

## Approximation by light maps and parametric Lelek maps

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### ABSTRACT

The class of metrizable spaces  $M$  with the following approximation property is introduced and investigated:  $M \in AP(n, 0)$  if for every  $\varepsilon > 0$  and a map  $g : \mathbb{I}^n \rightarrow M$  there exists a 0-dimensional map  $g' : \mathbb{I}^n \rightarrow M$  which is  $\varepsilon$ -homotopic to  $g$ . It is shown that this class has very nice properties. For example, if  $M_i \in AP(n_i, 0)$ ,  $i = 1, 2$ , then  $M_1 \times M_2 \in AP(n_1 + n_2, 0)$ . Moreover,  $M \in AP(n, 0)$  if and only if each point of  $M$  has a local base of neighborhoods  $U$  with  $U \in AP(n, 0)$ . Using the properties of  $AP(n, 0)$ -spaces, we generalize some results of Levin and Kato–Matsushashi concerning the existence of residual sets of  $n$ -dimensional Lelek maps.

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## 1. Introduction

All spaces in the paper are assumed to be metrizable and all maps continuous. By  $C(X, M)$  we denote all maps from  $X$  into  $M$ . Unless stated otherwise, all function spaces are endowed with the source limitation topology.

One of the important properties of the  $n$ -dimensional cube  $\mathbb{I}^n$  or the Euclidean space  $\mathbb{R}^n$ , widely exploited in dimension theory, is that any map from an  $n$ -dimensional compactum  $X$  into  $\mathbb{I}^n$  (resp.,  $\mathbb{R}^n$ ) can be approximated by maps with 0-dimensional fibers, see [8] or [12]. The aim of this article is to introduce and investigate the class of spaces having that property. More precisely, we say that a space  $M$  has the  $(n, 0)$ -approximation property (br.,  $M \in AP(n, 0)$ ) if for every  $\varepsilon > 0$  and a map  $g : \mathbb{I}^n \rightarrow M$  there exists a 0-dimensional map  $g' : \mathbb{I}^n \rightarrow M$  which is  $\varepsilon$ -homotopic to  $g$ . Here,  $g'$  is  $\varepsilon$ -homotopic to  $g$  means that there is an  $\varepsilon$ -small homotopy  $h : \mathbb{I}^n \times \mathbb{I} \rightarrow M$  connecting  $g$  and  $g'$ . It is easily seen that this definition doesn't depend on the metric generating the topology of  $M$ . If  $M$  is  $LC^n$ , then  $M \in AP(n, 0)$  if and only if for every  $\varepsilon > 0$  and  $g \in C(\mathbb{I}^n, M)$  there exists a 0-dimensional map  $g'$  which is  $\varepsilon$ -close to  $g$ . We show that the class of  $AP(n, 0)$ -spaces is quite large and it has very nice properties. For example, if  $M_i \in AP(n_i, 0)$ ,  $i = 1, 2$ , then  $M_1 \times M_2 \in AP(n_1 + n_2, 0)$ . We also prove that  $M \in AP(n, 0)$  if and only if each point of  $M$  has a local base of neighborhoods  $U$  with  $U \in AP(n, 0)$ . Moreover, every path-connected compactum  $M \in AP(n, 0)$  is a  $(V^n)$ -continuum in the sense of P. Alexandroff [1]. In particular, according to [10], any such  $M$  is a Cantor  $n$ -manifold, as well as a strong Cantor  $n$ -manifold in the sense of Hadžiivanov [9]. All complete  $LC^0$ -spaces without isolated points are  $AP(1, 0)$ , as well as, every manifold modeled on the  $n$ -dimensional Menger cube or the  $n$ -dimensional Nöbeling space has the  $AP(n, 0)$ -property.

The class of  $AP(n, 0)$ -spaces is very natural for obtaining results about approximation by dimensionally restricted maps. We present such a result about  $n$ -dimensional Lelek maps. Recall that a map  $g : X \rightarrow M$  between compact spaces is said to

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be an  $n$ -dimensional Lelek map [18] if the union of all non-trivial continua contained in the fibers of  $g$  is of dimension  $\leq n$ . For convenience, if  $n \leq 0$ , by an  $n$ -dimensional Lelek map we simply mean a 0-dimensional map. Every  $n$ -dimensional Lelek map  $g$  between compacta is at most  $n$ -dimensional because the components of each fiber of  $g$  is at most  $n$ -dimensional (we say that  $g$  is  $n$ -dimensional if all fibers of  $g$  are at most  $n$ -dimensional). Lelek [16] constructed such a map from  $\mathbb{I}^{n+1}$  onto a dendrite. Levin [18] improved this result by showing that the space  $C(X, \mathbb{I}^n)$  of all maps from  $X$  into  $\mathbb{I}^n$  contains a dense  $G_\delta$ -subset consisting of  $(m - n)$ -dimensional Lelek maps for any compactum  $X$  with  $\dim X = m \geq n$ . Recently Kato and Matsushashi [13] established a version of the Levin result with  $\mathbb{I}^n$  replaced by more general class of spaces. This class consists of complete metric ANR-spaces having a piecewise embedding dimension  $\geq n$ .

In the present paper we generalize the result of Kato–Matsushashi in two directions. First, we establish a parametric version of their theorem and second, we show that this parametric version holds for  $AP(n, 0)$ -spaces (see Section 6 where it is shown that the class of  $AP(n, 0)$ -spaces contains properly the class of ANR's with piecewise embedding dimensions  $\geq n$ ). Here is our result about parametric Lelek maps (a more general version is established in Section 3; moreover we also present a version of Theorem 1.1 when  $M$  has a property weaker than  $AP(n, 0)$ , see Theorem 4.5).

**Theorem 1.1.** *Let  $f : X \rightarrow Y$  be a perfect map with  $\dim_\Delta(f) \leq m$ , where  $X$  and  $Y$  are metric spaces. If  $M \in AP(n, 0)$  is completely metrizable, then there exists a  $G_\delta$ -set  $\mathcal{H} \subset C(X, M)$  such that every simplicially factorizable map in  $C(X, M)$  is homotopically approximated by maps from  $\mathcal{H}$  and for every  $g \in \mathcal{H}$  and  $y \in Y$  the restriction  $g|_{f^{-1}(y)}$  is an  $(m - n)$ -dimensional Lelek map.*

Let us explain the notions in Theorem 1.1. A map  $g \in C(X, M)$  is homotopically approximated by maps from  $\mathcal{H}$  means that for every function  $\varepsilon \in C(X, (0, 1])$  there exists  $g' \in \mathcal{H}$  which is  $\varepsilon$ -homotopic to  $g$ . Here, the maps  $g$  and  $g'$  are said to be  $\varepsilon$ -homotopic, if there is a homotopy  $h : X \times \mathbb{I} \rightarrow M$  connecting  $g$  and  $g'$  such that each set  $h(\{x\} \times \mathbb{I})$  has a diameter  $< \varepsilon(x)$ ,  $x \in X$ .

The function space  $C(X, M)$  appearing in this theorem is endowed with the source limitation topology whose neighborhood base at a given function  $f \in C(X, M)$  consists of the sets

$$B_\rho(f, \varepsilon) = \{g \in C(X, M) : \rho(g, f) < \varepsilon\},$$

where  $\rho$  is a fixed compatible metric on  $M$  and  $\varepsilon : X \rightarrow (0, 1]$  runs over continuous positive functions on  $X$ . The symbol  $\rho(f, g) < \varepsilon$  means that  $\rho(f(x), g(x)) < \varepsilon(x)$  for all  $x \in X$ . Since  $X$  is metrizable, the source limitation topology doesn't depend on the metric  $\rho$  [15] and it has the Baire property provided  $M$  is completely metrizable [21].

We say that a map  $g : X \rightarrow M$  is simplicially factorizable [2] if there exists a simplicial complex  $L$  and two maps  $g_1 : X \rightarrow L$  and  $g_2 : L \rightarrow M$  such that  $g = g_2 \circ g_1$ . In each of the following cases the set of simplicially factorizable maps is dense in  $C(X, M)$  (see [2, Proposition 4]): (i)  $M$  is an ANR; (ii)  $\dim X \leq k$  and  $M$  is  $LC^{k-1}$ ; (iii)  $X$  is a  $C$ -space and  $M$  is locally contractible.

The dimension  $\dim_\Delta(f)$  was defined in [2]:  $\dim_\Delta(f)$  of a map  $f : X \rightarrow Y$  is equal to the smallest cardinal number  $\tau$  for which there is a map  $g : X \rightarrow \mathbb{I}^\tau$  such that the diagonal product  $f \Delta g : X \rightarrow Y \times \mathbb{I}^\tau$  is a 0-dimensional map. For any perfect map  $f : X \rightarrow Y$  between metric spaces we have: (i)  $\dim(f) \leq \dim_\Delta(f)$ ; (ii)  $\dim_\Delta(f) = \dim(f)$  if  $Y$  is a  $C$ -space, see [22] and [28]; (iii)  $\dim_\Delta(f) \leq \dim(f) + 1$  if the spaces  $X, Y$  are compact, see [17].

Since every metric space admitting a perfect finite-dimensional map onto a  $C$ -space is also a  $C$ -space [11], Theorem 1.1 implies the following (here,  $\mathcal{H}$  is homotopically dense in  $C(X, M)$  if every  $g \in C(X, M)$  is homotopically approximated by maps from  $\mathcal{H}$ ):

**Corollary 1.2.** *Let  $f : X \rightarrow Y$  be a perfect  $m$ -dimensional map between metric spaces with  $Y$  being a  $C$ -space. If  $M \in A(n, 0)$  is completely metrizable, then in each of the following cases there exists a homotopically dense  $G_\delta$ -subset  $\mathcal{H} \subset C(X, M)$  consisting of maps  $g$  such that  $g|_{f^{-1}(y)}$  is an  $(m - n)$ -dimensional Lelek map for every  $y \in Y$ :*

- $M$  is locally contractible;
- $X$  is finite-dimensional and  $M \in LC^{k-1}$  for  $k = \dim X$ .

The paper is organized as follows. In Section 2 some preliminary results about  $AE(n, 0)$ -spaces are presented. In Section 3 we establish a generalized version of Theorem 1.1. Section 4 is devoted to almost  $AP(n, 0)$ -spaces. The final two sections contain some interesting properties of  $AP(n, 0)$ -spaces. In Section 5 we prove that  $AP(n, 0)$ -property has a local nature, as well as,  $M_1 \times M_2 \in AP(n_1 + n_2, 0)$  provided each  $M_i \in AP(n_i, 0)$  is completely metrizable. It is also established in this section that every path connected compactum is a  $(V^n)$ -continuum. Section 6 is devoted to the interplay of  $AP(n, 0)$ -property and the general position properties  $m\text{-}\overline{DD}^{(n,k)}$ -properties. In particular, it is shown that every point of a locally path-connected  $AP(n, 0)$ -space  $X$  is a homological  $Z_{n-1}$ -point in  $X$ . Another result from this section states that every completely metrizable space possessing the disjoint  $(n - 1)$ -disks property is an  $AP(n, 0)$ -space.

## 2. Preliminary results about $AP(n, 0)$ -spaces

In this section we established some preliminary results on  $AP(n, 0)$ -spaces which are going to be used in next sections.

Suppose  $M$  is a metric space and  $\varepsilon > 0$ . We write  $d_n(M) < \varepsilon$  if  $M$  can be covered by an open family  $\gamma$  such that  $\text{diam } U < \varepsilon$  for all  $U \in \gamma$  and  $\text{ord}(\gamma) \leq n + 1$  (the last inequality means that at most  $n + 1$  elements of  $\gamma$  can have a common point). It is easily seen that if  $F \subset M$  is closed and  $d_n(F) < \varepsilon$ , then  $F$  is covered by an open in  $M$  family  $\gamma$  of mesh  $< \varepsilon$  and  $\text{ord} \leq n + 1$ . We also agree to denote by  $\text{cov}(M)$  the family of all open covers of  $M$ . Let us begin with the following technical lemma.

**Lemma 2.1.** *Let  $Z = A \cup B$  be a compactum, where  $A$  and  $B$  are closed subsets of  $Z$ . Suppose  $C \subset Z$  is closed such that  $\dim C \cap A \leq 0$  and  $d_0(C \cap B) < \varepsilon$ . Then  $d_0(C) < \varepsilon$ .*

**Proof.** Since  $d_0(C \cap B) < \varepsilon$ , there exists a disjoint open family  $\gamma = \{W_1, \dots, W_k\}$  in  $B$  of mesh  $< \varepsilon$ . We extend every  $W_i$  to an open set  $\tilde{W}_i$  in  $Z$  such that  $\tilde{W}_i \cap B = W_i$  and  $\tilde{\gamma} = \{\tilde{W}_i: i = 1, \dots, k\}$  is a disjoint family of mesh  $< \varepsilon$  (this can be done because  $B$  is closed in  $Z$ ). Observe that  $C_1 = C \setminus \bigcup_{i=1}^k \tilde{W}_i$  is a closed subset of  $C \cap A$  disjoint from  $B$ . Since  $\dim C \cap A \leq 0$ , there exists a clopen set  $C_2$  in  $C \cap A$  disjoint from  $B$  and containing  $C_1$ . Obviously,  $C_2$  is clopen in  $C$ . Moreover,  $\dim C_2 \leq 0$ . Hence, there exists a cover  $\omega = \{V_j: j = 1, \dots, m\} \in \text{cov}(C_2)$  consisting of clopen subsets of  $C_2$  with  $\text{diam } V_j < \varepsilon$  for every  $j$ . Then  $\omega_1 = \omega \cup \{\tilde{W}_i \cap C \setminus C_2\}$  is a disjoint open cover of  $C$  and  $\text{mesh}(\omega_1) < \varepsilon$ . So,  $d_0(C) < \varepsilon$ .  $\square$

The proof of the next lemma is extracted from [2, Theorem 4].

**Lemma 2.2.** *Let  $G \subset C(X, M)$ , where  $(M, \rho)$  is a complete metric space. Suppose  $(U(i))_{i \geq 1}$  is a sequence of open subsets of  $C(X, M)$  such that*

- for any  $g \in G$ ,  $i \geq 1$  and any function  $\eta \in C(X, (0, 1])$  there exists  $g_i \in B_\rho(g, \eta) \cap U(i) \cap G$  which is  $\eta$ -homotopic to  $g$ .

Then, for any  $g \in G$  and  $\varepsilon : X \rightarrow (0, 1]$  there exists  $g' \in \bigcap_{i=1}^\infty U(i)$  and an  $\varepsilon$ -homotopy connecting  $g$  and  $g'$ . Moreover,  $g'|_A = g_0|_A$  for some  $g_0 \in C(X, M)$  and  $A \subset X$  provided  $g_i|_A = g_0|_A$  for all  $i$ .

**Proof.** For fixed  $g \in G$  and  $\varepsilon \in C(X, (0, 1])$  let  $g_0 = g$  and  $\varepsilon_0 = \varepsilon/3$ . We shall construct by induction a sequence  $(g_i : X \rightarrow M)_{i \geq 1} \subset G$ , a sequence  $(\varepsilon_i)_{i \geq 1}$  of positive functions, and a sequence  $(H_i : X \times [0, 1] \rightarrow M)_{i \geq 1}$  of  $\varepsilon_{i-1}$ -homotopies satisfying the conditions:

- $H_{i+1}(x, 0) = g_i(x)$  and  $H_{i+1}(x, 1) = g_{i+1}(x)$  for every  $x \in X$ ;
- $g_{i+1} \in B_\rho(g_i, \varepsilon_i) \cap U(i+1) \cap G$ ;
- $\varepsilon_{i+1} \leq \varepsilon_i/2$ ;
- $B_\rho(g_{i+1}, 3\varepsilon_{i+1}) \subset U(i+1)$ .

Assume that, for some  $i$ , we have already constructed maps  $g_1, \dots, g_i$ , positive numbers  $\varepsilon_1, \dots, \varepsilon_i$ , and homotopies  $H_1, \dots, H_i$  satisfying the above conditions. According to the hypotheses, there exists a map  $g_{i+1} \in B_\rho(g_i, \varepsilon_i) \cap U(i+1) \cap G$  such that  $g_{i+1}$  is  $\varepsilon_i$ -homotopic to  $g_i$ . Let  $H_{i+1} : X \times [0, 1] \rightarrow M$  be an  $\varepsilon_i$ -homotopy connecting the maps  $g_i$  and  $g_{i+1}$ . Since the set  $U(i+1)$  is open in  $C(X, M)$ , there is a positive function  $\varepsilon_{i+1} \leq \varepsilon_i/2$  such that  $B_\rho(g_{i+1}, 3\varepsilon_{i+1}) \subset U(i+1)$ . This completes the inductive step.

It follows from the construction that the function sequence  $(g_i)_{i \geq 1}$  converges uniformly to some continuous function  $g' : X \rightarrow M$ . Obviously,  $\rho(g', g_i) \leq \sum_{j=i}^\infty \varepsilon_j \leq 2\varepsilon_i$  for every  $i$ . Hence, according to the choice of the sequences  $(\varepsilon_i)$  and  $(g_i)$ ,  $g' \in U(i)$  for every  $i \geq 1$ . So,  $g' \in \bigcap_{i=1}^\infty U(i)$ . Moreover, the  $\varepsilon_{i-1}$ -homotopies  $H_i$  compose an  $\varepsilon$ -homotopy  $H : X \times [0, 1] \rightarrow M$

$$H(x, t) = \begin{cases} H_i(x, 2^i(t - 1 + \frac{1}{2^{i-1}})) & \text{if } t \in [1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}], i \geq 1; \\ g'(x) & \text{if } t = 1, \end{cases}$$

connecting  $g_0 = g$  and  $g'$ .

It follows from our construction that  $g'|_A = g_0|_A$  if  $g_i|_A = g_0|_A$  for all  $i \geq 1$ .  $\square$

**Proposition 2.3.** *Let  $M$  be a completely metrizable space with the  $AP(n, 0)$ -property and  $X$  be an  $n$ -dimensional compactum. Then*

$$\mathcal{E} = \{g \in C(X, M): \dim g \leq 0\}$$

is a  $G_\delta$ -subset of  $C(X, M)$  such that every simplicially factorizable map is homotopically approximated by maps from  $\mathcal{E}$ .

**Proof.** We fix a metric on  $X$  and a complete metric  $\rho$  on  $M$ . For every  $\varepsilon > 0$  let  $C(X, M; \varepsilon)$  be the set of all maps  $g \in C(X, M)$  with  $d_0(g^{-1}(a)) < \varepsilon$  for every  $a \in M$ . It is easily seen that  $C(X, M; \varepsilon)$  is open in  $C(X, M)$  and  $\mathcal{E} = \bigcap_{i=1}^\infty C(X, M; 1/i)$ . So,  $\mathcal{E}$  is a  $G_\delta$ -subset of  $C(X, M)$ .

**Claim.** For every simplicially factorizable map  $h \in C(X, M)$  and positive numbers  $\eta$  and  $\delta$  there exists a simplicially factorizable map  $h' \in C(X, M; \eta)$  which is  $\delta$ -homotopic to  $h$ .

To prove the claim, fix a simplicially factorizable map  $h \in C(X, M)$  and positive numbers  $\delta, \eta$ . Then there exists a simplicial complex  $L$  and maps  $q_1 : X \rightarrow L, q_2 : L \rightarrow M$  with  $h = q_2 \circ q_1$ . We can assume that  $L$  is finite and  $q_2(L) \subset \bigcup \gamma$ , where  $\gamma = \{B_\rho(y, \delta/4) : y \in h(X)\}$ . Because  $L$  is an ANR and  $\dim X \leq n$ , there is a polyhedron  $K$  with  $\dim K \leq n$  and two maps  $f : X \rightarrow K, \alpha : K \rightarrow L$  such that  $f$  is an  $\eta$ -map and  $q_1$  and  $\alpha \circ f$  are  $q_2^{-1}(\gamma)$ -homotopic. So,  $h$  and  $q_2 \circ \alpha \circ f$  are  $\delta/2$ -homotopic. Moreover, there is a cover  $\omega \in \text{cov}(K)$  such that  $\text{diam } f^{-1}(W) < \eta$  for every  $W \in \omega$ . Let  $\theta$  be a Lebesgue number of  $\omega$ . It remains to find a map  $h^* : K \rightarrow M$  which is  $\delta/2$ -homotopic to  $q_2 \circ \alpha$  and  $h^* \in C(K, M; \theta)$  (then  $h' = h^* \circ f$  would be a simplicially factorizable map  $\delta$ -homotopic to  $h$  and  $h' \in C(X, M; \eta)$ ).

To find such a map  $h^* : K \rightarrow M$ , let  $\{\sigma_1, \dots, \sigma_m\}$  be an enumeration of the simplexes of  $K$  and  $K_i = \bigcup_{j=1}^{i-1} \sigma_j$ . We are going to construct by induction maps  $h_i : K \rightarrow M, i = 0, \dots, m$ , satisfying the following conditions:

- $h_0 = q_2 \circ \alpha$ ;
- $h_i|_{K_i}$  belongs to  $C(K_i, M; \theta), 1 \leq i \leq m$ ;
- $h_i$  and  $h_{i+1}$  are  $(\delta/2m)$ -homotopic,  $i = 0, \dots, m - 1$ .

Let  $\mathcal{V} = \{B_\rho(y, \delta/4m) : y \in M\} \in \text{cov}(M)$  and assume that  $h_i$  has already been constructed. Since  $h_i|_{K_i} \in C(K_i, M; \theta)$ , every fiber  $h_i^{-1}(y) \cap K_i$  of  $h_i|_{K_i}, y \in h_i(K_i)$ , is covered by a finite open and disjoint family  $\Omega(y)$  in  $K_i$  with  $\text{mesh}(\Omega(y)) < \theta$ . Using that  $h_i|_{K_i}$  is a perfect map, we find a cover  $\mathcal{V}_i \in \text{cov}(M)$  such that  $\mathcal{V}_i$  is a star-refinement of  $\mathcal{V}$  and if  $St(y, \mathcal{V}_i) \cap h_i(K_i) \neq \emptyset$  for some  $y \in M$ , then there is  $z \in h_i(K_i)$  with  $h_i^{-1}(St(y, \mathcal{V}_i)) \cap K_i \subset \Omega(z)$ . Since  $M$  has the  $AP(n, 0)$ -property, there exists a 0-dimensional map  $p_i : \sigma_{i+1} \rightarrow M$  which is  $\mathcal{V}_i$ -homotopic to  $h_i|_{\sigma_{i+1}}$ . By the Homotopy Extension Theorem,  $p_i$  can be extended to a map  $h_{i+1} : K \rightarrow M$  being  $\mathcal{V}_i$ -homotopic to  $h_i$ . Then  $h_{i+1}$  is  $(\delta/2m)$ -homotopic to  $h_i$ . To show that  $h_{i+1}|_{K_{i+1}}$  belongs to  $C(K_{i+1}, M; \theta)$ , we observe that  $h_{i+1}^{-1}(y) \cap K_{i+1} = (h_{i+1}^{-1}(y) \cap K_i) \cup (h_{i+1}^{-1}(y) \cap \sigma_{i+1}), y \in M$ . According to our construction, we have  $h_i(h_{i+1}^{-1}(y) \cap K_i) \subset St(y, \mathcal{V}_i)$ . Hence,  $h_{i+1}^{-1}(y) \cap K_i$  is contained in  $\Omega(z)$  for some  $z \in h_i(K_i)$ . Therefore,  $d_0(h_{i+1}^{-1}(y) \cap K_i) < \theta$ . Since  $h_{i+1}^{-1}(y) \cap \sigma_{i+1}$  is 0-dimensional, Lemma 2.1 implies that  $d_0(h_{i+1}^{-1}(y) \cap K_{i+1}) < \theta$ . Obviously the map  $h^* = h_m$  is  $\delta/2$ -homotopic to  $q_2 \circ \alpha$  and  $h^* \in C(K, M; \theta)$ . This completes the proof of the claim.

To finish the proof of the proposition, we apply Lemma 2.2 with  $G$  being the set of all simplicially factorizable maps from  $C(X, M)$  and  $U(i) = C(X, M; 1/i), i \geq 1$  (we can apply Lemma 2.2 because of the claim). Hence, for every simplicially factorizable map  $g \in C(X, M)$  and a positive number  $\varepsilon$  there exists a map  $g' \in \bigcap_{i=1}^\infty U(i)$  which is  $\varepsilon$ -homotopic to  $g$ . Finally, we observe that  $\bigcap_{i=1}^\infty U(i)$  consists of 0-dimensional maps.  $\square$

**Proposition 2.4.** Let  $M$  be a completely metrizable space with the  $AP(n, 0)$ -property,  $X$  a compactum and  $Z$  an  $F_\sigma$ -subset of  $X$  with  $\dim Z \leq n - 1$ . Then there exists a  $G_\delta$ -subset  $\mathcal{H} \subset C(X, M)$  with the following properties:

- $Z$  is contained in the union of trivial components of the fibers of  $g$  for all  $g \in \mathcal{H}$ ;
- for every simplicially factorizable map  $g \in C(X, M)$  and every  $\varepsilon > 0$  there exists  $g' \in \mathcal{H}$  which is  $\varepsilon$ -homotopic to  $g$ .

**Proof.** We represent  $Z$  as the union of an increasing sequence  $(Z_i)_{i \geq 1}$  with all  $Z_i$  being closed in  $Z$ . For every  $\varepsilon > 0$  let

$$\mathcal{H}(Z_i, \varepsilon) = \{g \in C(X, M) : F(g, \varepsilon) \cap Z_i = \emptyset\},$$

where

$$F(g, \varepsilon) = \bigcup \{C : C \text{ is a component of a fiber of } g \text{ with } \text{diam } C \geq \varepsilon\}.$$

It is easily seen that each  $\mathcal{H}(Z_i, \varepsilon)$  is open in  $C(X, M)$ . So,  $\mathcal{H} = \bigcap_{i=1}^\infty \mathcal{H}(Z_i, 1/i)$  is a  $G_\delta$ -set in  $C(X, M)$  and  $Z$  is contained in the union of trivial components of the fibers of  $g$  for every  $g \in \mathcal{H}$ .

To prove the second item of our proposition, we first consider the particular case when  $X$  is a polyhedron and  $Z$  is a (compact) subpolyhedron of  $X$ .

**Claim 1.** Suppose that, in addition to hypotheses of Proposition 2.4,  $X$  is a polyhedron and  $Z$  is a subpolyhedron of  $X$ . Then, for every  $g_0 \in C(X, M)$  and  $\delta > 0$  there exists  $g \in \mathcal{H} = \bigcap_{i=1}^\infty \mathcal{H}(Z, 1/i)$  which is  $\delta$ -homotopic to  $g_0$ .

The proof of Claim 1 is a slight modification of the proof of [13, Theorem 2.2]. Let  $g_0 \in C(X, M)$  and  $\delta > 0$ . We take an open neighborhood  $W$  of  $Z$  in  $X$ , a retraction  $r : W \rightarrow Z$  and a function  $\alpha : X \rightarrow \mathbb{I}$  such that  $\alpha^{-1}(0) = Z, \alpha^{-1}(1) = X \setminus W$ . Since  $X$  is a compact ANR, we can choose  $W$  so small that  $g_0|_W$  is  $\delta/2$ -homotopic to  $g_0 \circ r$ . Next, denote by  $\pi : Z \times \mathbb{I} \rightarrow Z$  the projection and consider the map  $\varphi : W \rightarrow Z \times \mathbb{I}, \varphi(x) = (r(x), \alpha(x))$ . Obviously,  $g_0 \circ r = g_0 \circ \pi \circ \varphi$ . So,  $g_0 \circ \pi \circ \varphi$  is  $\delta/2$ -homotopic to  $g_0|_W$ . Because  $Z$  is a polyhedron, so is  $Z \times \mathbb{I}$ . Hence, every map from  $C(Z \times \mathbb{I}, M)$  is simplicially factorizable. Moreover,  $\dim Z \times \mathbb{I} \leq n$  and  $M \in AP(n, 0)$ . Therefore, by Proposition 2.3, there exists a 0-dimensional map  $h : Z \times \mathbb{I} \rightarrow M$

which is  $\delta/2$ -homotopic to the map  $g_0 \circ \pi$ . Then  $h \circ \varphi$  is  $\delta/2$ -homotopic to  $g_0 \circ \pi \circ \varphi$ . Consequently,  $h \circ \varphi$  is  $\delta$ -homotopic to  $g_0|_W$ . By the Homotopy Extension Theorem, there exists a map  $g \in C(X, M)$  such that  $g$  is  $\delta$ -homotopic to  $g_0$  and  $g|_{\bar{U}} = (h \circ \varphi)|_{\bar{U}}$ , where  $U$  is an open neighborhood of  $Z$  in  $X$  with  $\bar{U} \subset W$ .

To finish the proof of Claim 1, it remains to show that  $g \in \mathcal{H}$ . Let  $C$  be a subcontinuum of  $g^{-1}(y)$  for some  $y \in M$  and let  $Z \cap C \neq \emptyset$ . We are going to prove that  $C \subset Z$ . Otherwise, there would be a subcontinuum  $C' \subset C \cap U$  such that  $C' \cap Z \neq \emptyset$  and  $C' \setminus Z \neq \emptyset$ . Then  $g(C') = h(\varphi(C')) = y$  and, according to the definition of  $\alpha$ ,  $\varphi(C')$  is a non-degenerate continuum in  $h^{-1}(y)$ . Since  $h$  is 0-dimensional, this is a contradiction. Hence,  $C \subset Z$ . Using again that  $\dim(h) \leq 0$ , we conclude that  $\varphi(C)$  is a point. On the other hand  $\varphi(C) = C \times \{0\} \subset Z \times \mathbb{I}$ . Therefore,  $C$  should be a trivial continuum.

Now, consider the general case of Proposition 2.4.

**Claim 2.** Let  $g_0 \in C(X, M)$  be a simplicially factorizable map and  $\delta, \eta$  positive numbers. Then for any natural  $i$  there exists a simplicially factorizable map  $g \in \mathcal{H}(Z_i, \eta)$  which is  $\delta$ -homotopic to  $g_0$ .

Since  $g_0$  is simplicially factorizable, there exist a finite simplicial complex  $L$  and maps  $q_1 : X \rightarrow L, q_2 : L \rightarrow M$  with  $g_0 = q_2 \circ q_1$ . Let  $\mathcal{V} = \{B_\rho(y, \delta/4) : y \in M\} \in \text{cov}(M)$  and  $\mathcal{W} = q_2^{-1}(\mathcal{V}) \in \text{cov}(L)$ . Next, choose a finite cover  $\mathcal{U} \in \text{cov}(X)$  of  $X$  with  $\text{mesh}(\mathcal{U}) < \eta$  such that:

- at most  $n$  elements of the family  $\gamma = \{U \in \mathcal{U} : U \cap Z_i \neq \emptyset\}$  can have a common point;
- there exists a map  $h : \mathcal{N}(\mathcal{U}) \rightarrow L$  such that  $h \circ f_{\mathcal{U}}$  is  $\mathcal{W}$ -homotopic to  $q_1$ , where  $\mathcal{N}(\mathcal{U})$  is the nerve of  $\mathcal{U}$  and  $f_{\mathcal{U}} : X \rightarrow \mathcal{N}(\mathcal{U})$  is the natural map.

Then  $q_2 \circ h \circ f_{\mathcal{U}}$  is  $\delta/2$ -homotopic to  $g_0$  and the subpolyhedron  $K$  of  $\mathcal{N}(\mathcal{U})$  generated by the family  $\gamma$  is of dimension  $\leq n - 1$ . So, according to Claim 1, there exists a map  $g_1 : \mathcal{N}(\mathcal{U}) \rightarrow M$  such that  $g_1$  is  $\delta/2$ -homotopic to  $q_2 \circ h$  and  $K$  is contained in the union of trivial components of the fibers of  $g_1$ . Since all fibers of  $f_{\mathcal{U}}$  are of diameter  $< \eta$  and  $f_{\mathcal{U}}(Z_i) \subset K$ , the map  $g = g_1 \circ f_{\mathcal{U}}$  belongs to  $\mathcal{H}(Z_i, \eta)$ . Moreover,  $g$  is obviously simplicially factorizable and  $\delta$ -homotopic to  $g_0$ . This completes the proof of Claim 2.

Finally, the proof of Proposition 2.4 follows from Lemma 2.2 (with  $G$  being the set of all simplicially factorizable maps from  $C(X, M)$  and  $U(i) = \mathcal{H}(Z_i, 1/i)$  for every  $i \geq 1$ ) and Claim 2.  $\square$

### 3. Parametric Lelek maps

In this section we are going to prove Theorem 1.1. Everywhere in this section we suppose that  $(M, \rho)$  is a given complete metric space and  $\mu = \{W_\nu : \nu \in \Lambda\}, \mu_1 = \{G_\nu : \nu \in \Lambda\}$  are locally finite open covers of  $M$  such that

(\*)  $\bar{G}_\nu \subset W_\nu$  and  $W_\nu \in AP(n_\nu, 0)$  with  $0 \leq n_\nu$  for every  $\nu \in \Lambda$ .

Obviously, Theorem 1.1 follows directly from next theorem.

**Theorem 3.1.** Let  $f : X \rightarrow Y$  be a perfect map between metrizable spaces with  $\dim_\Delta(f) \leq m$ . Suppose  $(M, \rho)$  is a complete metric space and  $\mu, \mu_1$  two locally finite open covers of  $M$  satisfying condition (\*). Then there is a  $G_\delta$ -set  $\mathcal{H} \subset C(X, M)$  such that any simplicially factorizable map in  $C(X, M)$  can be homotopically approximated by maps from  $\mathcal{H}$  and every  $g \in \mathcal{H}$  has the following property: for any  $y \in Y$  and  $\nu \in \Lambda$  the restriction  $g|_{(f^{-1}(y) \cap g^{-1}(\bar{G}_\nu))}$  is an  $(m - n_\nu)$ -dimensional Lelek map from  $f^{-1}(y) \cap g^{-1}(\bar{G}_\nu)$  into  $\bar{G}_\nu$ .

The proof of Theorem 3.1 consists of a few propositions. We can assume that  $n_\nu \leq m$  for all  $\nu \in \Lambda$ . For every  $y \in Y, \varepsilon > 0, \nu \in \Lambda$  and  $g \in C(X, M)$  we denote by  $F_\nu(g, \varepsilon, y)$  the union of all continua  $C$  of diameter  $\geq \varepsilon$  such that  $C \subset f^{-1}(y) \cap g^{-1}(z_C)$  for some  $z_C \in \bar{G}_\nu$ . Note that each  $F_\nu(g, \varepsilon, y)$  is compact as a closed subset of  $f^{-1}(y)$ . Then, for a fixed  $\eta > 0$ , let  $\mathcal{H}(y, \varepsilon, \eta)$  be the set of all  $g \in C(X, M)$  such that  $d_{m-n_\nu}(F_\nu(g, \varepsilon, y)) < \eta$  for every  $\nu \in \Lambda$ . The last inequality means that  $F_\nu(g, \varepsilon, y)$  can be covered by an open family  $\gamma$  in  $X$  such that  $\text{mesh}(\gamma) < \eta$  and no more  $m - n_\nu + 1$  elements of  $\gamma$  have a common point. If  $V \subset Y$ , then  $\mathcal{H}(V, \varepsilon, \eta)$  denotes the intersection of all  $\mathcal{H}(y, \varepsilon, \eta), y \in V$ .

**Lemma 3.2.** For every  $y \in Y$  and every  $g \in \mathcal{H}(y, \varepsilon, \eta)$  there exists a neighborhood  $V_y$  of  $y$  in  $Y$  and  $\delta_y > 0$  such that if  $y' \in V_y$  and  $\rho(h(x), g(x)) < \delta_y$  for all  $x \in f^{-1}(y')$ , then  $h \in \mathcal{H}(y', \varepsilon, \eta)$ .

**Proof.** Since  $\mu_1$  is locally finite and  $f^{-1}(y)$  is compact, for every  $y \in Y$  and  $g \in C(X, M)$  there exists a neighborhood  $O_g(y)$  of  $g(f^{-1}(y))$  such that the family  $A_g(y) = \{\nu \in \Lambda : O_g(y) \cap G_\nu \neq \emptyset\}$  is finite.

Assume the lemma is not true for some  $y \in Y$  and  $g \in \mathcal{H}(y, \varepsilon, \eta)$ . Then there exists a local base of neighborhoods  $V_i$  of  $y$ , points  $y_i \in V_i$  and functions  $g_i \in C(X, M)$  such that  $g_i|_{f^{-1}(y_i)}$  is  $1/i$ -close to  $g|_{f^{-1}(y_i)}$  but  $g_i \notin \mathcal{H}(y_i, \varepsilon, \eta)$ . It is easily seen that for some  $k$  and all  $i \geq k$  we have  $g_i(f^{-1}(y_i)) \subset O_g(y)$ . Consequently, for every  $i \geq k$  there exists  $\nu(i) \in A_g(y)$  such that  $d_{m-n_{\nu(i)}}(F_{\nu(i)}(g_i, \varepsilon, y_i)) \geq \eta$  (because  $g_i \notin \mathcal{H}(y_i, \varepsilon, \eta)$ ). So, each  $F_{\nu(i)}(g_i, \varepsilon, y_i)$  doesn't have any open cover of mesh  $< \eta$

and order  $\leq m - n_{v(i)} + 1$ . Since the family  $\Lambda_g(y)$  is finite, there exists  $v(0) \in \Lambda_g(y)$  with  $v(i) = v(0)$  for infinitely many  $i$ . Without loss of generality, we may suppose that  $v(i) = v(0)$  for all  $i \geq k$ . This implies that  $g_i(f^{-1}(y_i)) \cap \bar{G} \neq \emptyset$ ,  $i \geq k$ , where  $G = G_{v(0)}$ . Consequently,  $g(f^{-1}(y)) \cap \bar{G} \neq \emptyset$ . Since  $g \in \mathcal{H}(Y, \varepsilon, \eta)$ ,  $F_{v(0)}(g, \varepsilon, y)$  can be covered by an open family  $\gamma$  in  $X$  of order  $\leq m - n_{v(0)} + 1$  and  $\text{mesh}(\gamma) < \eta$ . Let  $U = \bigcup \gamma$ . To obtain a contradiction, it suffice to show that  $F_{v(i)}(g_i, \varepsilon, y_i) \subset U$  for some  $i \geq k$ . Indeed, otherwise for every  $i \geq k$  there would exist points  $x_i \in F_{v(i)}(g_i, \varepsilon, y_i) \setminus U$ ,  $z_i \in \bar{G}$ , and a component  $C_i$  of  $f^{-1}(y_i) \cap g_i^{-1}(z_i)$  containing  $x_i$  with  $\text{diam} C_i \geq \varepsilon$ . Using that  $P = f^{-1}(\{y_i\}_{i=k}^\infty \cup \{y\})$  is a compactum, we can suppose that  $\{x_i\}_{i=k}^\infty$  converges to a point  $x_0 \in f^{-1}(y)$ ,  $\{z_i\}_{i=k}^\infty$  converges to a point  $z_0 \in \bar{G}$  and  $\{C_i\}_{i=k}^\infty$  (considered as a sequence in the space of all closed subsets of  $P$  equipped with the Vietoris topology) converges to a closed set  $C \subset f^{-1}(y) \cap g^{-1}(z_0)$ . It is easily seen that  $C$  is connected and  $\text{diam} C \geq \varepsilon$ . Hence,  $C \subset F_{v(0)}(g, \varepsilon, y) \subset U$ . So,  $x_i \in C_i \subset U$  for some  $i$ , a contradiction.  $\square$

Now, we are in a position to show that the sets  $\mathcal{H}(Y, \varepsilon, \eta)$  are open in  $C(X, M)$ .

**Proposition 3.3.** *For any closed set  $F \subset Y$  and any  $\varepsilon, \eta > 0$ , the set  $\mathcal{H}(F, \varepsilon, \eta)$  is open in  $C(X, M)$ .*

**Proof.** Let  $g_0 \in \mathcal{H}(F, \varepsilon, \eta)$ . Then, by Lemma 3.2, for every  $y \in F$  there exist a neighborhood  $V_y$  and a positive  $\delta_y \leq 1$  such that  $h \in \mathcal{H}(y', \varepsilon, \eta)$  provided  $y' \in V_y$  and  $h|f^{-1}(y')$  is  $\delta_y$ -close to  $g_0|f^{-1}(y')$ . The family  $\{V_y \cap F : y \in F\}$  can be supposed to be locally finite in  $F$ . Consider the set-valued lower semi-continuous map  $\varphi : F \rightarrow (0, 1]$ ,  $\varphi(y) = \bigcup \{(0, \delta_z] : y \in V_z\}$ . By [23, Theorem 6.2, p. 116],  $\varphi$  admits a continuous selection  $\beta : F \rightarrow (0, 1]$ . Let  $\bar{\beta} : Y \rightarrow (0, 1]$  be a continuous extension of  $\beta$  and  $\alpha = \bar{\beta} \circ f$ . It remains only to show that if  $g \in C(X, M)$  with  $\rho(g_0(x), g(x)) < \alpha(x)$  for all  $x \in X$ , then  $g \in \mathcal{H}(F, \varepsilon, \eta)$ . So, we take such a  $g$  and fix  $y \in F$ . Then there exists  $z \in F$  with  $y \in V_z$  and  $\alpha(x) \leq \delta_z$  for all  $x \in f^{-1}(y)$ . Hence,  $\rho(g(x), g_0(x)) < \delta_z$  for each  $x \in f^{-1}(y)$ . According to the choice of  $V_z$  and  $\delta_z$ ,  $g \in \mathcal{H}(y, \varepsilon, \eta)$ . Therefore,  $\mathcal{H}(F, \varepsilon, \eta)$  is open in  $C(X, M)$ .  $\square$

To prove Theorem 3.1 it suffices to show that if  $g \in C(X, M)$  is a simplicially factorizable map and  $\delta \in C(X, (0, 1])$ , then for any  $\varepsilon, \eta > 0$  there exists a simplicially factorizable map  $g_{\varepsilon\eta} \in \mathcal{H}(Y, \varepsilon, \eta)$  which is  $\delta$ -homotopic to  $g$ . Indeed, since any set  $\mathcal{H}(Y, \varepsilon, \eta)$  is open, Lemma 2.2 would imply that every simplicially factorizable map is homotopically approximated by simplicially factorizable maps from  $\mathcal{H} = \bigcap_{i,j=1}^\infty \mathcal{H}(Y, 1/i, 1/j)$ . But for every  $g \in \mathcal{H}$ ,  $y \in Y$ ,  $v \in \Lambda$  and  $i \geq 1$  we have  $d_{m-n_v}(F_v(g, 1/i, y)) = 0$ . So,  $\dim F_v(g, 1/i, y) \leq m - n_v$ . Then  $\dim F_v(g, y) \leq m - n_v$ , where  $F_v(g, y) = \bigcup_{i=1}^\infty F_v(g, 1/i, y)$ . On the other hand,  $F_v(g, y)$  is the union of all non-trivial continua contained in the fibers of  $g|g^{-1}(\bar{G}_v) \cap f^{-1}(y)$ . Therefore,  $\mathcal{H}$  consists of maps  $g$  such that  $g|g^{-1}(\bar{G}_v) \cap f^{-1}(y) : g^{-1}(\bar{G}_v) \cap f^{-1}(y) \rightarrow \bar{G}_v$  is  $(m - n_v)$ -dimensional Lelek map for every  $v \in \Lambda$  and  $y \in Y$ .

Next proposition shows that all simplicially factorizable maps in  $C(X, M)$  can be homotopically approximated by simplicially factorizable maps from  $\mathcal{H}(Y, \varepsilon, \eta)$  provided the set  $\mathcal{H}_p(L)$  is homotopically dense in  $C(N, M)$ , where  $p : N \rightarrow L$  is any perfect  $m$ -dimensional PL-map between two simplicial complexes  $N, L$  equipped with the CW-topology and  $\mathcal{H}_p(L)$  is the set of all maps  $q : N \rightarrow M$  such that  $q|q^{-1}(\bar{G}_v) \cap p^{-1}(z) : q^{-1}(\bar{G}_v) \cap p^{-1}(z) \rightarrow \bar{G}_v$  is an  $(m - n_v)$ -dimensional Lelek map for every  $v \in \Lambda$  and  $z \in L$ . Recall that  $p : N \rightarrow L$  is a PL-map (resp., a simplicial map) if  $p$  maps every simplex  $\sigma$  of  $N$  into (resp., onto) some simplex of  $L$  and  $p$  is linear on  $\sigma$ .

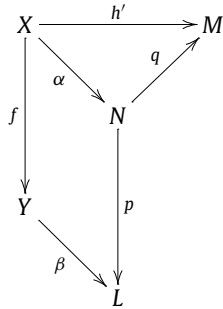
**Proposition 3.4.** *Let  $X, Y, f$  and  $M$  satisfy the hypotheses of Theorem 3.1. Suppose the set  $\mathcal{H}_p(L)$  is homotopically dense in  $C(N, M)$  for any perfect  $m$ -dimensional PL-map  $p : N \rightarrow L$ . Then for any simplicially factorizable map  $g \in C(X, M)$  and any  $\delta \in C(X, (0, 1])$ ,  $\varepsilon, \eta > 0$  there exists a simplicially factorizable  $h \in \mathcal{H}(Y, \varepsilon, \eta)$  such that  $h$  is  $\delta$ -homotopic to  $g$ .*

**Proof.** For fixed  $\delta \in C(X, (0, 1])$  and a simplicially factorizable map  $g \in C(X, M)$  we are going to find a simplicially factorizable  $h \in \mathcal{H}(Y, \varepsilon, \eta)$  with  $\rho(g(x), h(x)) < \delta(x)$  for all  $x \in X$ , where  $\varepsilon$  and  $\eta$  are arbitrary positive reals. Since  $g$  is simplicially factorizable, there exists a simplicial complex  $D$  and maps  $g_D : X \rightarrow D$ ,  $g^D : D \rightarrow M$  such that  $g = g^D \circ g_D$ . The metric  $\rho$  induces a continuous pseudometric  $\rho_D$  on  $D$ ,  $\rho_D(x, y) = \rho(g^D(x), g^D(y))$ . By [5] and [25],  $D$  being a stratifiable ANR is a neighborhood retract of a locally convex space. Hence, we can apply [2, Lemma 8.1] to find an open cover  $\mathcal{U}$  of  $X$  satisfying the following condition: if  $\alpha : X \rightarrow K$  is a  $\mathcal{U}$ -map into a paracompact space  $K$  (i.e.,  $\alpha^{-1}(\omega)$  refines  $\mathcal{U}$  for some  $\omega \in \text{cov}(K)$ ), then there exists a map  $q' : G \rightarrow D$ , where  $G$  is an open neighborhood of  $\alpha(X)$  in  $K$ , such that  $g_D$  and  $q' \circ \alpha$  are  $\delta/2$ -homotopic with respect to the metric  $\rho_D$ . Let  $\mathcal{U}_1$  be an open cover of  $X$  refining  $\mathcal{U}$  with  $\text{mesh} \mathcal{U}_1 < \min\{\varepsilon, \eta\}$  and  $\text{inf} \delta(U) > 0$  for all  $U \in \mathcal{U}_1$ .

Next, according to [2, Theorem 6], there exists an open cover  $\mathcal{V}$  of  $Y$  such that: for any  $\mathcal{V}$ -map  $\beta : Y \rightarrow L$  into a simplicial complex  $L$  we can find a  $\mathcal{U}_1$ -map  $\alpha : X \rightarrow K$  into a simplicial complex  $K$  and a perfect  $m$ -dimensional PL-map  $p : K \rightarrow L$  with  $\beta \circ f = p \circ \alpha$ . We can assume that  $\mathcal{V}$  is locally finite. Take  $L$  to be the nerve of the cover  $\mathcal{V}$  and  $\beta : Y \rightarrow L$  the corresponding natural map. Then there are a simplicial complex  $K$  and maps  $p$  and  $\alpha$  satisfying the above conditions. Hence, the following diagram is commutative:

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & K \\
 f \downarrow & & \downarrow p \\
 Y & \xrightarrow{\beta} & L
 \end{array}$$

Since  $K$  is paracompact, the choice of the cover  $\mathcal{U}$  guarantees the existence of a map  $q_D : G \rightarrow D$ , where  $G \subset K$  is an open neighborhood of  $\overline{\alpha(X)}$ , such that  $g_D$  and  $h_D = q_D \circ \alpha$  are  $\delta/2$ -homotopic with respect to  $\rho_D$ . Then, according to the definition of  $\rho_D$ ,  $h' = g^D \circ q_D \circ \alpha$  is  $\delta/2$ -homotopic to  $g$  with respect to  $\rho$ . Replacing the triangulation of  $K$  by a suitable subdivision, we may additionally assume that no simplex of  $K$  meets both  $\overline{\alpha(X)}$  and  $K \setminus G$ . So, the union  $N$  of all simplexes  $\sigma \in K$  with  $\sigma \cap \overline{\alpha(X)} \neq \emptyset$  is a subcomplex of  $K$  and  $N \subset G$ . Moreover, since  $N$  is closed in  $K$ ,  $p : N \rightarrow L$  is a perfect  $m$ -dimensional PL-map. Therefore, we have the following commutative diagram, where  $q = g^D \circ q_D$ :



Now, we shall construct a continuous function  $\delta_1 : N \rightarrow (0, 1]$  with  $\delta_1 \circ \alpha \leq \delta$ . Since  $\alpha$  is a  $\mathcal{U}_1$ -map, there is an open cover  $\mathcal{V}_1$  of  $N$  such that the cover  $\alpha^{-1}(\mathcal{V}_1) = \{\alpha^{-1}(V) : V \in \mathcal{V}_1\}$  refines  $\mathcal{U}_1$ . Because  $\inf \delta(U) > 0$  for any  $U \in \mathcal{U}_1$ ,  $\inf \delta(\alpha^{-1}(V)) > 0$  for any  $V \in \mathcal{V}_1$ . We can assume that  $\mathcal{V}_1$  is locally finite and consider the lower semi-continuous set-valued map  $\varphi : N \rightarrow (0, 1]$  defined by  $\varphi(z) = \bigcup \{(0, \inf \delta(\alpha^{-1}(V))\} : z \in V \in \mathcal{V}_1\}$ . Then, by [23, Theorem 6.2, p. 116],  $\varphi$  admits a continuous selection  $\delta_1 : N \rightarrow (0, 1]$ . Obviously,  $\delta_1(z) \leq \inf \delta(\alpha^{-1}(z))$  for all  $z \in N$ . Hence,  $\delta_1 \circ \alpha \leq \delta$ .

Since, according to our assumption,  $\mathcal{H}_p(L)$  is homotopically dense in  $C(N, M)$ , there exists a map  $q_1 \in \mathcal{H}_p(L)$  such that  $q_1$  is  $\delta_1/2$ -homotopic to  $q$ . Let  $h = q_1 \circ \alpha$ . Then  $h$  and  $q \circ \alpha$  are  $\delta/2$ -homotopic because  $\delta_1 \circ \alpha \leq \delta$ . On the other hand,  $q \circ \alpha = h'$  is  $\delta/2$ -homotopic to  $g$ . Hence,  $g$  and  $h$  are  $\delta$ -homotopic. Moreover,  $h$  is obviously simplicially factorizable.

It remains to show that  $h \in \mathcal{H}(Y, \varepsilon, \eta)$ . To this end, we fix  $y \in Y$  and  $v \in \Lambda$ , and consider the set  $F_v(h, \varepsilon, y)$ . Recall that  $F_v(h, \varepsilon, y)$  is the union of all continua of diameter  $\geq \varepsilon$  such that  $C \subset f^{-1}(y) \cap h^{-1}(a_C)$  for some  $a_C \in \overline{G}_v$ . For any such continuum  $C$  we have  $\alpha(C) \subset p^{-1}(\beta(y)) \cap q_1^{-1}(a_C)$ . Since the diameters of all fibers of  $\alpha$  are  $< \varepsilon$  (recall that  $\alpha$  is a  $\mathcal{U}_1$ -map),  $\alpha(C)$  is a non-trivial continuum in  $p^{-1}(\beta(y)) \cap q_1^{-1}(a_C)$ . Therefore,  $\alpha(C) \subset F_v(q_1, \beta(y))$ , where  $F_v(q_1, \beta(y))$  denotes the union of all non-trivial continua which are contained in the fibers of the restriction  $q_{yv} = q_1|_{(p^{-1}(\beta(y)) \cap q_1^{-1}(\overline{G}_v))}$ . Actually, we proved that  $\alpha(F_v(h, \varepsilon, y)) \subset F_v(q_1, \beta(y))$ . Since  $q_1 \in \mathcal{H}_p(L)$ ,  $q_{yv} : p^{-1}(\beta(y)) \cap q_1^{-1}(\overline{G}_v) \rightarrow \overline{G}_v$  is  $(m - n_v)$ -dimensional Lelek map. Consequently,  $\dim F_v(q_1, \beta(y)) \leq m - n_v$ . So, there exists an open cover  $\gamma$  of  $F_v(q_1, \beta(y))$  of order  $\leq m - n_v + 1$  (such a cover  $\gamma$  exists because  $F_v(q_1, \beta(y))$  is metrizable as a subset of the metrizable compactum  $p^{-1}(\beta(y))$ ). We can suppose that  $\gamma$  is so small that  $\alpha^{-1}(\gamma)$  refines  $\mathcal{U}_1$ . But  $\text{mesh}(\mathcal{U}_1) < \eta$ . Consequently,  $\alpha^{-1}(\gamma) \cap F_v(h, \varepsilon, y)$  is an open cover of  $F_v(h, \varepsilon, y)$  of order  $\leq m - n_v + 1$  and  $\text{mesh} < \eta$ . This means that  $d_{m-n_v}(F_v(h, \varepsilon, y)) < \eta$ . Therefore, we found a simplicially factorizable map  $h \in \mathcal{H}(Y, \varepsilon, \eta)$  which is  $\delta$ -homotopic to  $g$ .  $\square$

In next two lemmas we suppose that  $p : N \rightarrow L$  is an  $m$ -dimensional PL-map between finite simplicial complexes. As everywhere in this section,  $(M, \rho)$  is a complete metric space possessing two locally finite open covers  $\mu = \{W_v : v \in \Lambda\}$  and  $\mu_1 = \{G_v : v \in \Lambda\}$  such that  $\overline{G}_v \subset W_v$ ,  $W_v \in AP(n_v, 0)$  and  $0 \leq n_v \leq m$  for every  $v \in \Lambda$ . For given  $\varepsilon, \eta > 0$  and  $y \in L$  we denote by  $\mathcal{H}_p(y, \varepsilon, \eta)$  the set of  $g \in C(N, M)$  such that  $d_{m-n_v}(F_v(g, \varepsilon, y)) < \eta$  for every  $v \in \Lambda$ . Here,  $F_v(g, \varepsilon, y)$  is the union of all continua  $C \subset p^{-1}(y) \cap g^{-1}(z_C)$  with  $z_C \in \overline{G}_v$  and  $\text{diam } C \geq \varepsilon$ . If  $B \subset L$ , then  $\mathcal{H}_p(B, \varepsilon, \eta)$  stands for the intersection of all  $\mathcal{H}_p(y, \varepsilon, \eta)$ ,  $y \in B$ .

**Lemma 3.5.** *Let  $B \subset L$  be a subcomplex of  $L$  and  $g_0 \in C(N, M)$  be a map such that  $g_0 \in \mathcal{H}_p(B) = \bigcap_{i \geq 1} \mathcal{H}_p(B, 1/i, 1/i)$ . Then, for every  $\varepsilon > 0$ ,  $q \geq 1$  and  $g \in C(N, M)$  with  $g|_{p^{-1}(B)} = g_0|_{p^{-1}(B)}$  there exists a map  $g_q \in C(N, M)$  extending  $g_0|_{p^{-1}(B)}$  such that  $g_q$  is  $\varepsilon$ -homotopic to  $g$  and  $g_q \in \mathcal{H}_p(y, 1/q, 1/q)$  for all  $y \in L$ .*

**Proof.** We fix  $q$  and  $g \in C(N, M)$  with  $g|_{p^{-1}(B)} = g_0|_{p^{-1}(B)}$ . Then, by Lemma 3.2, for every  $y \in B$  there exists a neighborhood  $V_y$  in  $L$  and  $\delta_y > 0$  such that any  $h \in C(N, M)$  belongs to  $\mathcal{H}_p(V_y, 1/q, 1/q)$  provided  $h|_{p^{-1}(V_y)}$  is  $\delta_y$ -close to  $g|_{p^{-1}(V_y)}$  (we can apply Lemma 3.2 because  $g|_{p^{-1}(B)} = g_0|_{p^{-1}(B)}$  yields  $g \in \mathcal{H}_p(y, 1/q, 1/q)$  for every  $y \in B$ ). Let  $\{V_{y(i)}\}_{i \leq s}$  be a finite subfamily of  $\{V_y : y \in B\}$  covering  $B$  and  $V = \bigcup_{1 \leq i \leq s} V_{y(i)}$ .

Since  $\mu$  is locally finite in  $M$ ,  $g(N)$  meets only finitely many  $W_j = W_{v(j)}$ ,  $j = 1, \dots, k$ . For any  $j \leq k$  let  $P_j = g^{-1}(\overline{G}_{v(j)})$  and  $U_j^1, U_j^2$  be open subsets of  $N$  such that  $P_j \subset U_j^1 \subset \overline{U}_j^1 \subset U_j^2 \subset \overline{U}_j^2 \subset g^{-1}(W_j)$ . We also choose  $\varepsilon_0 > 0$  satisfying the following condition:

- If  $h \in C(N, M)$  with  $\rho(g(x), h(x)) < \varepsilon_0$  for every  $x \in N$ , then  $\{v \in \Lambda : h(N) \cap W_v \neq \emptyset\}$  is contained in  $\{v(j) : j = 1, \dots, k\}$ , and for all  $j$  we have  $h^{-1}(\overline{G}_{v(j)}) \subset U_j^1$  and  $h(\overline{U}_j^2) \subset W_j$ .

Let  $\delta_0 = \min\{\varepsilon, \varepsilon_0, \delta_{y(i)} : i \leq s\}$ . Considering suitable subdivisions of  $N$  and  $L$ , we can suppose that  $p$  is a simplicial map and the following conditions hold:

- Every simplex  $\sigma \in L$  intersecting the set  $L \setminus V$  does not meet  $B$ .
- Every simplex  $\tau \in N$  intersecting the set  $\bar{U}_j^1$  does not meet  $N \setminus U_j^2$ ,  $j = 1, \dots, k$ .

Let  $L_0 = \bigcup\{\sigma \in L : \sigma \cap L \setminus V \neq \emptyset\}$  and  $N_j = \bigcup\{\tau \in N : \tau \cap \bar{U}_j^1 \neq \emptyset\}$ ,  $1 \leq j \leq k$ . Obviously  $L_0$  is a subpolyhedron of  $L$  disjoint from  $B$  and containing  $L \setminus V$ . Moreover, each  $N_j$  is a subpolyhedron of  $N$  such that  $\bar{U}_j^1 \subset N_j \subset U_j^2$ . Now, for every  $j \leq k$  consider the map  $p_j = p|_{N_j} : N_j \rightarrow p(N_j)$ . Since  $p$  is  $m$ -dimensional, by [22] or [27], there exists an  $(n_{v(j)} - 1)$ -dimensional sigma-compact set  $Z_j = \bigcup_{i \geq 1} Z_{ij} \subset N_j$  such that each  $Z_{ij}$  is compact and  $\dim p_j^{-1}(y) \setminus Z_j \leq m - n_{v(j)}$  for all  $y \in p(N_j)$ . Denote

$$T_{ij} = \{h \in C(N, M) : F^j(h, 1/q) \cap Z_{ij} = \emptyset\},$$

where  $F^j(h, 1/q)$  is the union of all continua  $C \subset N_j$  of diameter  $\geq 1/q$  which are contained in fibers of the map  $h|_{N_j}$ . It is easily seen that  $T_{ij}$  are open in  $C(N, M)$ .

**Claim.** For any  $h \in B_\rho(g, \delta_0)$ ,  $\eta > 0$  and  $i, j \geq 1$  there exists  $h_{ij} \in T_{ij} \cap B_\rho(h, \eta) \cap B_\rho(g, \delta_0)$  which is  $\eta$ -homotopic to  $h$ .

Indeed,  $h \in B_\rho(g, \delta_0)$  and  $\delta_0 \leq \varepsilon_0$  imply that  $h(N_j) \subset W_j$  and  $h(N) \cap W_v = \emptyset$  if  $v \neq v(j)$  for all  $j$ . Let  $\eta_1 = \min\{\eta, \delta_0 - \rho(g(x), h(x)) : x \in N\}$ . Since  $h|_{N_j}$  is simplicially factorizable (as a map whose domain is a polyhedron),  $W_j \in AP(n_{v(j)}, 0)$  and  $Z_{ij}$  is a compact subset of  $N_j$  with  $\dim Z_{ij} \leq n_{v(j)} - 1$ , according to Proposition 2.4, there is a map  $h' : N_j \rightarrow W_j$  which is  $\eta_1$ -homotopic to  $h|_{N_j}$  and the union of all non-trivial components of the fibers of  $h'$  is disjoint from  $Z_{ij}$ . By the Homotopy Extension Theorem,  $h'$  admits an extension  $h_{ij} \in C(N, M)$  with  $h_{ij}$  being  $\eta_1$ -homotopic to  $h$ . Obviously,  $h_{ij} \in T_{ij}$  and  $h_{ij} \in B_\rho(g, \delta_0)$ .

The above claim allows us to apply Lemma 2.2 for the set  $B_\rho(g, \delta_0)$  and the sequence  $\{T_{ij}\}_{i,j \geq 1}$  to obtain a map  $h_1 \in C(N, M)$  such that  $h_1 \in \bigcap_{i,j=1}^\infty T_{ij}$  and  $h_1$  is  $\delta_0$ -homotopic to  $g$ . Let  $h_2 : p^{-1}(L_0 \cup B) \rightarrow M$  be defined by  $h_2(x) = g_0(x)$  if  $x \in p^{-1}(B)$  and  $h_2(x) = h_1(x)$  if  $x \in p^{-1}(L_0)$ . Obviously,  $h_2$  is  $\delta_0$ -homotopic to  $g$ . Since  $L_0 \cup B$  is a subpolyhedron of  $L$  and  $p$  is a simplicial map,  $p^{-1}(L_0 \cup B)$  is a subpolyhedron of  $N$ . Then, by the Homotopy Extension Theorem, there exists a map  $g_q \in C(N, M)$  extending  $h_2$  with  $g_q$  being  $\delta_0$ -homotopic to  $g$ .

It remains only to show that  $g_q \in \mathcal{H}_p(y, 1/q, 1/q)$  for all  $y \in L$ . This is true if  $y \in V$ . Indeed, then  $y$  belongs to some  $V_{y(i)}$ . Since  $g_q|_{V_{y(i)}}$  is  $\delta_0$ -close to  $g|_{V_{y(i)}}$  and  $\delta_0 \leq \delta_{y(i)}$ ,  $g_q \in \mathcal{H}_p(y, 1/q, 1/q)$  according to the choice of  $V_{y(i)}$  and  $\delta_{y(i)}$ . If  $y \in L_0$ , then  $g_q|_{p^{-1}(y)} = h_1|_{p^{-1}(y)}$ . Since  $h_1$  is  $\delta_0$ -close to  $g$  and  $\delta_0 \leq \varepsilon_0$ ,  $h_1^{-1}(\bar{G}_{v(j)}) \subset U_j^1 \subset N_j$  for every  $j \leq k$ . So,  $p^{-1}(y) \cap g_q^{-1}(\bar{G}_{v(j)}) = p^{-1}(y) \cap h_1^{-1}(\bar{G}_{v(j)}) = p_j^{-1}(y) \cap g_q^{-1}(\bar{G}_{v(j)})$ ,  $j \leq k$ . On the other hand,  $h_1 \in \bigcap_{i,j=1}^\infty T_{ij}$  implies that every restriction  $h_1|_{h_1^{-1}(\bar{G}_{v(j)})} : h_1^{-1}(\bar{G}_{v(j)}) \rightarrow \bar{G}_{v(j)}$  has the following property: the union of all non-trivial components of the fibers of  $h_1|_{h_1^{-1}(\bar{G}_{v(j)})}$  is contained in  $N_j \setminus Z_j$ . Hence, the union of all non-trivial components of the fibers of  $g_q|_{p^{-1}(y) \cap g_q^{-1}(\bar{G}_{v(j)})}$  is contained in  $p_j^{-1}(y) \setminus Z_j$ . Since  $\dim p_j^{-1}(y) \setminus Z_j \leq m - n_{v(j)}$ , every  $g_q|_{p^{-1}(y) \cap g_q^{-1}(\bar{G}_{v(j)})}$  is an  $(m - n_{v(j)})$ -dimensional Lelek map. But  $g_q(N)$  doesn't meet any  $\bar{G}_v$  except for  $v \in \{v(j) : j = 1, \dots, k\}$ . Therefore,  $g_q \in \mathcal{H}_p(y, 1/i, 1/i)$  for every  $i \geq 1$ .  $\square$

**Lemma 3.6.** Let  $B \subset L$  be a subcomplex of  $L$  and  $g_0 \in C(N, M)$  be a map such that  $g_0 \in \mathcal{H}_p(B) = \bigcap_{i \geq 1} \mathcal{H}_p(B, 1/i, 1/i)$ . Then, for every  $\delta > 0$  there exists  $\bar{g}_0 \in \mathcal{H}_p(L)$  which is  $\delta$ -homotopic to  $g_0$  and  $\bar{g}_0|_{p^{-1}(B)} = g_0|_{p^{-1}(B)}$ .

**Proof.** Each set  $\mathcal{H}_p(L, 1/i, 1/i)$  is open in  $C(N, M)$  according to Proposition 3.3. So, by Lemma 3.5, we can apply Lemma 2.2 (with  $U(i)$  being  $\mathcal{H}_p(L, 1/i, 1/i)$  and  $A = p^{-1}(B)$ ) to find a map  $\bar{g}_0 \in \bigcap_{i \geq 1} \mathcal{H}_p(L, 1/i, 1/i)$  which is  $\delta$ -homotopic to  $g_0$  and  $\bar{g}_0|_{p^{-1}(B)} = g_0|_{p^{-1}(B)}$ . Finally,  $\bar{g}_0 \in \mathcal{H}_p(L)$  because  $\mathcal{H}_p(L) = \bigcap_{i \geq 1} \mathcal{H}_p(L, 1/i, 1/i)$ .  $\square$

Next proposition completes the proof of Theorem 3.1. We suppose that  $p : N \rightarrow L$  is a perfect  $m$ -dimensional PL-map between simplicial complexes,  $(M, \rho)$  a complete metric space and  $\mu = \{W_\nu : \nu \in \Lambda\}$ ,  $\mu_1 = \{G_\nu : \nu \in \Lambda\}$  are locally finite open covers of  $M$  with  $\bar{G}_\nu \subset W_\nu$  and  $W_\nu \in AP(n_\nu, 0)$  for every  $\nu \in \Lambda$ , where  $n_\nu \leq m$  are integers. If  $B \subset A \subset L$ , denote by  $\mathcal{H}_p(A, B)$  the set of the maps  $g \in C(p^{-1}(A), M)$  such that  $g|_{g^{-1}(\bar{G}_\nu) \cap p^{-1}(y)} : g^{-1}(\bar{G}_\nu) \cap p^{-1}(y) \rightarrow \bar{G}_\nu$  is  $(m - n_\nu)$ -dimensional Lelek map for every  $\nu \in \Lambda$  and  $y \in B$ . When  $A = B$ , we write  $\mathcal{H}_p(B)$  instead of  $\mathcal{H}_p(A, B)$ .

**Proposition 3.7.** Let  $p : N \rightarrow L$  and  $(M, \rho)$  be as above. Then the set  $\mathcal{H}_p(L)$  is homotopically dense in  $C(N, M)$ .

**Proof.** As usual, the simplicial complexes  $N$  and  $L$  are equipped with the CW-topology. But when consider a diameter of any subset of  $N$  we mean the diameter with respect to the standard metric generating the metric topology of  $N$ . According

to the notations in this section, for every sets  $B \subset A \subset L$  and  $\varepsilon, \eta > 0$ , let  $\mathcal{H}_p(A, B, \varepsilon, \eta)$  be the set of all  $g \in C(p^{-1}(A), M)$  such that  $d_{m-n_\nu}(F_\nu(g, \varepsilon, y)) < \eta$  for all  $y \in B$  and  $\nu \in \Lambda$ . Although the domain of  $g$  is the set  $A$  (not the whole space  $N$ ), we use the same notation  $F_\nu(g, \varepsilon, y)$  to denote the union of all continua  $C \subset p^{-1}(y) \cap g^{-1}(z_C)$  with  $z_C \in \overline{G}_\nu$  and  $\text{diam } C \geq \varepsilon$ . It follows from the definition of  $\mathcal{H}_p(A, B)$  that this is the set of the maps  $g \in C(p^{-1}(A), M)$  such that  $\dim F_\nu(g, y) \leq m - n_\nu$  for any  $y \in B$  and  $\nu \in \Lambda$ , where  $F_\nu(g, y) = \bigcup_{i=1}^\infty F_\nu(g, 1/i, y)$ . It is easily seen that  $\mathcal{H}_p(A, B) = \bigcap_{i \geq 1} \mathcal{H}_p(A, B, 1/i, 1/i)$ .

Now, let us finish the proof of Proposition 3.7. Fix  $g \in C(N, M)$  and  $\delta \in C(N, (0, 1])$ . We are going to find  $h \in \mathcal{H}_p(L)$  which is  $\delta$ -homotopic to  $g$ . To this end, let  $L^{(i)}, i \geq 0$ , denote the  $i$ -dimensional skeleton of  $L$  and  $L^{(-1)} = \emptyset$ . We put  $h_{-1} = g$  and construct inductively a sequence  $(h_i : N \rightarrow M)_{i \geq 0}$  of maps such that

- $h_i|_{p^{-1}(L^{(i-1)})} = h_{i-1}|_{p^{-1}(L^{(i-1)})}$ ;
- $h_i$  is  $\frac{\delta}{2^{i+2}}$ -homotopic to  $h_{i-1}$ ;
- $h_i \in \mathcal{H}_p(L, L^{(i)})$ .

Assuming that the map  $h_{i-1} : N \rightarrow M$  has been constructed, consider the complement  $L^{(i)} \setminus L^{(i-1)} = \bigsqcup_{j \in J_i} \overset{\circ}{\sigma}_j$ , which is the discrete union of open  $i$ -dimensional simplexes. Since  $h_{i-1}|_{\sigma_j}$  belongs to  $\mathcal{H}_p(\sigma_j, \sigma_j^{(i-1)})$  for any simplex  $\sigma_j \in L^{(i)}$ , we can apply Lemma 3.6 to find a map  $g_j : p^{-1}(\sigma_j) \rightarrow M$  such that

- $g_j$  coincides with  $h_{i-1}$  on the set  $p^{-1}(\sigma_j^{(i-1)})$ ;
- $g_j$  is  $\frac{\delta}{2^{i+2}}$ -homotopic to  $h_{i-1}$ ;
- $g_j \in \mathcal{H}_p(\sigma_j, \sigma_j)$ .

Next, define a map  $q_i : p^{-1}(L^{(i)}) \rightarrow M$  by the formula

$$q_i(x) = \begin{cases} h_{i-1}(x) & \text{if } x \in p^{-1}(L^{(i-1)}); \\ g_j(x) & \text{if } x \in p^{-1}(\sigma_j). \end{cases}$$

It can be shown that  $q_i$  is  $\frac{\delta}{2^{i+2}}$ -homotopic to  $h_{i-1}|_{p^{-1}(L^{(i)})}$ . Since  $p^{-1}(L^{(i)})$  is a subpolyhedron of  $N$ , we can apply the Homotopy Extension Theorem to find a continuous extension  $h_i : N \rightarrow M$  of the map  $q_i$  which is  $\frac{\delta}{2^{i+2}}$ -homotopic to  $h_{i-1}$ . Moreover,  $h_i \in \mathcal{H}_p(L, L^{(i)})$  because  $h_{i-1} \in \mathcal{H}_p(L, L^{(i-1)})$  and  $g_j \in \mathcal{H}_p(\sigma_j, \sigma_j)$  for any  $j$ . This completes the inductive step.

Then the limit map  $h = \lim_{i \rightarrow \infty} h_i : N \rightarrow M$  is well defined, continuous and  $\delta$ -homotopic to  $g$  (the last two properties of  $h$  hold because  $h$  has this properties for any simplex from  $N$  and because of the definition of the CW-topology on  $N$ ). Finally, since  $h|_{p^{-1}(L^{(i)})} = h_i|_{p^{-1}(L^{(i)})}$  and  $h_i \in \mathcal{H}_p(L, L^{(i)})$  for every  $i$ ,  $h \in \mathcal{H}_p(L)$ .  $\square$

#### 4. Almost AE(n, 0)-spaces

We already observed that if  $M$  is an  $LC^n$ -space, then  $M \in AP(n, 0)$  if and only if  $M$  has the following property:

- for every map  $g \in C(\mathbb{I}^n, M)$  and every  $\varepsilon > 0$  there exists a 0-dimensional map  $g' \in C(\mathbb{I}^n, M)$  which is  $\varepsilon$ -close to  $g$ .

Any space having the above property will be referred as *almost AP(n, 0)*. Obviously, every  $LC^{n-1}$  almost  $AP(n, 0)$ -space has the  $AP(n-1, 0)$ -property.

We are going to establish an analogue of Theorem 3.1 for almost  $AP(n, 0)$ -spaces.

**Lemma 4.1.** *Every complete  $LC^{n-1}$ -space  $M$  admits a complete metric  $\rho$  generating its topology and satisfying the following condition: If  $Z$  is an  $n$ -dimensional space,  $A \subset Z$  its closed set and  $h : Z \rightarrow M$ , then for every function  $\alpha : Z \rightarrow (0, 1]$  and every map  $g : A \rightarrow M$  with  $\rho(g(z), h(z)) < \alpha(z)/8$  for all  $z \in A$  there exists a map  $\bar{g} : Z \rightarrow M$  extending  $g$  such that  $\rho(\bar{g}(z), h(z)) < \alpha(z)$  for all  $z \in Z$ .*

**Proof.** We embed  $M$  in a Banach space  $E$  as a closed subset. Since the Hilbert cube is the image of the  $n$ -dimensional Menger compactum under an  $n$ -invertible map [7], we can find a metric space  $E(n)$  with  $\dim E(n) \leq n$  and a perfect  $n$ -invertible surjection  $p : E(n) \rightarrow E$ . Here,  $p$  is  $n$ -invertible means that every map from at most  $n$ -dimensional space into  $E$  can be lifted to a map into  $E(n)$ .

Since  $M \in LC^{n-1}$ , there exist a neighborhood  $W$  of  $M$  in  $E$  and a map  $q : p^{-1}(W) \rightarrow M$  extending the restriction  $p|_{p^{-1}(M)}$ . For every open  $U \subset M$  let  $T(U) = W \setminus p(q^{-1}(M \setminus U))$ . Obviously,  $T(U) \subset \overline{W}$  is open,  $T(U) \cap M = U$  and  $q(p^{-1}(T(U))) = U$ . Let  $\mathcal{T}$  be the collection of all pairs  $(U, V)$  of open sets in  $M$  such that  $\overline{\text{conv}(V)} \subset T(U)$ , where  $\overline{\text{conv}(V)}$  is the closed convex hull of  $V$  in  $E$ . Now, consider the family

$$\mathcal{T} = \{(U, V) : U, V \text{ are open in } M \text{ and } \overline{\text{conv}(V)} \subset T(U)\}.$$

The family  $\mathcal{T}$  has the following properties: (i) for any  $z \in M$  and its neighborhood  $U$  in  $M$  there is a neighborhood  $V \subset U$  of  $z$  with  $(U, V) \in \mathcal{T}$ ; (ii) for any  $(U, V) \in \mathcal{T}$  and open sets  $U', V' \subset M$  we have  $(U', V') \in \mathcal{T}$  provided  $U \subset U'$  and  $V' \subset V$ .

By [6, Proposition 2.3], there exists a complete metric  $\rho$  on  $M$  such that for every  $z \in M$  and  $r \in (0, 1)$  the pair of open balls  $(B_\rho(z, r), B_\rho(z, r/8))$  belongs to  $\mathcal{T}$ .

Suppose we are given an  $n$ -dimensional space  $Z$ , its closed subset  $A \subset Z$  and two maps  $h : Z \rightarrow M$  and  $g : A \rightarrow M$  such that  $\rho(g(z), h(z)) < \alpha(z)/8$  for all  $z \in A$ , where  $\alpha \in C(Z, (0, 1])$ . Consider the set-valued map  $\phi : Z \rightarrow E$ ,  $\phi(z) = g(z)$  if  $z \in A$  and  $\phi(z) = \text{conv}(B_\rho(h(z), \alpha(z)/8))$  if  $z \notin A$ . Then  $\phi$  is lower semi-continuous and has closed and convex values in  $E$ . So, by the Michael convex-valued selection theorem [20],  $\phi$  has a continuous selection  $g_1$ . Next, we lift  $g_1$  to a map  $g_2 : Z \rightarrow E(n)$ . According to the definition of  $\mathcal{T}$ , every  $p^{-1}(\text{conv}(B_\rho(h(z), \alpha(z)/8)))$  is contained in  $q^{-1}(B_\rho(h(z), \alpha(z)))$ . Hence,  $\bar{g} = q \circ g_2$  is the required extension of  $g$ .  $\square$

We also need the following lemma whose proof is similar to that one of Lemma 2.2.

**Lemma 4.2.** *Let  $G \subset C(X, M)$ , where  $(M, \rho)$  is a complete metric space. Suppose  $(U(i))_{i \geq 1}$  is a sequence of open subsets of  $C(X, M)$  such that*

- for any  $g \in G$ ,  $i \geq 1$  and any function  $\eta \in C(X, (0, 1])$  there exists  $g_i \in B_\rho(g, \eta) \cap U(i) \cap G$  which is  $\eta$ -close to  $g$ .

Then, for any  $g \in G$  and  $\varepsilon : X \rightarrow (0, 1]$  there exists  $g' \in \bigcap_{i=1}^\infty U(i)$  which is  $\varepsilon$ -close to  $g$ . If, in addition, all  $g_i$  can be chosen such that  $g_i|_A = g_0|_A$  for some  $g_0 \in C(X, M)$  and  $A \subset X$ , then  $g'|_A = g_0|_A$ .

Next two propositions are analogues of Propositions 2.3 and 2.4.

**Proposition 4.3.** *Let  $M$  be a complete  $\text{LC}^{n-1}$  almost  $AP(n, 0)$ -space. Then for every  $n$ -dimensional compactum  $X$  there exists a dense  $G_\delta$ -subset  $\mathcal{H} \subset C(X, M)$  of 0-dimensional maps.*

**Proof.** First of all, let us note that since  $\dim X \leq n$  and  $M$  is  $\text{LC}^{n-1}$ , the set of all simplicially factorizable maps from  $C(X, M)$  is dense in  $C(X, M)$ . Analyzing the proof of Proposition 2.3 and using Lemma 4.2 instead of Lemma 2.2, one can see that it suffices to establish the following claim:

**Claim.** *If  $K$  is a finite  $n$ -dimensional polyhedron, then every map  $g : K \rightarrow M$  can be approximated by a 0-dimensional map  $g' : K \rightarrow M$ .*

Recall that  $C(K, M; \eta)$ , where  $\eta > 0$ , consists of all maps  $g \in C(K, M)$  with  $d_0(g^{-1}(g(x))) < \eta$  for every  $x \in K$ . Since any  $C(K, M; \eta)$  is open in  $C(K, M)$ , according to Lemma 4.2, it is enough to show that for every  $g \in C(K, M)$  and  $\eta > 0$  there exists  $g' \in C(K, M; \eta)$  which is  $\eta$ -close to  $g$ . To this end, we equipped  $M$  with a complete metric satisfying the hypotheses of Lemma 4.1 and fix  $0 < \eta \leq 1$  and  $g \in C(K, M)$ . Since  $M \in \text{LC}^{n-1}$  and  $M$  is almost  $AP(n, 0)$ ,  $M \in AP(n - 1, 0)$ . So, there exists a 0-dimensional map  $h : K^{(n-1)} \rightarrow M$  which is  $\eta/16$ -close to  $g|_{K^{(n-1)}}$ . Here,  $K^{(n-1)}$  is the  $(n - 1)$ -dimensional skeleton of  $K$ . By Lemma 4.1,  $h$  can be extended to a map  $\bar{h} \in C(K, M)$  with  $\rho(\bar{h}(x), g(x)) < \eta/2$  for all  $x \in K$ . Let  $\sigma_j$ ,  $j = 1, \dots, k$ , be all  $n$ -dimensional simplexes of  $K$ . Then  $K \setminus K^{(n-1)}$  is a disjoint union of the open simplexes  $\overset{\circ}{\sigma}_j$ . Let  $K_i = K^{(n-1)} \bigcup_{j=1}^i \sigma_j$ ,  $i = 1, \dots, k$ . We are going to construct by induction maps  $h_i : K \rightarrow M$ ,  $i = 0, \dots, k$ , satisfying the following conditions:

- $h_0 = \bar{h}$ ;
- $h_i|_{K_i}$  belongs to  $C(K_i, M; \eta)$ ,  $1 \leq i \leq k$ ;
- $h_i$  and  $h_{i+1}$  are  $(\eta/2k)$ -close,  $i = 0, \dots, k - 1$ .

Assume that  $h_i$  has already been constructed. Since  $h_i|_{K_i}$  is in  $C(K_i, M; \eta)$ , every fiber  $h_i^{-1}(y) \cap K_i$  of  $h_i|_{K_i}$ ,  $y \in h_i(K_i)$ , is covered by a finite open and disjoint family  $\Omega(y)$  in  $K_i$  with  $\text{mesh}(\Omega(y)) < \eta$ . Using that  $h_i(K_i)$  is compact, we find  $0 < \delta_i < \eta/2k$  such that if  $\rho(y, h_i(K_i)) < \delta_i$  for some  $y \in M$ , then there is  $z \in h_i(K_i)$  with  $h_i^{-1}(B_\rho(y, \delta_i)) \cap K_i \subset \Omega(z)$ . Since  $M$  is almost  $AP(n, 0)$ , there exists a 0-dimensional map  $p_i : \sigma_{i+1} \rightarrow M$  which is  $\delta_i/8$ -close to  $h_i|_{\sigma_{i+1}}$ . By Lemma 4.1,  $p_i$  can be extended to a map  $h_{i+1} : K \rightarrow M$  being  $\delta_i$ -close to  $h_i$ . To show that  $h_{i+1}|_{K_{i+1}}$  belongs to  $C(K_{i+1}, M; \eta)$ , we observe that  $h_{i+1}^{-1}(y) \cap K_{i+1} = (h_{i+1}^{-1}(y) \cap K_i) \cup (h_{i+1}^{-1}(y) \cap \sigma_{i+1})$ ,  $y \in M$ . According to our construction, we have  $h_i(h_{i+1}^{-1}(y) \cap K_i) \subset B_\rho(y, \delta_i) \cap h_i(K_i)$ . Hence,  $h_{i+1}^{-1}(y) \cap K_i \subset h_i^{-1}(B_\rho(y, \delta_i)) \cap K_i \subset \Omega(z)$  for some  $z \in h_i(K_i)$ . Therefore,  $d_0(h_{i+1}^{-1}(y) \cap K_i) < \eta$ . Since  $h_{i+1}^{-1}(y) \cap \sigma_{i+1}$  is 0-dimensional, Lemma 2.1 implies that  $d_0(h_{i+1}^{-1}(y) \cap K_{i+1}) < \eta$ . Obviously  $h_k \in C(K, M; \eta)$  and  $h_k$  is  $\eta/2$ -close to  $\bar{h}$ . Hence,  $g' = h_k$  is  $\eta$ -close to  $g$ . This completes the proof of the claim.  $\square$

**Proposition 4.4.** *Let  $M$  be a complete  $\text{LC}^{n-1}$  almost  $AP(n, 0)$ -space. Then for every  $n$ -dimensional compactum  $X$  and its  $F_\sigma$ -subset  $Z$  with  $\dim Z \leq n - 1$  there exists a dense  $G_\delta$ -subset  $\mathcal{H} \subset C(X, M)$  of maps  $g$  such that  $Z$  is contained in the union of trivial components of the fibers of  $g$ .*

**Proof.** Following the proof of Proposition 2.4 and using Lemma 4.2 and Proposition 4.3 instead of Lemma 2.2 and Proposition 2.3, respectively, it suffices to prove the following analogue of Claim 1 from Proposition 2.4.

**Claim.** Suppose  $X$  is a finite  $n$ -dimensional polyhedron and  $Z$  a subpolyhedron of  $X$  with  $\dim Z \leq n - 1$ . Then for every  $g_0 \in C(X, M)$  and  $\delta > 0$  there exists  $g \in \mathcal{H} = \bigcap_{i=1}^{\infty} \mathcal{H}(Z, 1/i)$  which is  $\delta$ -close to  $g_0$ .

Here,  $\mathcal{H}(Z, 1/i)$  is the set of all  $g \in C(X, M)$  with  $F(g, 1/i) \cap Z = \emptyset$ . We follow the proof of Claim 1, Proposition 2.4 and use the same notations. The first difference is that we take  $W$  to be a neighborhood of  $Z$  in  $X$  such that  $g_0|W$  and  $g_0 \circ r$  are  $\delta/16$ -close. Then,  $g_0 \circ \pi \circ \varphi$  is  $\delta/16$ -close to  $g_0|W$ . Next, we use by Proposition 4.3 to choose a 0-dimensional map  $h : Z \times \mathbb{I} \rightarrow M$  which is  $\delta/16$ -close to  $g_0 \circ \pi$ . So,  $h \circ \varphi$  is  $\delta/8$ -close to  $g_0|W$ . Finally, take a neighborhood  $U$  of  $Z$  in  $X$  with  $\bar{U} \subset W$ , and use Lemma 4.1 to find an extension  $g \in C(X, M)$  of  $(h \circ \varphi)|\bar{U}$  with  $g$  being  $\delta$ -close to  $g_0$ . Then  $g \in \mathcal{H}$ .  $\square$

We establish now an analogue of Theorem 3.1 for almost  $AP(n, 0)$ -spaces.

**Theorem 4.5.** Let  $f : X \rightarrow Y$  be a perfect map between metrizable spaces with  $\dim X \leq n$  and  $M$  is a complete  $LC^{n-1}$ -space. Suppose  $\mu = \{W_\nu : \nu \in \Lambda\}$  and  $\mu_1 = \{G_\nu : \nu \in \Lambda\}$  are locally finite open covers of  $M$  such that  $\bar{G}_\nu \subset W_\nu$  and each  $W_\nu$  is almost  $AP(m_\nu, 0)$  for every  $\nu \in \Lambda$ . Then there is a dense  $G_\delta$ -set  $\mathcal{H} \subset C(X, M)$  of maps  $g$  such that for any  $y \in Y$  and  $\nu \in \Lambda$  the restriction  $g|(f^{-1}(y) \cap g^{-1}(\bar{G}_\nu))$  is an  $(n - m_\nu)$ -dimensional Lelek map.

**Proof.** Following the notations and the proof of Theorem 3.1, we can assume that  $m_\nu \leq n$  for all  $\nu$ . Let  $\mathcal{H}(y, \varepsilon, \eta)$  denote the set of all  $g \in C(X, M)$  such that  $d_{n-m_\nu}(F_\nu(g, \varepsilon, y)) < \eta$  for every  $\nu \in \Lambda$ , where  $y \in Y$  and  $\varepsilon, \eta > 0$  are fixed. Moreover, for any  $F \subset Y$ , let  $\mathcal{H}(F, \varepsilon, \eta)$  be the intersection of all  $\mathcal{H}(y, \varepsilon, \eta)$ ,  $y \in F$ . One can establish a lemma analogical to Lemma 3.2. Then, as in Proposition 3.3, we can show that any  $\mathcal{H}(F, \varepsilon, \eta)$  is open in  $C(X, M)$ , where  $F \subset Y$  is closed.

Observe that  $\dim_\Delta(f) \leq m$  because  $X$ , as a space of dimension  $\leq n$ , admits a uniformly 0-dimensional map into  $\mathbb{I}^n$ , see [14]. Moreover, we can assume that the simplicial complex  $K$  in Proposition 3.4 is  $n$ -dimensional. So, we can apply Proposition 3.4 in the present situation to show that any  $\mathcal{H}(Y, \varepsilon, \eta)$  is dense in  $C(X, M)$  provided for any perfect  $PL$  map  $p : N \rightarrow L$  with  $\dim N \leq n$  the set  $\mathcal{H}_p(L)$  is dense in  $C(N, M)$ . Here,  $\mathcal{H}_p(L)$  is the set of all maps  $q : N \rightarrow M$  such that  $q|q^{-1}(\bar{G}_\nu) \cap p^{-1}(z)$  is an  $(n - m_\nu)$ -dimensional Lelek map for every  $\nu \in \Lambda$  and  $z \in L$ . Hence, it remains to show that  $\mathcal{H}_p(L)$  is dense in  $C(N, M)$  for any  $PL$ -map  $p : N \rightarrow L$  with  $\dim N \leq n$ . And this follows from the proof of Lemmas 3.5, 3.6 and Proposition 3.7 with the only difference that now we replace the application of Lemma 2.2, Propositions 2.3 and 2.4 by Lemma 4.2, Propositions 4.3 and 4.4, respectively.  $\square$

**Corollary 4.6.** Let  $M$  be a complete  $LC^{n-1}$ -space such that each point  $z \in M$  has a neighborhood which is almost  $AP(n, 0)$ . Then, for every perfect map  $f : X \rightarrow Y$  between metric spaces with  $\dim X \leq n$ , there is a dense  $G_\delta$ -set  $\mathcal{H} \subset C(X, M)$  consisting of maps  $g$  such that every restriction  $g|f^{-1}(y)$ ,  $y \in Y$ , is a 0-dimensional map.

Any manifold  $M$  modeled on the  $n$ -dimensional Menger cube has the  $AP(n, 0)$ -property, see Corollary 6.5 below. So, Theorem 1.1 holds for such a space  $M$ . But Theorem 1.1 does not provide any information about the density of the set  $\mathcal{H}$  in  $C(X, M)$  except that every simplicially factorizable map in  $C(X, M)$  can be approximated by maps from  $\mathcal{H}$ . Next proposition shows that, in this special case, the set  $\mathcal{H}$  is dense in  $C(X, M)$  with respect to the uniform convergence topology.

**Proposition 4.7.** Let  $f : X \rightarrow Y$  be a perfect map between metrizable spaces with  $\dim_\Delta(f) \leq m$  and  $M$  be a manifold modeled on the  $n$ -dimensional Menger cube. Then there is a  $G_\delta$ -set  $\mathcal{H} \subset C(X, M)$  dense in  $C(X, M)$  with respect to the uniform convergence topology such that for any  $g \in \mathcal{H}$  and  $y \in Y$  the restriction  $g|f^{-1}(y)$  is an  $(m - n)$ -dimensional Lelek map.

**Proof.** Let  $\mathcal{H}$  be the set from the proof of Theorem 3.1. To show that  $\mathcal{H}$  is dense in  $C(X, M)$  equipped with the uniform convergence topology, we used an idea from the proof of [13, Corollary 2.8]. According to [4], for any  $\varepsilon > 0$  there exists an  $n$ -dimensional polyhedron  $P \subset M$  of piecewise embedding dimension  $n$  and maps  $u : M \rightarrow P$  and  $v : P \rightarrow M$  such that  $u$  is a retraction,  $v$  is 0-dimensional,  $v \circ u$  is  $\varepsilon/2$ -close to the identity  $id_M$ . Since every ANR of piecewise embedding dimension  $n$  has the  $AP(n, 0)$ -property (see [13, Proposition 2.1]), according to Theorem 3.1, for every  $g \in C(X, M)$  there is  $g' : X \rightarrow P$  such that  $g'$  is  $\delta$ -close to  $u \circ g$  and  $g'|f^{-1}(y)$  is an  $(m - n)$ -dimensional Lelek map for all  $y \in Y$ . Here  $\delta > 0$  is chosen such that  $dist(v(x), v(y)) < \varepsilon/2$  for any  $x, y \in P$  which are  $\delta$ -close. Then  $v \circ g'$  is  $\varepsilon$ -close to  $g$  and since  $v$  is 0-dimensional,  $v \circ g' \in \mathcal{H}$ .  $\square$

### 5. Properties of $AP(n, 0)$ -spaces

In this section we investigate the class of  $AP(n, 0)$ -spaces.

**Lemma 5.1.** *Let  $K$  be a polyhedron of dimension  $\leq n$  and  $L \subset K$  a subpolyhedron. Suppose  $(X, \rho)$  is a complete metric space possessing the  $AP(n, 0)$ -property and  $g_0 \in C(K, X)$  with  $\dim g_0^{-1}(g_0(x)) \leq 0$  for all  $x \in L$ . Then for every  $\delta > 0$  there exists a 0-dimensional map  $g : K \rightarrow X$  which is  $\delta$ -homotopic to  $g_0$  and  $g|L = g_0|L$ .*

**Proof.** We already observed in Section 2 that all sets  $C(K, X; \varepsilon)$ ,  $\varepsilon > 0$ , consisting of maps  $h \in C(K, X)$  with  $d_0(h^{-1}(h(x))) < \varepsilon$  for every  $x \in K$  are open in  $C(K, X)$ . Since every map from  $C(K, X)$  is simplicially factorizable,  $C(K, X; \varepsilon)$  are homotopically dense in  $C(K, X)$  according to Proposition 2.3.

**Claim 1.** *For every  $x \in L$  and  $j \geq 1$  there exist  $\varepsilon_x > 0$  and a neighborhood  $U_x$  of  $x$  in  $K$  satisfying the following condition: If  $h \in C(K, X)$  and  $Z \subset K$  with  $\rho(g_0(y), h(y)) < \varepsilon_x$  for all  $y \in Z \cup \bar{U}_x$ , then  $d_0(h^{-1}(h(y)) \cap Z) < 1/j$  for any  $y \in \bar{U}_x$ .*

The proof of this claim is similar to the proof of Lemma 3.2.

**Claim 2.** *Let  $h \in C(K, X)$  with  $h|L = g_0|L$ . Then, for every  $\eta > 0$  and  $j \geq 1$  there exists  $h_j \in C(K, X; 1/j)$  such that  $h_j|L = g_0|L$  and  $h_j$  is  $\eta$ -homotopic to  $h$ .*

We fix  $j$  and  $\eta > 0$ . Choose finitely many points  $x(i) \in L$ ,  $i \leq k$ , positive reals  $\varepsilon_{x(i)}$  and neighborhoods  $U(x_i)$  in  $K$  satisfying the hypotheses of Claim 1 such that  $L \subset U = \bigcup_{i \leq k} U(x_i)$ . Taking a smaller neighborhood, if necessarily, we can suppose that  $\rho(g_0(x), h(x)) < \eta_1/2$  for all  $x \in \bar{U}$ , where  $\eta_1 = \min\{\eta, \varepsilon_{x(i)} : i \leq k\}$ . Consider a triangulation  $T$  of  $K$  such that  $\sigma \in T$  and  $\sigma \cap K \setminus U \neq \emptyset$  imply  $\sigma \cap L = \emptyset$ . Now, let  $N$  be the subpolyhedron of  $K$  given by  $N = \bigcup\{\sigma \in T : \sigma \cap K \setminus U \neq \emptyset\}$ . Obviously,  $N$  and  $L$  are disjoint. Since  $X \in AP(n, 0)$  and  $\dim N \leq n$ , by Proposition 2.3, there exists a 0-dimensional map  $g_N \in C(N, X)$  which is  $\eta_1/2$ -homotopic to  $h|N$  (we can apply Proposition 2.3 because  $h|N$  is simplicially factorizable as a map with a polyhedral domain). The map  $h' : N \cup L \rightarrow X$ ,  $h'|N = g_N$  and  $h'|L = g_0|L$ , is  $\eta_1/2$ -homotopic to  $h|(N \cup L)$ . So, by the Homotopy Extension Theorem,  $h'$  admits an extension  $h_j : K \rightarrow X$  which is  $\eta_1/2$ -homotopic to  $h$ . It remains only to show that  $d_0(h_j^{-1}(h_j(x))) < 1/j$  for any  $x \in K$ . To this end, observe that  $\rho(g_0(x), h_j(x)) < \eta_1$  for all  $x \in \bar{U}$ . Because  $K = N \cup \bar{U}$ , for every  $x \in K$  we have  $h_j^{-1}(h_j(x)) = (h_j^{-1}(h_j(x)) \cap N) \cup (h_j^{-1}(h_j(x)) \cap \bar{U})$ . Since  $h_j|N = g_N$  and  $g_N$  is 0-dimensional,  $\dim h_j^{-1}(h_j(x)) \cap N \leq 0$ . On the other hand,  $h_j^{-1}(h_j(x)) \cap \bar{U} = h_j^{-1}(h_j(y)) \cap \bar{U}$  for some  $y \in \bar{U}$ . So, there exists  $m \leq k$  with  $y \in \bar{U}_{x(m)}$ . Since  $h_j| \bar{U}$  is  $\eta_1$ -close to  $g_0| \bar{U}$ ,  $\rho(g_0(z), h_j(z)) < \varepsilon_{x(m)}$  for all  $z \in \bar{U}$ . Hence, according to the choice of  $U_{x(m)}$  and  $\varepsilon_{x(m)}$ ,  $d_0(h_j^{-1}(h_j(y)) \cap \bar{U}) < 1/j$ . Therefore,  $d_0(h_j^{-1}(h_j(x)) \cap \bar{U}) < 1/j$ . Finally, by Lemma 2.1,  $d_0(h_j^{-1}(h_j(x))) < 1/j$ . This completes the proof of Claim 2.

We are in a position to complete the proof of Lemma 5.1. Because of Claim 2, we can apply Lemma 2.2 (with  $G$  being in our case the set  $\{h \in C(K, X) : h|L = g_0|L\}$  and  $U(j) = C(K, X; 1/j)$ ) to obtain a map  $g \in \bigcap_{j=1}^\infty C(K, X; 1/j)$  such that  $g|L = g_0|L$  and  $g$  is  $\delta$ -homotopic to  $g_0$ . Obviously,  $g \in \bigcap_{j=1}^\infty C(K, X; 1/j)$  yields  $\dim g \leq 0$ .  $\square$

**Proposition 5.2.** *For any space  $X$  we have:*

- (1) *If  $X$  has the  $AP(n, 0)$ -property, then every open subset of  $X$  also has this property.*
- (2) *If  $X$  is completely metrizable, then  $X$  has the  $AP(n, 0)$ -property if and only if it admits a cover by open subsets with that property.*

**Proof.** To prove the first item, suppose  $U$  is an open subset of  $X \in AP(n, 0)$ ,  $\mathcal{U} \in \text{cov}(U)$  and  $g \in C(\mathbb{I}^n, U)$ . Let  $\mathcal{U}' = \mathcal{U} \cup \{X \setminus g(\mathbb{I}^n)\}$ . Since  $X \in AP(n, 0)$ , there is a 0-dimensional map  $g' \in C(\mathbb{I}^n, X)$  such that  $g'$  is  $\mathcal{U}'$ -homotopic to  $g$ . Then  $g'(\mathbb{I}^n) \subset U$  and there exists a  $\mathcal{U}$ -homotopy  $h : \mathbb{I}^n \rightarrow U$  joining  $g$  and  $g'$ .

According to Michael's theorem on local properties [19], the second item will be established if we show that: (i) A space has the  $AP(n, 0)$ -property provided it is a discrete sum of spaces with the same property; (ii) A completely metrizable space has the  $AP(n, 0)$ -property provided it is a union of two open subspaces with this property. Condition (i) trivially follows from the definition. Let us check condition (ii).

Suppose  $X$  is a completely metrizable space and  $X = X_1 \cup X_2$  is the union of two open subspaces  $X_1, X_2 \subset X$  with the  $AP(n, 0)$ -property. Fix an open cover  $\mathcal{U}$  of  $X$  and a map  $g : \mathbb{I}^n \rightarrow X$  and choose two open sets  $W_1, W_2 \subset X$  such that  $X = W_1 \cup W_2$  and  $W_i \subset \bar{W}_i \subset X_i$  for  $i \in \{1, 2\}$ . Next, find a complete metric  $\rho$  on  $X$  such that

- $\rho(X \setminus W_1, X \setminus W_2) \geq 1$ ;
- $B_\rho(\bar{W}_i, 1) \subset X_i$  for  $i \in \{1, 2\}$ , and
- each set of diameter  $< 1$  in  $(X, \rho)$  lies in some  $U \in \mathcal{U}$ .

Let  $V = V_1 \cap V_2$ , where  $V_i = B_\rho(\bar{W}_i, 1/2)$ ,  $i = 1, 2$ , and choose a triangulation  $T$  of  $\mathbb{I}^n$  such that for any simplex  $\sigma \in T$  we have  $\text{diam } g(\sigma) < 1/4$ . Now, consider the polyhedra

$$K_i = \bigcup\{\sigma \in T : g(\sigma) \cap \bar{W}_i \neq \emptyset\}, \quad i \in \{1, 2\},$$

and

$$L_2 = \bigcup \{ \sigma \in T : g(\sigma) \cap \bar{V}_2 \neq \emptyset \}.$$

Obviously,  $g(K_i) \subset V_i \subset \bar{V}_i \subset X_i$ ,  $i = 1, 2$ . So,  $g(K_0) \subset V$ , where  $K_0 = K_1 \cap K_2$ . Moreover, we have

$$K_2 \subset g^{-1}(V_2) \subset \overline{g^{-1}(V_2)} \subset L_2 \subset g^{-1}(X_2)$$

and

$$K_0 \subset g^{-1}(V) \subset L_2.$$

Choose now positive  $\delta \leq \min\{\rho(g(L_2), X \setminus X_2), \rho(g(K_1), X \setminus X_1), 1/2\}$  such that

$$h^{-1}(h(K_0)) \subset g^{-1}(V)$$

for any  $h \in C(\mathbb{I}^n, X)$  which is  $\delta$ -close to  $g$ .

Since  $X_2 \in AP(n, 0)$  and  $L_2$  is a polyhedron of dimension  $\leq n$ , by Proposition 2.3, there exists a 0-dimensional map  $g_2 : L_2 \rightarrow X_2$  which is  $\delta$ -homotopic to  $g|_{L_2}$ . Next, by the Homotopy Extension Theorem,  $g_2$  can be extended to a map  $\bar{g}_2 \in C(\mathbb{I}^n, X)$   $\delta$ -homotopic to  $g$ . According to the choice of  $\delta$ ,  $\bar{g}_2(L_2) \subset X_2$ ,  $\bar{g}_2(K_1) \subset X_1$  and  $(\bar{g}_2)^{-1}(\bar{g}_2(K_0)) \subset g^{-1}(V) \subset L_2$ . Hence, the restriction  $g_0 = \bar{g}_2|_{K_1}$  is a map from  $K_1$  into  $X_1$  and, for every  $x \in K_0$ , we have  $g_0^{-1}(g_0(x)) \subset g_2^{-1}(g_2(x))$ . The last inclusion implies that  $\dim g_0^{-1}(g_0(x)) \leq 0$  for all  $x \in K_0$  because  $g_2$  is 0-dimensional. Since  $X_1 \in AP(n, 0)$ , we can apply Lemma 5.1 (for the polyhedra  $K_0 \subset K_1$  and the map  $g_0$ ) to obtain a 0-dimensional map  $g_1 \in C(K_1, X_1)$  such that  $g_1|_{K_0} = g_0$  and  $g_1$  is 1/2-homotopic to  $g_0$ . Finally, consider the map  $g_{12} \in C(\mathbb{I}^n, X)$  defined by  $g_{12}|_{K_1} = g_1$  and  $g_{12}|_{K_2} = g_2|_{K_2}$ . Since both  $g_1$  and  $g_2$  are 0-dimensional, so is  $g_{12}$ . Moreover,  $g_{12}$  is 1-homotopic to  $g$  because  $\delta \leq 1/2$ . So, according to the choice of  $\rho$ ,  $g_{12}$  is  $\mathcal{U}$ -homotopic to  $g$ .  $\square$

**Proposition 5.3.** *If  $X$  and  $Y$  are complete metric spaces such that  $X \in AP(n, 0)$  and  $Y \in AP(m, 0)$ , then  $X \times Y \in AP(n + m, 0)$ .*

**Proof.** Let  $\rho_X$  and  $\rho_Y$  be complete metrics on  $X$  and  $Y$ , respectively. We fix  $g \in C(\mathbb{I}^{n+m}, X \times Y)$  and consider the complete metric  $\rho = \max\{\rho_X, \rho_Y\}$  on  $X \times Y$ . It suffices to show that for every  $\varepsilon > 0$  there exists a 0-dimensional map  $h \in C(\mathbb{I}^{n+m}, X \times Y)$  which is  $\varepsilon$ -homotopic to  $g$ . Let  $g_X = \pi_X \circ g$  and  $g_Y = \pi_Y \circ g$ , where  $\pi_X$  and  $\pi_Y$  are the projections from  $X \times Y$  onto  $X$  and  $Y$ , respectively. We represent  $\mathbb{I}^{n+m}$  as the union  $\bigcup_{i=1}^{n+m+1} A_i$  with each  $A_i$  being a 0-dimensional  $G_\delta$ -subset of  $\mathbb{I}^{n+m}$ . Then  $Z_X = \mathbb{I}^{n+m} \setminus \bigcup_{i=n+1}^{n+m+1} A_i$  is an  $F_\sigma$ -subset of  $\mathbb{I}^{n+m}$  which is contained in  $\bigcup_{i=1}^n A_i$ . So,  $\dim Z_X \leq n - 1$ . Since  $X \in AE(n, 0)$  and every map from  $C(\mathbb{I}^{n+m}, X)$  is simplicially factorizable, Proposition 2.4 yields the existence of a map  $h_X \in C(\mathbb{I}^{n+m}, X)$  such that  $h_X$  is  $\varepsilon$ -homotopic to  $g_X$  and  $F(h_X) = \bigcup_{j=1}^\infty F(h_X, 1/j) \subset \mathbb{I}^{n+m} \setminus Z_X$ , where  $F(h_X, 1/j)$  is the union of all components  $C$  of fibers of  $h_X$  with  $\text{diam } C \geq 1/j$ . Obviously,  $F(h_X)$  is an  $F_\sigma$ -set in  $\mathbb{I}^{n+m}$  with  $F(h_X) \subset \bigcup_{i=n+1}^{n+m+1} A_i$ . Hence,  $Z_Y = F(h_X) \setminus A_{n+m+1}$  is also an  $F_\sigma$ -set in  $\mathbb{I}^{n+m}$  with  $\dim Z_Y \leq m - 1$ . Now, since  $Y \in AE(m, 0)$ , we may apply again Proposition 2.4 to obtain a map  $h_Y \in C(\mathbb{I}^{n+m}, Y)$  which is  $\varepsilon$ -homotopic to  $g_Y$  and  $F(h_Y) = \bigcup_{j=1}^\infty F(h_Y, 1/j) \subset \mathbb{I}^{n+m} \setminus Z_Y$ . Then the diagonal product  $h = h_X \Delta h_Y : \mathbb{I}^{n+m} \rightarrow X \times Y$  is  $\varepsilon$ -homotopic to  $g$ . It remains only to show that  $h$  is 0-dimensional. If  $C$  is a non-trivial component of a fiber of  $h$ , then  $C \subset F(h_X) \cap F(h_Y) \subset F(h_X) \setminus Z_Y \subset A_{m+n+1}$ . Since  $A_{m+n+1}$  is 0-dimensional,  $C$  should be a point. Therefore, all components of the fibers of  $h$  are trivial, i.e.  $\dim h = 0$ .  $\square$

Finally, we are going to show that every arc-wise connected  $AP(n, 0)$ -compactum is a continuum ( $V^n$ ) in the sense of P. Alexandroff. Recall that a compact metric space  $(M, \rho)$  is a ( $V^n$ )-continuum [1] if for any pair of disjoint closed subsets  $A$  and  $B$  of  $M$  both having non-empty interiors there exists  $\varepsilon > 0$  such that  $d_{n-2}(C) > \varepsilon$  for every partition  $C$  in  $M$  between  $A$  and  $B$ . It is easily seen that this is a topological property, i.e., it doesn't depend on the metric  $\rho$ . Obviously, every continuum ( $V^n$ ) is a Cantor  $n$ -manifold (a compactum which is not disconnected by any  $(n - 2)$ -dimensional closed subset). Moreover, any ( $V^n$ ) continuum has a stronger property [10]: it cannot be decomposed into a countable union of proper closed subsets  $F_i$  with  $\dim F_i \cap F_j \leq n - 2$ . The compacta with the last property are called strong Cantor  $n$ -manifolds [9].

**Proposition 5.4.** *Every path-connected compactum  $M \in AP(n, 0)$  is a continuum ( $V^n$ ).*

**Proof.** Let  $A$  and  $B$  be disjoint closed subsets of  $M$  with non-empty interior and  $\rho$  be a metric on  $M$ . Since  $M$  is path-connected, we can choose a map  $g : \mathbb{I}^n \rightarrow M$  such that  $g(\mathbb{I}^n) \cap \text{Int}(A) \neq \emptyset \neq g(\mathbb{I}^n) \cap \text{Int}(B)$ . Then there exists a 0-dimensional map  $g_1 : \mathbb{I}^n \rightarrow M$  which is so close to  $g$  that  $g_1(\mathbb{I}^n)$  meets both  $\text{Int}(A)$  and  $\text{Int}(B)$ . Thus,  $A_1 = g_1^{-1}(A)$  and  $B_1 = g_1^{-1}(B)$  are disjoint closed subsets of  $\mathbb{I}^n$  with non-empty interiors. Since  $\mathbb{I}^n$  is a continuum ( $V^n$ ) [1], there exists  $\varepsilon > 0$  such that  $d_{n-2}(C_1) > \varepsilon$  for every partition  $C_1 \subset \mathbb{I}^n$  between  $A_1$  and  $B_1$ . Because  $g_1$  is 0-dimensional, every  $y \in M$  has a neighborhood  $W_y$  such that  $g_1^{-1}(W_y)$  splits into a finite disjoint family of open subsets of  $\mathbb{I}^n$  each of diameter  $< \varepsilon$ . Let  $\delta$  be the Lebesgue number of the cover  $\{W_y : y \in M\}$ . Then  $d_{n-2}(C) \geq \delta$  for any partition  $C \subset M$  between  $A$  and  $B$ . Indeed, otherwise there would be a partition  $C$  and an open family  $\gamma_C$  in  $M$  of order  $\leq n + 1$  such that  $\gamma_C$  covers  $C$  and  $\text{diam}(W) < \delta$  for

every  $W \in \gamma_C$ . Hence,  $g_1^{-1}(\gamma_C)$  is an open family in  $\mathbb{I}^n$  of order  $\leq n + 1$  and covering  $g_1^{-1}(C)$ . Moreover, each  $g_1^{-1}(W)$ ,  $W \in \gamma_C$ , splits into a finite disjoint open family consisting of sets with diameter  $< \varepsilon$ . Therefore,  $d_{n-2}(g_1^{-1}(C)) < \varepsilon$ . This is a contradiction because  $g_1^{-1}(C)$  is a partition in  $\mathbb{I}^n$  between  $A_1$  and  $B_1$ .  $\square$

## 6. AP(n, 0)-spaces and general position properties

The parametric general position properties were introduced in [2]. We say that a space  $M$  has the  $m\text{-}\overline{DD}^{(n,k)}$ -property, where  $m, n, k \geq 0$  are integers or  $\infty$ , if for any open cover  $\mathcal{U}$  of  $M$  and any two maps  $f: \mathbb{I}^m \times \mathbb{I}^n \rightarrow M$ ,  $g: \mathbb{I}^m \times \mathbb{I}^k \rightarrow M$  there exist maps  $f': \mathbb{I}^m \times \mathbb{I}^n \rightarrow M$ ,  $g': \mathbb{I}^m \times \mathbb{I}^k \rightarrow M$  which are  $\mathcal{U}$ -homotopic to  $f$  and  $g$ , respectively, and  $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \emptyset$  for all  $z \in \mathbb{I}^m$ . It is clear that this is exactly the well-known disjoint  $n$ -disks property provided  $m = 0$ ,  $n = k$  and  $M$  is  $\text{LC}^n$ . When  $m = 0$ , we simply write  $\overline{DD}^{(n,k)}$  instead of  $0\text{-}\overline{DD}^{(n,k)}$ .

**Lemma 6.1.** *Let  $M$  be completely metrizable having the  $\overline{DD}^{(n,k)}$ -property. Suppose  $X$  is a compactum and  $A, B \subset X$  closed disjoint subsets with  $\dim A \leq n$  and  $\dim B \leq k$ . Then every simplicially factorizable map  $g: X \rightarrow M$  can be homotopically approximated by maps  $g' \in C(X, M)$  such that  $g'(A) \cap g'(B) = \emptyset$ .*

**Proof.** Let  $g \in C(X, M)$  be simplicially factorizable and  $\delta > 0$ . As in Claim 2 from the proof of Proposition 2.4, we can find a finite open cover  $\mathcal{U}$  of  $X$  such that:

- at most  $n + 1$  elements of the family  $\gamma_A = \{U \in \mathcal{U}: U \cap A \neq \emptyset\}$  intersect;
- at most  $k + 1$  elements of the family  $\gamma_B = \{U \in \mathcal{U}: U \cap B \neq \emptyset\}$  intersect;
- $\bigcup \gamma_A \cap \bigcup \gamma_B = \emptyset$ ;
- there exists a map  $h: \mathcal{N}(\mathcal{U}) \rightarrow M$  such that  $h \circ f_{\mathcal{U}}$  is  $\delta/2$ -homotopic to  $g$ , where  $\mathcal{N}(\mathcal{U})$  is the nerve of  $\mathcal{U}$  and  $f_{\mathcal{U}}: X \rightarrow \mathcal{N}(\mathcal{U})$  is the natural map.

Let  $K_A$  and  $K_B$  be the subpolyhedra of  $\mathcal{N}(\mathcal{U})$  generated by the families  $\gamma_A$  and  $\gamma_B$ , respectively. Then  $\dim K_A \leq n$ ,  $\dim K_B \leq k$  and  $K_A \cap K_B = \emptyset$ . For any simplexes  $\sigma \in K_A$  and  $\tau \in K_B$  let

$$G(\sigma, \tau) = \{p \in C(\mathcal{N}(\mathcal{U}), M): p(\sigma) \cap p(\tau) = \emptyset\}.$$

Obviously, each  $G(\sigma, \tau)$  is open in  $C(\mathcal{N}(\mathcal{U}), M)$ . Using  $M \in \overline{DD}^{(n,k)}$  and the Homotopy Extension Theorem, one can show that all  $G(\sigma, \tau)$  are homotopically dense in  $C(\mathcal{N}(\mathcal{U}), M)$ . So, by Lemma 2.2, there exists  $h' \in \bigcap \{G(\sigma, \tau): \sigma \in K_A, \tau \in K_B\}$  which is  $\delta/2$ -homotopic to  $h$ . Then  $g' = h' \circ f_{\mathcal{U}}$  is  $\delta$ -homotopic to  $g$  and  $h'(A) \cap h'(B) = \emptyset$ .  $\square$

**Proposition 6.2.** *For a space  $X$  we have:*

1. If  $X$  is  $\text{LC}^0$ , then  $X$  has the  $\overline{DD}^{(0,0)}$ -property if and only if  $X$  has no isolated point.
2. If  $X$  is completely metrizable and  $X \in \overline{DD}^{(0,0)}$ , then  $X$  has the  $\text{AP}(1, 0)$ -property.

**Proof.** For the first item, see [2, Proposition 7]. To prove the second item, we fix countably many closed 0-dimensional subsets  $P_i$  of  $\mathbb{I}$  consisting of irrational numbers such that  $\dim \mathbb{I} \setminus \bigcup_{i \geq 1} P_i = 0$ . For every rational  $t \in \mathbb{Q}$  and  $i \geq 1$  let  $U_i(t) = \{g \in C(\mathbb{I}, M): g(t) \notin g(P_i)\}$ . By Lemma 6.1, each  $U_i(t)$  is homotopically dense in  $C(\mathbb{I}, M)$ . On the other hand, obviously all  $U_i(t)$  are open in  $C(\mathbb{I}, M)$ . Therefore, by Lemma 2.2, the set  $U = \bigcap \{U_i(t): t \in \mathbb{Q}, i \geq 1\}$  is homotopically dense in  $C(\mathbb{I}, X)$ . Moreover, for every  $g \in U$  and  $x \in \mathbb{I}$  we have the following: if  $g^{-1}(g(x)) \cap \mathbb{Q} \neq \emptyset$ , then  $g^{-1}(g(x)) \subset \mathbb{I} \setminus \bigcup_{i \geq 1} P_i$ ; if  $g^{-1}(g(x)) \cap \mathbb{Q} = \emptyset$ , then  $g^{-1}(g(x)) \subset \mathbb{I} \setminus \mathbb{Q}$ . Hence,  $U$  consists of 0-dimensional maps.  $\square$

Next corollary of Propositions 5.3 and 6.2 provides more examples of  $\text{AP}(n, 0)$ -spaces.

**Corollary 6.3.** *If  $M_i$ ,  $i = 1, \dots, n$ , are completely metrizable  $\text{LC}^0$ -spaces without isolated points, then  $\prod_{i=1}^n M_i$  has the  $\text{AP}(n, 0)$ -property.*

Here is a characterization of the  $\overline{DD}^{(n,k)}$ -property.

**Proposition 6.4.** *A completely metrizable space  $M$  has the  $\overline{DD}^{(n,k)}$ -property, where  $n \leq k \leq \infty$ , if and only if  $M$  satisfies the following condition  $(n, k)$ :*

- If  $X$  is a compactum and  $A \subset B \subset X$  its  $\sigma$ -compact subsets with  $\dim A \leq n$  and  $\dim B \leq k$ , then any simplicially factorizable map  $g \in C(X, M)$  can be homotopically approximated by maps  $h \in C(X, M)$  such that  $h^{-1}(h(x)) \cap B = x$  for all  $x \in A$ .

**Proof.** Suppose  $M \in \overline{DD}^{(n,k)}$  and let  $A \subset B \subset X$  be two  $\sigma$ -compact subsets of a compactum  $X$  with  $\dim A \leq n$  and  $\dim B \leq k$ . Then  $A = \bigcup_{p \geq 1} A_p$  and  $B = \bigcup_{m \geq 1} B_m$ , where  $A_p$  and  $B_m$  are compact sets of dimension  $\dim A_p \leq n$  and  $\dim B_m \leq k$ . For every  $p, i \geq 1$  let  $\omega_i(p) = \{A_{pj}(i) : j = 1, 2, \dots, s(p, i)\}$  be a family of closed subsets of  $A_p$  such that  $\omega_i(p)$  covers  $A_p$  and  $\text{mesh}(\omega_i(p)) < 1/i$ . We also choose sequences  $\{B_{mq}(p, i, j)\}_{q=1}^\infty$  of closed sets  $B_{mq}(p, i, j) \subset B_m$  with  $B_m \setminus A_{pj}(i) = \bigcup_{q=1}^\infty B_{mq}(p, i, j)$ , where  $p, i \geq 1$  and  $j = 1, 2, \dots, s(p, i)$ . Then the sets

$$G(p, i, j, m, q) = \{g \in C(X, M) : g(A_{pj}(i)) \cap g(B_{mq}(p, i, j)) = \emptyset\}$$

are open in  $C(X, M)$  and the intersection  $G$  of all  $G(p, i, j, m, q)$  consists of maps  $g$  with  $g^{-1}(g(x)) \cap B = x$  for any  $x \in A$ . Hence, according to Lemma 2.2, every simplicially factorizable map is homotopically approximated by maps from  $G$  provided the following is true: For any simplicially factorizable map  $g \in C(X, M)$ ,  $\varepsilon > 0$  and  $(p, i, j, m, q)$  with  $p, i, m, q \geq 1$  and  $1 \leq j \leq s(i, p)$ , there exists a simplicially factorizable map  $g(p, i, j, m, q) \in G(p, i, j, m, q)$  which is  $\varepsilon$ -homotopic to  $g$ . Since  $M \in \overline{DD}^{(n,k)}$  and any couple  $A_{pj}(i), B_{mq}(p, i, j)$  consists of disjoint closed sets in  $X$  of dimension  $\leq n$  and  $\leq k$ , respectively, the last statement follows from Lemma 6.1. Hence  $M$  satisfies condition  $(n, k)$ .

Suppose now that  $M$  satisfies condition  $(n, k)$ . To show that  $M \in \overline{DD}^{(n,k)}$ , let  $f : \mathbb{I}^n \rightarrow M$  and  $g : \mathbb{I}^k \rightarrow M$  be two maps. If  $k < \infty$ , we denote by  $X$  the disjoint sum  $\mathbb{I}^n \uplus \mathbb{I}^k$  and consider the map  $h : X \rightarrow M$ ,  $h|_{\mathbb{I}^n} = f$  and  $h|_{\mathbb{I}^k} = g$ . Since  $h$  is simplicially factorizable (as a map with a polyhedral domain) and  $M$  has the  $(n, k)$ -property, for every  $\varepsilon > 0$  there is a map  $h_1 \in C(X, M)$  such that  $h_1^{-1}(h_1(x)) = x$  for all  $x \in \mathbb{I}^n$  and  $h_1$  is  $\varepsilon$ -homotopic to  $h$ . Then the maps  $f_1 = h_1|_{\mathbb{I}^n}$  and  $g_1 = h_1|_{\mathbb{I}^k}$  are  $\varepsilon$ -homotopic to  $f$  and  $g$ , respectively, and  $f_1(\mathbb{I}^n) \cap g_1(\mathbb{I}^k) = \emptyset$ . So,  $M \in \overline{DD}^{(n,k)}$ . If  $n < k = \infty$ , we choose  $k(1) < \infty$  such that  $n \leq k(1)$  and the map  $g' = g \circ r_{k(1)}$  is  $\varepsilon/2$ -homotopic to  $g$ , where  $r_{k(1)} : \mathbb{I}^\infty \rightarrow \mathbb{I}^{k(1)}$  is the retraction of  $\mathbb{I}^\infty$  onto  $\mathbb{I}^{k(1)} \subset \mathbb{I}^\infty$  defined by  $r_{k(1)}((x_1, x_2, \dots)) = (x_1, x_2, \dots, x_{k(1)}, 0, 0, \dots)$ . Then the map  $h : \mathbb{I}^n \uplus \mathbb{I}^\infty \rightarrow M$ ,  $h|_{\mathbb{I}^n} = f$  and  $h|_{\mathbb{I}^\infty} = g'$ , is simplicially factorizable. Hence, as in the previous case, we can use the  $(n, \infty)$ -property of  $M$  to show that  $M \in \overline{DD}^{(n,\infty)}$ . If  $n = k = \infty$ , we homotopically approximate both  $f$  and  $g$  by maps  $f' = f \circ r_{n(1)}$  and  $g' = g \circ r_{k(1)}$ , respectively, and proceed as in the first case.  $\square$

**Corollary 6.5.** Every completely metrizable space  $M \in \overline{DD}^{(n-1,n-1)}$  has the  $AP(n, 0)$ -property. In particular, every manifold modeled on the  $n$ -dimensional Menger cube or the  $n$ -dimensional Nöbeling space has the  $AP(n, 0)$ -property.

**Proof.** Let  $\mathbb{I}^n = A \cup B$ , where  $B = \mathbb{I}^n \setminus A$  is 0-dimensional and  $A = \bigcup_{i \geq 1} A_i$  is  $\sigma$ -compact with  $\{A_i\}$  being a sequence of closed  $(n - 1)$ -dimensional subsets of  $\mathbb{I}^n$ . Since  $C(\mathbb{I}^n, M)$  consists of simplicially factorizable maps, by Proposition 6.4, every map in  $C(\mathbb{I}^n, M)$  is homotopically approximated by maps  $g \in C(\mathbb{I}^n, M)$  such that  $g^{-1}(g(x)) \cap A = x$  for all  $x \in A$ . Let us show that any such  $g$  is 0-dimensional. Indeed, since for every  $x \in \mathbb{I}^n$  the intersection  $g^{-1}(g(x)) \cap A$  can have at most one point,  $g^{-1}(g(x)) = (g^{-1}(g(x)) \cap A) \cup (g^{-1}(g(x)) \cap B)$  is 0-dimensional.

The second part of the corollary follows from the fact that any  $n$ -dimensional Menger manifold, as well as any manifold modeled on the  $n$ -dimensional Nöbeling space, is a complete  $LC^{n-1}$ -space with the disjoint  $n$ -disks property [4]. So, any such a space has the  $\overline{DD}^{(n-1,n-1)}$ -property.  $\square$

The following notions were introduced in [2] and [3]. Let  $A$  be a closed subset of a space  $X$ . We say that  $A$  is:

- a (homotopical)  $Z_n$ -set in  $X$  if for any an open cover  $\mathcal{U}$  of  $X$  and a map  $f : \mathbb{I}^n \rightarrow X$  there is a map  $g : \mathbb{I}^n \rightarrow X$  such that  $g(\mathbb{I}^n) \cap A = \emptyset$  and  $g$  is  $\mathcal{U}$ -near ( $\mathcal{U}$ -homotopic) to  $f$ ;
- homological  $Z_n$ -set in  $X$  if  $H_k(U, U \setminus A) = 0$  for all open sets  $U \subset X$  and all  $k < n + 1$ .

Here,  $H_*(U, U \setminus A) = 0$  are the relative singular homology groups with integer coefficients. Each homotopical  $Z_n$ -set in  $X$  is both a  $Z_n$ -set and a homological  $Z_n$ -set in  $X$ . The converse is not always true.

**Theorem 6.6.** Let  $X$  be a locally path-connected  $AP(n, 0)$ -space. Then every  $x \in X$  is a homological  $Z_{n-1}$ -point in  $X$ .

**Proof.** By [3, Corollary 8.4], it suffices to check that  $(x, 0)$  is a homotopical  $Z_n$ -point in  $X \times [-1, 1]$  for every  $x \in X$ . Given a map  $f : \mathbb{I}^n \rightarrow X \times [-1, 1]$  we are going to homotopically approximate  $f$  by a map  $g : \mathbb{I}^n \rightarrow X \times [-1, 1]$  with  $(x, 0) \notin g(\mathbb{I}^n)$ . Let  $f = (f_1, f_2)$  where  $f_1 : \mathbb{I}^n \rightarrow X$  and  $f_2 : \mathbb{I}^n \rightarrow [-1, 1]$  are the components of  $f$ . Since  $X \in AP(n, 0)$ , there is a 0-dimensional map  $g_1 : \mathbb{I}^n \rightarrow X$  that homotopically approximates  $f_1$ . Because  $Z = g_1^{-1}(x)$  is a 0-dimensional subset of  $\mathbb{I}^n$  and  $\{0\}$  is a homotopical  $Z_0$ -set in  $[-1, 1]$ , the map  $f_2$  can be homotopically approximated by a map  $g_2 : \mathbb{I}^n \rightarrow [-1, 1]$  such that  $0 \notin f_2(Z)$ . Then the map  $g = (g_1, g_2) : \mathbb{I}^n \rightarrow X \times [-1, 1]$  homotopically approximates  $f$  and  $(x, 0) \notin g(\mathbb{I}^n)$ .  $\square$

**Proposition 6.7.** If  $X \in AP(n, 0)$  and  $Y \in \infty\text{-}\overline{DD}^{(0,0)}$ , then  $X \times Y \in \overline{DD}^{(n,n)}$ .

**Proof.** Let  $f = (f_X, f_Y) : \mathbb{I}^n \oplus \mathbb{I}^n \rightarrow X \times Y$  be a given map. Since  $X \in AP(n, 0)$ , we homotopically approximate the first component  $f_X$  by a 0-dimensional map  $g_X : \mathbb{I}^n \oplus \mathbb{I}^n \rightarrow X$ . Next, use  $\infty\text{-}\overline{DD}^{(0,0)}$ -property of  $Y$  to homotopically approximate

the second component  $f_Y$  by a map  $g_Y : \mathbb{I}^n \oplus \mathbb{I}^n \rightarrow Y$  that is injective on the fibers of  $g_X$ . Then the map  $g = (g_X, g_Y) : \mathbb{I}^n \oplus \mathbb{I}^n \rightarrow X \times Y$  is injective, witnessing that  $X \times Y$  has the  $\overline{DD}^{(n,n)}$ -property.  $\square$

We say that a space  $X$  has the  $AP(\infty, 0)$ -property if  $X \in AP(n, 0)$  for every  $n \geq 1$ .

**Corollary 6.8.** *If  $X, Y$  are two locally path-connected spaces both possessing the  $AP(\infty, 0)$ -property and  $Y$  contains a dense set of homotopical  $Z_2$ -points, then  $X \times Y$  has the  $\overline{DD}^{(\infty, \infty)}$ -property.*

**Proof.** By Theorem 6.6, each point of  $Y$  is a homological  $Z_\infty$ -point. Taking into account that  $Y$  has a dense set of homotopical  $Z_2$ -points and applying Theorem 26(4,5) of [2], we conclude that the space  $Y \in \infty\text{-}\overline{DD}^{(0,0)}$ . Finally, by Proposition 6.7,  $X \times Y \in \overline{DD}^{(\infty, \infty)}$ .  $\square$

**Corollary 6.9.** *If  $X, Y$  are two compact AR's both possessing the  $AP(\infty, 0)$ -property, then  $X \times Y \times \mathbb{I}$  is homeomorphic to the Hilbert cube.*

**Proof.** Theorem 6.6 implies that every  $y \in Y$  is a homological  $Z_\infty$ -point in  $Y$ . Then, according to [2, Theorem 26(5)],  $Y \in \overline{DD}^{(1,1)}$ . Consequently, by [2, Theorem 10(5)], each point of  $Y$  is a homotopical  $Z_1$ -point in  $Y$ . Now, applying the Multiplication Formula for homological  $Z$ -sets [2, Theorem 17(1)], we conclude that each point of  $Y \times \mathbb{I}$  is a homotopical  $Z_2$ -point. Therefore, Corollary 6.8 yields that the product  $X \times (Y \times \mathbb{I})$  has the  $\overline{DD}^{(\infty, \infty)}$ -property. Being a compact AR, this product is homeomorphic to the Hilbert cube according to the Toruńczyk characterization theorem of  $Q$ -manifolds.  $\square$

In light of the preceding corollary, it is interesting to remark that there exists a compact space  $X \in AR \cap AP(\infty, 0)$  which is not homeomorphic to the Hilbert cube. To present such an example, we first prove that the  $AP(n, 0)$ -property is preserved by a special type of maps.

A map  $\pi : X \rightarrow Y$  is called an elementary cell-like map if:

- (1)  $\pi$  is a fine homotopy equivalence, i.e., for every open cover  $\mathcal{U}$  of  $Y$  there exists a map  $s : Y \rightarrow X$  with  $\pi \circ s$  being  $\mathcal{U}$ -homotopic to the identity on  $Y$ ;
- (2) the non-degeneracy set  $N_\pi = \{y \in Y : |\pi^{-1}(y)| \neq 1\}$  is at most countable;
- (3) each fiber  $\pi^{-1}(y)$ ,  $y \in N_\pi$ , is an arc.

**Proposition 6.10.** *Let  $\pi : X \rightarrow Y$  be an elementary cell-like map between metric spaces such that  $X$  is complete and every  $x \in \pi^{-1}(N_\pi)$  is a homotopical  $Z_n$ -point in  $X$ . Then  $Y \in AP(n, 0)$  provided  $X \in AP(n, 0)$ .*

**Proof.** Suppose  $X \in AP(n, 0)$ . To prove that  $Y \in AP(n, 0)$ , fix an open cover  $\gamma$  of  $Y$  and a map  $f : \mathbb{I}^n \rightarrow Y$ . Let  $\gamma_1$  be an open cover of  $Y$  which is star-refinement of  $\gamma$ . Since  $\pi$  is a fine homotopy equivalence, there is a map  $s : Y \rightarrow X$  such that  $\pi \circ s$  is  $\gamma_1$ -homotopic to the identity map on  $Y$ . Let  $D = \{x_i : i \geq 1\} \subset \pi^{-1}(N_\pi)$  be a sequence such that  $\pi^{-1}(y) \setminus D$  is 0-dimensional for every  $y \in N_\pi$ . Then the sets  $W_i = \{g \in C(\mathbb{I}^n, X) : x_i \notin g(\mathbb{I}^n)\}$  are open in  $C(\mathbb{I}^n, X)$  equipped with the uniform convergence topology. Moreover, each  $W_i$  is homotopically dense in  $C(\mathbb{I}^n, X)$  because  $x_i$  is a homotopical  $Z_n$ -point in  $X$ . We also consider the sets  $U_i = \{g \in C(\mathbb{I}^n, X) : d_0(g^{-1}(x)) < 1/i \text{ for all } x \in X\}$ . Since  $X \in AP(n, 0)$  and all maps from  $C(\mathbb{I}^n, X)$  are simplicially factorizable, it follows from Proposition 2.3 that any  $U_i$  is open and homotopically dense in  $C(\mathbb{I}^n, X)$ . This easily implies that the intersections  $V_i = W_i \cap U_i$  are open and homotopically dense in  $C(\mathbb{I}^n, X)$ . Consequently, by Lemma 2.2 (with  $G$  being  $C(\mathbb{I}^n, X)$ ), there exists a map  $g \in \bigcap_{i=1}^\infty V_i$  which is  $\pi^{-1}(\gamma_1)$ -homotopic to  $s \circ f$ . Obviously,  $g$  is 0-dimensional and  $g(\mathbb{I}^n) \subset X \setminus D$ . Consequently, the map  $f_1 = \pi \circ g : \mathbb{I}^n \rightarrow Y$  is also 0-dimensional and  $\gamma_1$ -homotopic to  $\pi \circ s \circ f$ . Since  $\pi \circ s \circ f$  is  $\gamma_1$ -homotopic to  $f$ ,  $f_1$  is  $\gamma$ -homotopic to  $f$ . Hence,  $Y \in AP(n, 0)$ .  $\square$

Singh [24] constructed an elementary cell-like map  $f : Q \rightarrow X$  from the Hilbert cube  $Q$  onto a compact  $X \in AR$  such that  $X \times \mathbb{I}$  is homeomorphic to  $Q$  but  $X$  contains no proper ANR-subspace of dimension  $\geq 2$ . By the preceding proposition, Singh's space has the  $AP(\infty, 0)$ -property. Thus we have:

**Corollary 6.11.** *There is a compact  $X \in AR$  with  $X \in AP(\infty, 0)$  such that  $X \times \mathbb{I}$  is homeomorphic to the Hilbert cube but  $X$  contains no proper ANR-subspace of dimension  $\geq 2$ .*

Now, we consider the spaces with piecewise embedding dimension  $n$ . According to [13], a map  $h : P \rightarrow M$  from a finite polyhedron  $P$  is said to be a piecewise embedding if there is a triangulation  $T$  of  $P$  such that  $h$  embeds each simplex  $\sigma \in T$  and  $h(P)$  is an ANR. For a space  $M$ , the piecewise embedding dimension  $ped(M)$  is the maximum  $k$  such that for any  $\varepsilon > 0$  and any map  $g : P \rightarrow M$  from a finite polyhedron  $P$  with  $\dim P \leq k$  there exists a piecewise embedding  $g' : P \rightarrow M$  which is  $\varepsilon$ -close to  $g$ . If  $y \in M$ , then  $ped_y(M)$  is the maximum of all  $ped(U)$ , where  $U$  is a neighborhood of  $y$  in  $M$ . Obviously,  $ped(M) \leq \min\{ped_y(M) : y \in M\}$ .

As we noted, by [13, Proposition 2.1], every complete ANR-space  $M$  has the  $AP(n, 0)$ -property provided  $\text{ped}(M) \geq n$ . But there are even compact ANR's having the  $AP(n, 0)$ -property with  $\text{ped} \leq n - 1$ . For example, according to Corollary 6.3, any product  $M = \prod_{i=1}^{l=n} M_i$  of dendrites with dense set of endpoints is an  $AP(n, 0)$ . On the other hand, by [26, Theorem 3.4],  $\text{ped}(M) \leq n - 1$ . Next proposition also shows that the property  $\text{ped} = n$  is quite restrictive.

**Proposition 6.12.** *If  $y \in M$  with  $\text{ped}_y(M) = n + 1$ , where  $(M, \rho)$  is a complete space, then  $y$  is a  $Z_n$ -point in  $M$ .*

**Proof.** Since  $\text{ped}_y(M) = n + 1$ , there exists a neighborhood  $U_y$  in  $M$  with  $\text{ped}(U_y) = n + 1$ . It is easily seen that if  $y$  is a  $Z_n$ -point in  $U_y$ , it is also a  $Z_n$ -point in  $M$ . So, we can suppose that  $U_y = M$ , and let  $\varepsilon > 0$  and  $g \in C(\mathbb{I}^n, M)$ . We identify  $\mathbb{I}^n$  with the set  $\{(x_1, \dots, x_{n+1}) \in \mathbb{I}^{n+1} : x_{n+1} = 0\}$  and let  $\pi : \mathbb{I}^{n+1} \rightarrow \mathbb{I}^n$  be the projection. Since  $\text{ped}(M) = n + 1$ , there exists a triangulation  $T$  of  $\mathbb{I}^{n+1}$  and a map  $g_1 \in C(\mathbb{I}^{n+1}, M)$  which is  $\varepsilon/2$ -close to  $g \circ \pi$  and  $g_1|_\sigma$  is an embedding for all  $\sigma \in T$ . Hence,  $g_1^{-1}(y)$  consists of finitely many points. Moreover, there is  $\delta > 0$  such that for any  $\delta$ -close points  $x', x''$  in  $\mathbb{I}^{n+1}$  we have  $\rho(g_1(x'), g_1(x'')) < \varepsilon/2$ . Since each  $x \in \mathbb{I}^{n+1}$  is a  $Z_n$ -point in  $\mathbb{I}^{n+1}$ ,  $g_1^{-1}(y)$  is a  $Z_n$ -set in  $\mathbb{I}^{n+1}$ . Hence, there exists a map  $h : \mathbb{I}^n \rightarrow \mathbb{I}^{n+1}$  such that  $h$  is  $\delta$ -close to the embedding  $i : \mathbb{I}^n \rightarrow \mathbb{I}^{n+1}$  and  $h(\mathbb{I}^n) \subset \mathbb{I}^{n+1} \setminus g_1^{-1}(y)$ . Then  $g' = g_1 \circ h \in C(\mathbb{I}^n, M)$  is  $\varepsilon$ -close to  $g$  and  $y \notin g'(\mathbb{I}^n)$ .  $\square$

Finally, let us complete the paper with the following question:

**Question 1.** Let  $X \in AR$  be a compact space such that  $X \in \overline{DD}^{[2,2]} \cap AP(\infty, 0)$ . Is  $X$  homeomorphic to the Hilbert cube?

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