta_z$, $x \in f^{-1}(y)$. Therefore, according to the choice of V_z and δ_z, $g \in \mathcal{R}^f_n(y,Z,k)$. ☐

Lemmas 2.1 and 2.3 imply the following proposition.

Proposition 2.1.

Let $H \subset Y$ be a closed set. Then $\mathcal{R}^f_n(H,Z)$ is a G_δ-subset of $C(X,M)$ in any of the cases (i) and (ii) from Lemma 2.3.

Next lemma is very useful when dealing with homotopically dense subsets of function spaces. Here, a set $U \subset C(X,M)$ is said to be *homotopically dense* in $C(X,M)$ if for every $g \in C(X,M)$ and $\varepsilon \in C(X,(0,1])$ there exists $g' \in U$ which is ε-homotopic to g.

Lemma 2.4 ([1, Lemma 2.2]).

Let X be a metric space and $G \subset C(X,M)$. Suppose $\{U(i)\}_{i\geq 1}$ is a sequence of open subsets of $C(X,M)$ such that

- *for any $h \in G$, $i \geq 1$ and any function $\eta \in C(X,(0,1])$ there exists $g_i \in B_\rho(h,\eta) \cap U(i) \cap G$ which is η-homotopic to h.*

Then, for any $g \in G$ and $\varepsilon\colon X \to (0,1]$ there exists $g' \in \bigcap_{i=1}^\infty U(i)$ and an ε-homotopy connecting g and g'. Moreover, $g'|A = g_0|A$ for some $g_0 \in C(X,M)$ and $A \subset X$ provided $g_i|A = g_0|A$ for all i.

Corollary 2.1.

Let X be a metric space and $\{G_i\}_{i\geq 1}$ a sequence of homotopically dense G_δ-subsets of $C(X,M)$. Then the set $\bigcap_{i=1}^\infty G_i$ is also homotopically dense in $C(X,M)$.

Proof. Each G_i is the intersection of a sequence $\{G_{ij}\}_{j\geq 1}$ of open sets in $C(X,M)$. Since G_i is homotopically dense in $C(X,M)$, so are all G_{ij}, $j \geq 1$. Then we apply Lemma 2.4 for the sequence $\{G_{ij}\}_{i,j\geq 1}$ with G being the whole space $C(X,M)$. ☐

We are going to show the local nature of the FAP(n)-properties.

Theorem 2.1.

A complete metric space M possesses the FAP(n)*-property if and only if every $z \in M$ has a neighborhood $U_z \in$* FAP(n).

Proof. It is easily seen that if $M \in$ FAP(n), then every open set $U \subset M$ also has the FAP(n)-property. Suppose every $z \in M$ has an open neighborhood $U_z \in$ FAP(n). Fix an integer $m \geq 0$ and consider the projection $\pi\colon \mathbb{I}^m \times \mathbb{I}^n \to \mathbb{I}^m$. We need to prove that the set $\mathcal{R}^\pi_n(\mathbb{I}^m, M)$ is homotopically dense in $C(\mathbb{I}^m \times \mathbb{I}^n, M)$. To this end, using an idea from the proof of [4, Theorem 3.6], for every $z \in M$ choose a positive ϵ_z such that U_z contains the closed ball $\overline{B(z,3\epsilon_z)}$ with center z and radius $3\epsilon_z$. Following the notations from the beginning of this section (with X replaced by $\mathbb{I}^m \times \mathbb{I}^n$ and Y by $\mathbb{I}^m$), we consider the sets $\mathcal{R}(z) = \mathcal{R}^\pi_n(\mathbb{I}^m, \overline{B(z,\epsilon_z)})$, $z \in M$.

Claim 2.1.

Every $\mathcal{R}(z)$, $z \in M$, is a homotopically dense G_δ-subset of $C(\mathbb{I}^m \times \mathbb{I}^n, M)$.

All $\mathcal{R}(z)$ are G_δ-subsets of $C(\mathbb{I}^m \times \mathbb{I}^n, M)$ by Proposition 2.1. To show their homotopical density in $C(\mathbb{I}^m \times \mathbb{I}^n, M)$, fix $z_0 \in M$, $g_0 \in C(\mathbb{I}^m \times \mathbb{I}^n, M)$ and $\epsilon > 0$ with $\epsilon < \epsilon_{z_0}$. Let $A_{z_0} = g_0^{-1}\big(\overline{B(z_0, 2\epsilon_{z_0})}\big)$ and $W_{z_0} = g_0^{-1}\big(B(z_0, 3\epsilon_{z_0})\big)$. Choose finitely many sets $K_i = A_i \times B_i$, $i = 1,2,..,k$, such that $A_i \subset \mathbb{I}^m$ and $B_i \subset \mathbb{I}^n$ are homeomorphic to $\mathbb{I}^m$ and $\mathbb{I}^n$, respectively, and $A_{z_0} \subset K = \bigcup_{i=1}^{i=k} K_i \subset W_{z_0}$. We can also suppose that there exists a polyhedron L such that $A_{z_0} \subset L \subset K$. For every i consider the set

$$\mathcal{R}_i = \{h \in C(K_i, U_{z_0}) : \forall\, y \in A_i \quad \dim h(\{y\} \times B_i) \le n\} \tag{2}$$

and let $p_i\colon C(K, U_{z_0}) \to C(K_i, U_{z_0})$ be the restriction map $g \to g|K_i$, $g \in C(K, U_{z_0})$. Obviously, p_i are continuous. By Proposition 2.1, each $\mathcal{R}_i$ is a G_δ-subset of $C(K_i, U_{z_0})$. Hence, all $p_i^{-1}(\mathcal{R}_i)$ are G_δ-subsets of $C(K, U_{z_0})$. Moreover, each $\mathcal{R}_i$ is homotopically dense in $C(K_i, U_{z_0})$ because $U_{z_0} \in \mathrm{FAP}(n)$. This, according to the Homotopy Extension Theorem, implies that $p_i^{-1}(\mathcal{R}_i)$ are also homotopically dense in $C(K, U_{z_0})$. So, by Corollary 2.1, $\mathcal{H} = \bigcap_{i=1}^{i=p} p_i^{-1}(\mathcal{R}_i)$ is homotopically dense in $C(K, U_{z_0})$. Then there exists a map $h \in \mathcal{H}$ which is ϵ-homotopic to $g_0|K$. Applying again the Homotopy Extension Theorem for the maps $h|L$ and g_0, we obtain a map $g^* \in C(\mathbb{I}^m \times \mathbb{I}^n, M)$ such that $g^*|L = h|L$ and g^* is ϵ-homotopic to g_0. Let us show that $g^* \in \mathcal{R}(z_0)$, or equivalently, $\dim g^*(\{y\} \times \mathbb{I}^n) \cap \overline{B(z_0, \epsilon_{z_0})} \le n$ for every $y \in \mathbb{I}^m$. It is easily seen that $(g^*)^{-1}(z) \subset A_{z_0}$ for every $z \in \overline{B(z_0, \epsilon_{z_0})}$. The last inclusion yields that $g^*(\{y\} \times \mathbb{I}^n) \cap \overline{B(z_0, \epsilon_{z_0})} \subset h((\{y\} \times \mathbb{I}^n) \cap A_{z_0})$ for any $y \in \pi(A_{z_0})$ and $g^*(\{y\} \times \mathbb{I}^n) \cap \overline{B(z_0, \epsilon_{z_0})} = \varnothing$ if $y \notin \pi(A_{z_0})$. Therefore, the proof of the claim is reduced to show that $\dim h((\{y\} \times \mathbb{I}^n) \cap A_{z_0}) \le n$ for any $y \in \pi(A_{z_0})$. And this is really true. Indeed, for any such y let $\Lambda(y) = \{i \le k : y \in A_i\}$. Then $(\{y\} \times \mathbb{I}^n) \cap A_{z_0} = \bigcup_{i \in \Lambda(y)} (\{y\} \times B_i) \cap A_{z_0}$. Since $h|K_i \in \mathcal{R}_i$, by (2) we have $\dim h((\{y\} \times B_i) \cap A_{z_0}) \le n$ for every $i \in \Lambda(y)$. Hence, $h((\{y\} \times \mathbb{I}^n) \cap A_{z_0})$ is the union of its closed sets $h((\{y\} \times B_i) \cap A_{z_0})$, $i \in \Lambda(y)$, each of dimension $\le n$. So, $\dim h((\{y\} \times \mathbb{I}^n) \cap A_{z_0}) \le n$ which completes the proof of the claim.

Now, we can show that $\mathcal{R}_n^\pi(\mathbb{I}^m, M)$ is homotopically dense in $C(\mathbb{I}^m \times \mathbb{I}^n, M)$. To this end, fix $g \in C(X, M)$ and $\eta > 0$, and choose finitely many points $z_i \in M$, $i = 1, \dots, q$, such that $g(\mathbb{I}^m \times \mathbb{I}^n) \subset \bigcup_{i=1}^{i=q} B(z_i, \epsilon_{z_i}/2)$. Let $\delta = \min\{\eta, \epsilon_{z_i}/2 : i \le q\}$. By the above claim, each $\mathcal{R}(z_i)$ is a homotopically dense G_δ-subset of $C(\mathbb{I}^m \times \mathbb{I}^n, M)$. Therefore, so is the set $\bigcap_{i \le q} \mathcal{R}(z_i)$ according to Corollary 2.1. Hence, there exists $g' \in \bigcap_{i \le q} \mathcal{R}(z_i)$ which is δ-homotopic to g. It is easily seen that $g'(\mathbb{I}^m \times \mathbb{I}^n) \subset \bigcup_{i=1}^{i=q} B(z_i, \epsilon_{z_i})$, so $g'(\{y\} \times \mathbb{I}^n) \subset \bigcup_{i \le q} g'(\{y\} \times \mathbb{I}^n) \cap \overline{B(z_i, \epsilon_{z_i})}$ for any $y \in \mathbb{I}^m$. Observe that each set $g'(\{y\} \times \mathbb{I}^n) \cap \overline{B(z_i, \epsilon_{z_i})}$, $i \le q$, is of dimension $\le n$ because $g' \in \mathcal{R}(z_i)$. Hence, $\dim g'(\{y\} \times \mathbb{I}^n) \le n$ for all $y \in \mathbb{I}^m$. Thus, $g' \in \mathcal{R}_n^\pi(\mathbb{I}^m, M)$. This completes the proof. ☐

Next proposition shows that in the definition of FAP(n)-spaces we can consider any product $\mathbb{I}^m \times \mathbb{I}^k$ where $m \ge 0$ and $k \le n$.

Proposition 2.2.

If a metrizable space M has the FAP(n)*-property, then any map $g\colon \mathbb{I}^m \times \mathbb{I}^k \to M$, where $m \ge 0$ and $k \le n$, can be approximated by a map $g'\colon \mathbb{I}^m \times \mathbb{I}^k \to M$ such that $\dim g'(\{z\} \times \mathbb{I}^k) \le n$ for all $z \in \mathbb{I}^m$.*

Proof. Suppose M has the FAP(n)-property. Let $\epsilon > 0$ and $g\colon \mathbb{I}^m \times \mathbb{I}^k \to M$ with $k \le m$. Take a retraction $r\colon \mathbb{I}^n \to \mathbb{I}^k$ and consider the maps $\pi_1\colon \mathbb{I}^m \times \mathbb{I}^n \to \mathbb{I}^m \times \mathbb{I}^k$ and $\pi_2\colon \mathbb{I}^m \times \mathbb{I}^k \to \mathbb{I}^m$ defined, respectively, by $\pi_1((z, x)) = (z, r(x))$ and $\pi_2(z, y) = z$. Then $\pi = \pi_2 \circ \pi_1\colon \mathbb{I}^m \times \mathbb{I}^n \to \mathbb{I}^m$ is the natural projection. Since $M \in \mathrm{FAP}(n)$, there exists $h \in C(\mathbb{I}^m \times \mathbb{I}^n, M)$ which is ϵ-homotopic to the map $g \circ \pi_1$ and $\dim h(\{z\} \times \mathbb{I}^n) \le n$ for all $z \in \mathbb{I}^m$. Consequently, the map $g' = h|(\mathbb{I}^m \times \mathbb{I}^k)$ is ϵ-homotopic to g and $\dim g'(\{z\} \times \mathbb{I}^k) \le n$, $z \in \mathbb{I}^m$. ☐

Next theorem provides a characterization of FAP(n)-spaces in terms of simplicial maps.

Theorem 2.2.

For a complete metric space M the following conditions are equivalent:

(i) M possesses the FAP(n)*-property;*

(ii) If $p\colon K \to L$ is at most n-dimensional simplicial map between finite simplicial complexes, then the set $\mathcal{R}_n^p(L, M)$ is homotopically dense in $C(K, M)$;

Proof. $(i) \Rightarrow (ii)$ Suppose $M \in \mathrm{FAP}(n)$ and $p\colon K \to L$ a simplicial map between finite simplicial complexes with $\dim p = k \le n$. Let $K^{(0)}$, $L^{(0)}$ be the set of vertices of K and L, respectively, and fix $g_0 \in C(K, M)$ and $\epsilon > 0$.
First, we assume that K is a simplex. Then L is also a simplex and, since $\dim p = k$, $p^{-1}(z) \cap K^{(0)}$ contains at most $k + 1$ points for every vertex $z \in L^{(0)}$. Consequently, we can find a map $e^{(0)}\colon K^{(0)} \to \sigma_k^{(0)}$ which is injective on each set $p^{-1}(z) \cap K^{(0)}$, $z \in L^{(0)}$. Here, σ_k is a k-dimensional simplex. This map induces an affine map $e\colon K \to \sigma_k$. Then

the diagonal map $h = p\triangle e\colon K \to L \times \sigma_k$ is an affine embedding. So, there exists a retraction $r\colon L \times \sigma_k \to K$ such that $h \circ r$ is the identity on $h(K)$. Consider the projection $\pi\colon : L \times \sigma_k \to L$. By Proposition 2.2, there exists a map $\overline{g}\colon L \times \sigma_k \to M$ ϵ-homotopic to $g_0 \circ r$ such that $\dim \overline{g}(\{z\} \times \sigma_k) \le n$ for every $z \in L$. Then for the map $g' = \overline{g} \circ h$ we have $\dim g'(p^{-1}(z)) \le n$ because $h(p^{-1}(z))$ is homeomorphic to a subset of $\{z\} \times \sigma_k$. Moreover, it follows that g' is ϵ-homotopic to g_0. Therefore, in this case $\mathcal{R}^p_n(L,M)$ is homotopically dense in $C(K,M)$.

Now, we can prove the general case. Let $\{K_i : i \le s\}$ be all simplexes of K and for each $i \le s$ we denote

$$\mathcal{H}_i = \{g \in C(K,M) : \dim g(p^{-1}(z) \cap K_i) \le n \ \forall\ z \in p(K_i)\}.$$

According to Proposition 2.1, $\mathcal{H}_i$ are G_δ in $C(K,M)$. It is easily seen that $\mathcal{R}^p_n(L,M) = \bigcap_{i=1}^{i=s}\mathcal{H}_i$. So, by Corollary 2.1, it suffices to show that each $\mathcal{H}_i$ is homotopically dense in $C(K,M)$. Using the previous case, each set $\mathcal{K}_i = \{g \in C(K_i,M) : \dim g(p^{-1}(z) \cap K_i) \le n \ \forall\ z \in p(K_i)\}$ is homotopically dense in $C(K_i,M)$. Therefore, there exists a map $g_i \in \mathcal{K}_i$ which is ϵ-homotopic to $g_0|K_i$. Then, by the Homotopy Extension Theorem, g_i can be extended to a map $\overline{g}_i \in C(K,M)$ ϵ-homotopic to g_0. Obviously, $\overline{g}_i \in \mathcal{H}_i$. So, each $\mathcal{H}_i$ is homotopically dense in $C(K,M)$ which completes the proof.

$(ii) \Rightarrow (i)$ This implication is trivial because any projection $\pi\colon \mathbb{I}^m \times \mathbb{I}^n \to \mathbb{I}^m$ is a simplicial map with respect to suitable triangulations of $\mathbb{I}^m$ and $\mathbb{I}^m \times \mathbb{I}^n$. ☐

3. Proof of Theorem 1.1 and Corollaries 1.1 - 1.2

In this section, following the notations from Section 2, we assume that (M,ρ) is a completely-metrizable FAP(n)-space. As we already observed, every $g \in C(X,M)$ is simplicially factorizable provided M is an ANR. Moreover, if $f\colon X \to Y$ is a perfect map between paracompact spaces, then $\dim f = \dim_\triangle(f)$ when either $\dim f = 0$ or Y is a C-space [1]. Let us also note that every ANR has the FAP(0)-property. Hence, the proofs of Corollaries 1.2 and 1.3 follow from Theorem 1.1.

By Proposition 2.1, $\mathcal{R}^f_n(Y,M)$ is a G_δ-subset of $C(X,M)$. So, to prove Theorem 1.1 it suffices to show that any simplicially factorizable map in $C(X,M)$ can be approximated by maps from $\mathcal{R}^f_n(Y,M)$. This will be done in Proposition 3.1 below.

Recall that a map $p\colon K \to L$ between two simplicial complexes is a PL-map if $p(\sigma)$ is contained in a simplex of L and p is linear on σ for every simplex $\sigma \in K$.

Lemma 3.1.

Let $p\colon K \to \sigma$ be a PL-map between a finite simplicial complex K and a simplex σ with $\dim p \le n$. Suppose $g_0 \in C(K,M)$ such that $\dim g_0(p^{-1}(y)) \le n$ for all $y \in \partial\sigma$, where $\partial\sigma$ is the boundary of σ. Then, for every $\epsilon > 0$ there exists a map $g \in \mathcal{R}^p_n(\sigma,M)$ which is ϵ-homotopic to g_0 and $g|p^{-1}(\partial\sigma) = g_0|p^{-1}(\partial\sigma)$.

Proof. We may assume that p is simplicial because any PL-map between finite simplicial complexes is simplicial with respect to some triangulations of the complexes. Let $\Omega = p^{-1}(\partial\sigma)$ and $G = \{g \in C(K,M) : g|\Omega = g_0|\Omega\}$. All sets $U(k) = \mathcal{R}^p_n(\sigma,M,k)$, $k \ge 1$, are open in $C(K,M)$ and their intersection is $\mathcal{R}^p_n(\sigma,M)$, see Lemmas 2.1 and 2.3. So, by Lemma 2.4, it suffices to show that each $U(k)$ has the following property: any $g \in G$ can be homotopically approximated by maps from $U(k) \cap G$.

So, fix $g \in G$, $k \ge 1$ and $\delta > 0$. We are going to find $h \in U(k) \cap G$ which is δ-homotopic to g. Since $g|\Omega = g_0|\Omega$, $g \in \mathcal{R}^p_n(y,M,k)$ for every $y \in \partial\sigma$. Consequently, each $y \in \partial\sigma$ has a neighborhood V_y in σ with corresponding $\delta_y > 0$ both satisfying the hypotheses of Lemma 2.2. Choose finitely many $y_i \in \partial\sigma$, $i \le s$, such that $V = \bigcup\limits_{i\le s} V_{y_i}$ covers $\partial\sigma$. Let $F = \sigma\backslash V$ and $\eta = \min\{\delta, \delta_{y_i} : i \le s\}$. We consider such a triangulation T of σ that the complex $L = \{\tau \in T : \tau \cap F \neq \varnothing\}$ is disjoint with $\partial\sigma$. Because K and σ are finite complexes, both they have triangulations T_K and T_σ such that T_σ is a subdivision of T and p remains simplicial with respect to T_K and T_σ. So, we can apply Theorem 2.2 to find a map $g_1 \in \mathcal{R}^p_n(\sigma,M)$ which is η-homotopic to g. Then the map $g_2\colon \Omega \cup p^{-1}(L) \to M$, $g_2|\Omega = g|\Omega$ and $g_2|p^{-1}(L) = g_1|p^{-1}(L)$, is η-homotopic to $g|\Omega \cup p^{-1}(L)$. Since $\Omega \cup p^{-1}(L)$ is a subcomplex of K, by the Homotopy Extension Theorem, g_2 can be extended to a map $h\colon K \to M$ which is η-homotopic to g. We have $h \in \mathcal{R}^p_n(y,M,k)$ for all $y \in \sigma$. Indeed, this follows from the choice of V_{y_i} and δ_i, $i \le s$ (when $y \in V$), and from $g_1 \in \mathcal{R}^p_n(\sigma,M)$ (when $y \in L$). Hence, $h \in U(k) \cap G$ which completes the proof. ☐

Next step is to prove that the set $\mathcal{R}^f_n(L,M)$ is homotopically dense in $C(N,M)$ for any perfect PL-map $f\colon N\to L$ between simplicial complexes with $\dim f\leq n$.

Lemma 3.2.

Let N,L be simplicial complexes and $f\colon N\to L$ a perfect PL-map with $\dim f\leq n$. Then $\mathcal{R}^f_n(L,M)$ is a homotopically dense subset of $C(N,M)$

Proof. We follow the arguments from the proof of [1, Lemma 11.3]. Fix $g\in C(N,M)$ and $\alpha\in C(N,(0,1])$. We are going to find $h\in\mathcal{R}^f_n(L,M)$ which is α-homotopic to g. Let $L^{(i)}$, $i\geq 0$, be the i-dimensional skeleton of L and put $L^{(-1)}=\varnothing$ and $h_{-1}=g$. Construct inductively a sequence $(h_i:N\to M)_{i\geq 0}$ of maps such that

- $h_i|f^{-1}(L^{(i-1)})=h_{i-1}|f^{-1}(L^{(i-1)})$;
- h_i is $\dfrac{\alpha}{2^{i+2}}$-homotopic to h_{i-1};
- $\dim h_i(f^{-1}(y))\leq n$ for every $y\in L^{(i)}$.

Assuming that the map $h_{i-1}:N\to M$ has been constructed, consider the complement $L^{(i)}\setminus L^{(i-1)}=\sqcup_{j\in J_i}\overset{\circ}{\sigma}_j$, which is the discrete union of open i-dimensional simplexes. Since, by [1, Lemma 4.1], each $f^{-1}(\sigma_j)$ is a finite subcomplex of N, and $\dim h_{i-1}(f^{-1}(y))\leq n$ for every $y\in L^{(i-1)}$, we can apply Lemma 3.1 to find a map $g_j:f^{-1}(\sigma_j)\to M$, $j\in J_i$, such that

- g_j coincides with h_{i-1} on the set $f^{-1}(\sigma_j^{(i-1)})$;
- g_j is $\dfrac{\alpha}{2^{i+2}}$-homotopic to h_{i-1};
- $\dim g_j(f^{-1}(y))\leq n$ for every $y\in\sigma_j$.

Define a map $\varphi_i:f^{-1}(L^{(i)})\to M$ by the formula

$$\varphi_i(x)=\begin{cases}h_{i-1}(x) & \text{if } x\in f^{-1}(L^{(i-1)});\\ g_j(x) & \text{if } x\in f^{-1}(\sigma_j).\end{cases}$$

It can be shown that φ_i is $\dfrac{\alpha}{2^{i+2}}$-homotopic to $h_{i-1}|f^{-1}(L^{((i)})$. Moreover, $f^{-1}(L^{(i)})$ is a subcomplex of N (according to [1, Lemma 4.1]). So, by the Homotopy Extension Theorem, there exists a continuous extension $h_i:N\to M$ of the map φ_i which is $\dfrac{\alpha}{2^{i+2}}$-homotopic to h_{i-1}. The map h_i satisfies the inductive conditions.
Then the limit map $h=\lim_{i\to\infty}h_i:N\to M$ is well-defined, continuous and α-homotopic to g. Finally, since $h|f^{-1}(L^{(i)})=h_i|f^{-1}(L^{(i)})$ for every $i\geq 0$, $h\in\mathcal{R}^f_n(L,M)$. ☐

Now, we can complete the proof of Theorem 1.1.

Proposition 3.1.

Let $f\colon X\to Y$ be a perfect map between paracompact spaces with $\dim_\triangle(f)\leq n$. Then every simplicially factorizable map $g\in C(X,M)$ can be homotopically approximated by simplicially factorizable maps $h\in C(X,M)$ such that $\dim h(f^{-1}(y))\leq n$ for every $y\in Y$.

Proof. We follow the construction from the proof of [2, Proposition 3.4]. Fix a simplicially factorizable map $g\in C(X,M)$ and $\epsilon\in C(X,(0,1])$. Then there exist a simplicial complex D and maps $g_D\colon X\to D$, $g^D\colon D\to M$ with $g=g^D\circ g_D$. The metric ρ induces a continuous pseudometric ρ_D on D, $\rho_D(x,y)=\rho(g^D(x),g^D(y))$. Since D is a neighborhood retract of a locally convex space (see [3] and [8]) and any sufficiently close maps from a given space into D are homotopic, we apply [1, Lemma 8.1] to find an open cover $\mathcal{U}$ of X satisfying the following condition: if $\alpha\colon X\to K$ is a $\mathcal{U}$-map into a paracompact space K (i.e., $\alpha^{-1}(\omega)$ refines $\mathcal{U}$ for some open cover ω of K), then there exists a map $q'\colon G\to D$, where G is an open neighborhood of $\overline{\alpha(X)}$ in K, such that g_D and $q'\circ\alpha$ are $\epsilon/2$-homotopic with respect to the pseudometric ρ_D. Let $\mathcal{U}_1$ be an open cover of X refining $\mathcal{U}$ with $\inf\{\epsilon(x):x\in U\}>0$ for all $U\in\mathcal{U}_1$.

Next, according to [1, Theorem 6], there exists a locally finite open cover $\mathcal{V}$ of Y such that: for any $\mathcal{V}$-map $\beta\colon Y \to L$ into a simplicial complex L we can find an $\mathcal{U}$-map $\alpha\colon X \to K$ into a simplicial complex K and a perfect PL-map $p\colon K \to L$ with $\beta \circ f = p \circ \alpha$ and $\dim p \leq \dim_{\triangle} f$. Take L to be the nerve of the cover $\mathcal{V}$ and $\beta\colon Y \to L$ the corresponding natural map. Then there are a simplicial complex K and maps p and α satisfying the above conditions. Hence, the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"'] & K \arrow[d, "p"] \\
Y \arrow[r, "\beta"'] & L
\end{tikzcd}$$

The choice of the cover $\mathcal{U}$ guarantees the existence of a map $\varphi_D\colon G \to D$, where $G \subset K$ is an open neighborhood of $\overline{\alpha(X)}$, such that g_D and $h_D = \varphi_D \circ \alpha$ are $\epsilon/2$-homotopic with respect to ρ_D. Then, according to the definition of ρ_D, $h' = g^D \circ \varphi_D \circ \alpha$ is $\epsilon/2$-homotopic to g with respect to ρ. Replacing the triangulation of K by a suitable subdivision, we may additionally assume that no simplex of K meets both $\overline{\alpha(X)}$ and $K \backslash G$. So, the union N of all simplexes $\sigma \in K$ with $\sigma \cap \overline{\alpha(X)} \neq \varnothing$ is a subcomplex of K and $N \subset G$. Moreover, since N is closed in K, $p_N = p|N\colon N \to L$ is a perfect map and $\dim p_N \leq \dim_{\triangle} f$. Therefore, we have the following commutative diagram, where N and L are finite complexes, p_N is a PL-map and $\varphi = g^D \circ \varphi_D$:

$$\begin{tikzcd}
N \arrow[d, "p_N"'] & X \arrow[l, "\alpha"'] \arrow[r, "\varphi\circ\alpha"] \arrow[d, "f"] & M \\
L & Y \arrow[l, "\beta"] \arrow[ur, "\varphi"']
\end{tikzcd}$$

Using that α is a $\mathcal{U}_1$-map and $\inf\{\epsilon(x) : x \in U\} > 0$ for all $U \in \mathcal{U}_1$, we can construct a continuous function $\epsilon_1 : N \to (0,1]$ with $\epsilon_1 \circ \alpha \leq \epsilon$. Then, by Lemma 3.2, there exists a map $\varphi_1 \in C(N, M)$ which is $\epsilon_1/2$-homotopic to φ and $\dim \varphi_1(p_N^{-1}(z)) \leq n$ for every $z \in L$. Let $g' = \varphi_1 \circ \alpha$. Obviously, g' is simplicially factorizable. It is easily seen that g' and g are ϵ-homotopic and $g'(f^{-1}(y)) \subset \varphi_1(p_N^{-1}(\beta(y)))$ for all $y \in Y$. So, $\dim g'(f^{-1}(y)) \leq \dim \varphi_1(p_N^{-1}(\beta(y))) \leq n$. The proof is completed. $\square$

4. Some more examples of FAP(n)-spaces

The class of $AP(n,0)$-spaces was introduced by the authors in [2]: we say that a metrizable space M has the AP$(n,0)$-*approximation property* (br., $M \in \mathrm{AP}(n,0)$) if for every $\varepsilon > 0$ and a map $g\colon \mathbb{I}^n \to M$ there exists a 0-dimensional map $g'\colon \mathbb{I}^n \to M$ which is ε-homotopic to g. Next proposition provides a wide class of spaces with the FAP(n)-property.

Proposition 4.1.
Let $M_1 \in \mathrm{AP}(n,0)$ be a completely-metrizable n-dimensional space, $n \geq 0$. Then $M_1 \times M_2$ has the FAP(n)-property for any completely-metrizable space M_2.

Proof. We are going to show that every map $g = (g_1, g_2)\colon \mathbb{I}^m \times \mathbb{I}^n \to M_1 \times M_2$, where $m \geq 0$, can be homotopically approximated by a map $h \in C(\mathbb{I}^m \times \mathbb{I}^n, M_1 \times M_2)$ with $\dim h(\{z\} \times \mathbb{I}^n) \leq n$ for all $z \in \mathbb{I}^m$. Denote by π the projection $\pi\colon \mathbb{I}^m \times \mathbb{I}^n \to \mathbb{I}^m$. Since M_1 has the AP$(n,0)$-property, the map $g_1\colon \mathbb{I}^m \times \mathbb{I}^n \to M_1$ can be homotopically approximated by a map $h_1\colon \mathbb{I}^m \times \mathbb{I}^n \to M_1$ such that all restrictions $h_1|(\{z\} \times \mathbb{I}^n)$, $z \in \mathbb{I}^m$, have 0-dimensional fibers, see [2, Theorem 1.1]. This means that the diagonal product $\pi \triangle h_1\colon \mathbb{I}^m \times \mathbb{I}^n \to \mathbb{I}^m \times M_1$ is a 0-dimensional map. It follows from our definition that every metrizable space has the FAP(0)-property. So, by Theorem 1.1, the map $g_2\colon \mathbb{I}^m \times \mathbb{I}^n \to M_2$ can be homotopically approximated by a map $h_2\colon \mathbb{I}^m \times \mathbb{I}^n \to M_2$ such that all images $h_2\big((\{z\} \times \mathbb{I}^n) \cap h_1^{-1}(z_1)\big)$, $(z, z_1) \in \mathbb{I}^m \times M_1$, are 0-dimensional. Then $h = (h_1, h_2)\colon \mathbb{I}^m \times \mathbb{I}^n \to M_1 \times M_2$ approximates g. For any $z \in \mathbb{I}^m$ consider the map $p_z : h(\{z\} \times \mathbb{I}^n) \to h_1(\{z\} \times \mathbb{I}^n)$, $p_z(h(z,t)) = h_1((z,t))$, $t \in \mathbb{I}^n$. Observe that $\dim h_1(\{z\} \times \mathbb{I}^n) \leq n$ (recall that $\dim M_1 = n$) and $p_z^{-1}(z_1) = h_2\big((\{z\} \times \mathbb{I}^n) \cap h_1^{-1}(z_1)\big)$ for any $z_1 \in h_1(\{z\} \times \mathbb{I}^n)$. So, $\dim p_z = 0$. According to the dimension-lowering Hurewicz theorem, $\dim h(\{z\} \times \mathbb{I}^n) \leq \dim h_1(\{z\} \times \mathbb{I}^n) + \dim p_z \leq n$. This completes the proof. $\square$

Since every space with the disjoint $(n-1)$-disks property $\mathrm{DD}^{n-1}\mathrm{P}$, in particular, every manifold modeled on the n-dimensional Menger cube or the n-dimensional Nobeling space, is an $\mathrm{AP}(n,0)$-space, see [2, Corollary 6.5], we have the following

Corollary 4.1.

Let X be a completely-metrizable n-dimensional space with the disjoint $(n-1)$-disks property. Then $X \times M$ has the $\mathrm{FAP}(n)$*-property for any completely-metrizable space M.*

Now, we introduced a subclass of the $\mathrm{FAP}(n)$-spaces: a metric space M is said to be a *strong* $\mathrm{FAP}(n)$*-space* if $M \in \mathrm{FAP}(k)$ for all $k \leq n$. This is equivalent to the following condition: any map $g \in C(\mathbb{I}^m \times \mathbb{I}^k, M)$, where $m \geq 0$ and $k \leq n$, can be homotopically approximated by a map $g' \in C(\mathbb{I}^m \times \mathbb{I}^k, M)$ with $\dim g'(\{z\} \times \mathbb{I}^k) \leq k$ for all $z \in \mathbb{I}^m$.
The local nature of strong $\mathrm{FAP}(n)$-spaces follows from Theorem 2.1.

Theorem 4.1.

A complete metric space M has the strong $\mathrm{FAP}(n)$*-property iff every $z \in M$ has a neighborhood with the same property.*

By [10], any Euclidean space possesses the strong $\mathrm{FAP}(n)$-property for all $n \geq 0$. More general examples of strong $\mathrm{FAP}(n)$-spaces are provided by next proposition.

Proposition 4.2.

The product $\prod_{i=1}^{n} M_i$ of completely-metrizable LC^0*-spaces without isolated points is a strong* $\mathrm{FAP}(n)$*-space.*

Proof. According to [2, Corollary 6.3], any product of k many completely-metrizable spaces LC^0-space without isolated points has the $\mathrm{AP}(k,0)$-property. Then Proposition 4.1 completes the proof. ☐

Corollary 4.2.

Any product of finitely many completely-metrizable 1-dimensional LC^0*-spaces without isolated points has the* $\mathrm{FAP}(n)$*-property for all $n \geq 0$.*

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