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Linear operators with compact supports, probability measures and Milyutin maps ☆

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ABSTRACT

The notion of a regular operator with compact supports between function spaces is introduced. On that base we obtain a characterization of absolute extensors for 0-dimensional spaces in terms of regular extension operators having compact supports. Milyutin maps are also considered and it is established that some topological properties, like paracompactness, metrizability and κ-metrizability, are preserved under Milyutin maps.

1. Introduction

In this paper we assume that all topological spaces are Tychonoff. The main concept is that one of a linear map between function spaces with compact supports. Let $u : C(X,E) \to C(Y,E)$ be a linear map, where $C(X,E)$ is the set of all continuous functions from X into a locally convex linear space E. We say that u *has compact supports* if for every $y \in Y$ the linear map $T(y) : C(X,E) \to E$, defined by $T(y)(h) = u(h)(y)$, $h \in C(X,E)$, has a compact support in X. Here, the support of a linear map $\mu : C(X,E) \to E$ is the set $s(\mu)$ of all $x \in \beta X$ such that for every neighborhood U of x in βX there exists $h \in C(X,E)$ with $(\beta h)(\beta X - U) = 0$ and $\mu(h) \neq 0$. Recall that βX is the Čech–Stone compactification of X and $\beta h : \beta X \to \beta E$ the extension of h. Obviously, $s(\mu) \subset \beta X$ is closed, so compact. When $s(\mu) \subset X$, μ is said to have a compact support. In a similar way we define a linear map with compact supports when consider the bounded function sets $C^*(X,E)$ and $C^*(Y,E)$ (if E is the real line $\mathbb{R}$, we simply write $C(X)$ and $C^*(X)$). If all $T(y)$ are *regular linear maps*, i.e., $T(y)(h)$ is contained in the closed convex hull $\overline{\operatorname{conv} h(X)}$ of $h(X)$ in E, then u is called a *regular operator*.

Haydon [19] proved that Dugundji spaces introduced by Pelczynski [26] coincide with the absolute extensors for 0-dimensional compact spaces (br., $X \in AE(0)$). Later Chigogidze [10] provided a more general definition of $AE(0)$-spaces in the class of all Tychonoff spaces. The notion of linear operators with compact supports arose from the attempt to find a characterization of $AE(0)$-spaces similar to the Pelczynski definition of Dugundji spaces. Here is this characterization (see Theorems 4.1–4.2). *For any space X the following conditions are equivalent*: (i) *X is an $AE(0)$-space*; (ii) *for every C-embedding of X in a space Y there exists a regular extension operator $u : C(X) \to C(Y)$ with compact supports*; (iii) *for every C-embedding of X in a space Y there exists a regular extension operator $u : C^*(X) \to C^*(Y)$ with compact supports*; (iv) *for any C-embedding of X in a space Y and any complete locally convex space E there exists a regular extension operator $u : C^*(X,E) \to C^*(Y,E)$ with compact supports*.

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It is easily seen that $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) is a regular extension operator with compact supports iff there exists a continuous map $T : Y \to P_c(X, E)$ (resp., $T : Y \to P_c^*(X, E)$) such that $T(y)$ is the Dirac measure δ_y at y for all $y \in X$. Here, $P_c(X, E)$ (resp., $P_c^*(X, E)$) is the space of all regular linear maps $\mu : C(X, E) \to E$ (resp., $\mu : C^*(X, E) \to E$) with compact supports equipped with the pointwise convergence topology (we write $P_c(X)$ and $P_c^*(X)$ when $E = \mathbb{R}$). Section 2 is devoted to properties of the functors P_c and P_c^* (actually, P_c^* is the well-known functor P_β [9] of all probability measures on βX whose supports are contained in X). It appears that *$P_c(X)$ is homeomorphic to the closed convex hull of $e_X(X)$ in $\mathbb{R}^{C(X)}$ provided X is realcompact, where e_X is the standard embedding of X into $\mathbb{R}^{C(X)}$* (Proposition 2.4), and *$P_c(X)$ is metrizable iff X is a metric compactum* (Proposition 2.5(ii)).

In Section 3 we consider regular averaging operators with compact support and Milyutin maps. Milyutin maps between compact spaces were introduced by Pelczynski [26]. There are different definitions of Milyutin maps in the non-compact case, see [1,28,37]. We say that a surjection $f : X \to Y$ is *a Milyutin map* if f admits a regular averaging operator $u : C(X) \to C(Y)$ having compact supports. This is equivalent to the existence of a map $T : Y \to P_c(X)$ such that $f^{-1}(y)$ contains the support of $T(y)$ for all $y \in Y$. It is shown, for example, that *for every product Y of metric spaces there is a 0-dimensional product X of metric spaces and a perfect Milyutin map $f : X \to Y$* (Corollary 3.10). Moreover, *every p-paracompact space is an image under a perfect Milyutin map of a 0-dimensional p-paracompact space* (Corollary 3.11).

In the last Section 5 we prove that some topological properties are preserved under Milyutin maps. These properties include paracompactness, collectionwise normality (complete) metrizability, stratifiability, δ-metrizability and κ-metrizability. In particular, we provide a positive answer to a question of Shchepin [31] whether *every AE(0)-space is κ-metrizable* (see Corollary 5.5).

Some of the result presented here were announced in [33] without proofs.

2. Measure spaces

Everywhere in this section E, F stand for locally convex linear topological spaces and $C(X, E)$ is the set of all continuous maps from a space X into E. By $C^*(X, E)$ we denote the bounded elements of $C(X, E)$. Let $\mu : C(X, E) \to F$ (resp., $\mu : C^*(X, E) \to F$) be a linear map. The support of μ is defined as the set $s(\mu)$ (resp., $s^*(\mu)$) of all $x \in \beta X$ such that for every neighborhood U of x in βX there exists $f \in C(X, E)$ (resp., $f \in C^*(X, E)$) with $(\beta f)(\beta X - U) = 0$ and $\mu(f) \neq 0$, see [36]. Obviously, $s(\mu)$ and $s^*(\mu)$ are closed in βX, so compact. Let us note that in the above definition $(\beta f)(\beta X - U) = 0$ is equivalent to $f(X - U) = 0$. We also use $s^*(\mu)$ to denote the support of the restriction $\mu|C^*(C, E)$ when μ is defined on $C(X, E)$ (in this case we have $s^*(\mu) \subset s(\mu)$).

Lemma 2.1. *Let μ be a linear map from $C(X, E)$ (resp., from $C^*(X, E)$) into F, where E and F are normed spaces.*

(i) *If V a neighborhood of $s(\mu)$ (resp., $s^*(\mu)$), then $\mu(f) = 0$ for every $f \in C(X, E)$ (resp., $f \in C^*(X, E)$) with $(\beta f)(V) = 0$.*
(ii) *If the restriction $\mu|C^*(X, E)$ is continuous when $C^*(X, E)$ is equipped with the uniform topology, then $\mu(f) = 0$ provided $f \in C(X, E)$ (resp., $f \in C^*(X, E)$) and $(\beta f)(s(\mu)) = 0$ (resp., $(\beta f)(s^*(\mu)) = 0$).*
(iii) *In each of the following two cases $s(\mu)$ coincides with $s^*(\mu)$: either $s(\mu) \subset X$ or μ is a non-negative linear functional on $C(X)$.*

Proof. When μ is a linear map on $C(X, E)$, items (i) and (ii) were established in [36, Lemma 2.1]; the case when μ is a linear map on $C^*(X, E)$ can be done by similar arguments. To prove (iii), we first suppose that $s(\mu) \subset X$. Then $s^*(\mu)$ is the support of the restriction $\mu|C^*(X, E)$ and $s^*(\mu) \subset s(\mu)$. So, we need to show that $s(\mu) \subset s^*(\mu)$. For a given point $x \in s(\mu)$ and its neighborhood U in βX there exists $g \in C(X, E)$ with $g(X - U) = 0$ and $\mu(g) \neq 0$. Because $g(s(\mu)) \subset E$ is compact, we can find $\epsilon > 0$ such that $s(\mu)$ is contained in the set $W = \{y \in X\colon \|g(y)\| < \epsilon\}$, where $\|.\|$ denotes the norm in E. Let $B_\epsilon = \{z \in E\colon \|z\| \leqslant \epsilon\}$ and $r : E \to B_\epsilon$ be a retraction (i.e., a continuous map with $r(z) = z$ for every $z \in B_\epsilon$). Then $h(y) = g(y)$ for every $y \in W$, where $h = r \circ g$. Hence, choosing an open set V in βX such that $V \cap X = W$, we have $(\beta(h - g))(V) = 0$. Since V is a neighborhood of $s(\mu)$, by (i), $\mu(h) = \mu(g) \neq 0$. Therefore, we found a map $h \in C^*(X, E)$ such that $\beta h(\beta X - U) = 0$ and $\mu(h) \neq 0$. This means that $x \in s^*(\mu)$. So, $s(\mu) = s^*(\mu)$.

Now, let $E = F = \mathbb{R}$ and μ be a non-negative linear functional on $C(X)$. Suppose there exists $x \in s(\mu)$ but $x \notin s^*(\mu)$. Then, for some neighborhood U of x in βX, we have

$$\mu(h) = 0 \quad \text{for every } h \in C^*(X) \text{ with } h(X - U) = 0. \tag{1}$$

Since $x \in s(\mu)$, there exists $f \in C(X)$ such that $f(X - U) = 0$ and $\mu(f) \neq 0$. Now, we use an idea from [21, proof of Theorem 1]. We represent f as the sum $f^+ + f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$. Since both f^+ and f^- are 0 outside U and $\mu(f) = \mu(f^+) + \mu(f^-) \neq 0$ implies that at least one of the numbers $\mu(f^+)$ and $\mu(f^-)$ is not 0, we can assume that $f \geqslant 0$. By (1), f is not bounded. Therefore, there is a sequence $\{x_n\} \subset X$ such that $\{t_n = f(x_n)\colon n \geqslant 1\}$ is an increasing and unbounded sequence. We set $t_0 = 0$ and for every $n \geqslant 1$ define the function $f_n \in C^*(X)$ as follows: $f_n(x) = 0$ if $f(x) \leqslant t_{n-1}$, $f_n(x) = f(x) - t_{n-1}$ if $t_{n-1} < f(x) \leqslant t_n$ and $f_n(x) = t_n - t_{n-1}$ provided $f(x) > t_n$. Let also $h_n = t_n \cdot f_n$ and $h = \sum_{n=1}^{\infty} h_n$. Then h is continuous and for every $n \geqslant 1$ we have

$$t_n(f - f_1 - f_2 - \cdots - f_n) \leqslant h - h_1 - h_2 - \cdots - h_n. \tag{2}$$

Since all f_n and h_n are bounded and continuous functions satisfying $f_n(X-U)=h_n(X-U)=0$, it follows from (1) that $\mu(h_n)=\mu(f_n)=0$, $n\geqslant 1$. So, by (2), $t_n\cdot\mu(f)\leqslant\mu(h)$ for every n. Hence, $\mu(f)=0$ which is a contradiction. Therefore, $s(\mu)=s^*(\mu)$. □

We say that a linear map μ on $C(X,E)$ (resp., on $C^*(X,E)$) has a *compact support* if $s(\mu)\subset X$ (resp., $s^*(\mu)\subset X$). If μ takes values in E, then it is called *regular* provided $\mu(f)$ belongs to the closure of the convex hull $\operatorname{conv} f(X)$ of $f(X)$ for every $f\in C(X,E)$ (resp., $f\in C^*(X,E)$). Below, $C_k(X,E)$ (resp., $C_k^*(X,E)$) stands for the space $C(X,E)$ (resp. $C^*(X,E)$) with the compact-open topology.

Proposition 2.2. *Let E be a normed space. A regular linear map μ on $C(X,E)$ (resp., $C^*(X,E)$) has a compact support in X if and only if μ is continuous on $C_k(X,E)$ (resp., $C_k^*(X,E)$).*

Proof. We consider only the case when μ is a map on $C(X,E)$, the other one is similar. Suppose $s(\mu)=H\subset X$. Since μ is regular, $\mu(f)\in\overline{\operatorname{conv} f(X)}$ for every $f\in C(X,E)$. This yields $\|\mu(f)\|\leqslant\|f\|$, $f\in C^*(X,E)$. Hence, the restriction $\mu|C^*(X,E)$ is continuous with respect to the uniform topology. So, by Lemma 2.1(ii), for every $f\in C(X,E)$ the value $\mu(f)$ depends only on the restriction $f|H$. Therefore, the linear map $\nu: C(H,E)\to E$, $\nu(g)=\mu(\tilde g)$, where $\tilde g\in C(X,E)$ is any continuous extension of g, is well defined. Note that such an extension $\tilde g$ always exists because $H\subset X$ is compact. Moreover, the restriction map $\pi_H: C_k(X,E)\to C_k(H,E)$ is surjective and continuous. Since $\mu=\nu\circ\pi_H$, μ would be continuous provided $\nu: C_k(H,E)\to E$ is so. Next claim implies that for every $g\in C(H,E)$ we have $\nu(g)\in\overline{\operatorname{conv} g(H)}$ and $\|\nu(g)\|\leqslant\|g\|$, which guarantee the continuity of ν.

Claim 1. $\mu(f)\in\overline{\operatorname{conv} f(H)}$ *for every* $f\in C(X,E)$.

Indeed, if $\mu(f)\notin\overline{\operatorname{conv} f(H)}$ for some $f\in C(X,E)$, then we can find a closed convex neighborhood W of $\overline{\operatorname{conv} f(H)}$ in E and a function $h\in C(X,E)$ such that $\mu(f)\notin W$, $h(X)\subset W$ and $h(x)=f(x)$ for all $x\in H$. As it was shown above, the last equality implies $\mu(f)=\mu(h)$. Hence, $\mu(f)=\mu(h)\in\overline{\operatorname{conv} h(X)}\subset W$, which is a contradiction.

Now, suppose $\mu: C_k(X,E)\to E$ is continuous. Then there exist a compact set $K\subset X$ and $\epsilon>0$ such that $\|\mu(f)\|<1$ for every $f\in C(X,E)$ with $\sup\{\|f(x)\|: x\in K\}<\epsilon$. We claim that $s(\mu)\subset K$. Indeed, otherwise there would be $x\in s(\mu)-K$, a neighborhood U of x in βX with $U\cap K=\emptyset$, and a function $g\in C(X,E)$ such that $g(X-U)=0$ and $\mu(g)\neq 0$. Choose an integer k with $\|\mu(kg)\|\geqslant 1$. On the other hand, $kg(x)=0$ for every $x\in K$. Hence, $\|\mu(kg)\|<1$, a contradiction. □

Now, for every space X and a locally convex space E let $P_c(X,E)$ (resp., $P_c^*(X,E)$) denote the set of all regular linear maps $\mu: C(X,E)\to E$ (resp., $\mu: C^*(X,E)\to E$) with compact supports equipped with the weak (i.e. pointwise) topology with respect to $C(X,E)$ (resp., $C^*(X,E)$). If E is the real line, we write $P_c(X)$ (resp., $P_c^*(X)$) instead of $P_c(X,\mathbb{R})$ (resp., $P_c^*(X,\mathbb{R})$). It is easily seen that a linear map $\mu: C(X)\to\mathbb{R}$ (resp., $\mu: C^*(X)\to\mathbb{R}$) is regular if and only if μ is non-negative and $\mu(1)=1$. If $h: X\to Y$ is a continuous map, then there exists a map $P_c(h): P_c(X)\to P_c(Y)$ defined by $P_c(h)(\mu)(f)=\mu(f\circ h)$, where $\mu\in P_c(X)$ and $f\in C(Y)$. Considering functions $f\in C^*(Y)$ in the above formula, we can define a map $P_c^*(h): P_c^*(X)\to P_c^*(Y)$. It is easily seen that $s(P_c(h)(\mu))\subset h(s(\mu))$ (resp., $s^*(P_c^*(h)(\mu))\subset h(s^*(\mu))$) for every $\mu\in P_c(X)$ (resp., $\mu\in P_c^*(X)$). Moreover, $P_c(h_2\circ h_1)=P_c(h_2)\circ P_c(h_1)$ and $P_c^*(h_2\circ h_1)=P_c^*(h_2)\circ P_c^*(h_1)$ for any two maps $h_1: X\to Y$ and $h_2: Y\to Z$. Therefore, both P_c and P_c^* are covariant functors in the category of all Tychonoff spaces and continuous maps. Let us also note that if X is compact then $P_c(X)$ and $P_c^*(X)$ coincide with the space $P(X)$ of all probability measures on X.

For every $x\in X$ we consider the Dirac's measure $\delta_x\in P_c(X,E)$ defined by $\delta_x(f)=f(x)$, $f\in C(X,E)$. In a similar way we define $\delta_x^*\in P_c^*(X,E)$. We also consider the maps $i_X: X\to P_c(X,E)$, $i_X(x)=\delta_x$, and $i_X^*: X\to P_c^*(X,E)$, $i_X(x)=\delta_x^*$. Next proposition is an easy exercise.

Proposition 2.3. *Let $h: X\to Y$ be a map.*

(i) *The map $i_X: X\to P_c(X)$ is a closed C-embedding, and $i_X^*: X\to P_c^*(X)$ is a closed C^*-embedding;*
(ii) *The map $P_c(h)$ is a (closed) C-embedding provided h is a (closed) C-embedding;*
(iii) *The map $P_c^*(h)$ is a (closed) C^*-embedding provided h is a (closed) C^*-embedding.*

There exists a natural embedding $e_X: X\to\mathbb{R}^{C(X)}$, $e_X(x)=(f(x))_{f\in C(X)}$. Denote by $M^+(X)$ the set of all regular linear functionals on $C(X)$ with the pointwise topology and consider the map $m_X: M^+(X)\to\mathbb{R}^{C(X)}$, $m_X(\mu)=(\mu(f))_{f\in C(X)}$. It easily seen that m_X is also an embedding extending e_X and $m_X(M^+(X))$ is a closed convex subset of $\mathbb{R}^{C(X)}$. Moreover, $P_c(X)\subset M^+(X)$. It is well known that for compact X the space $P(X)$ is homeomorphic with the convex closed hull of $e_X(X)$ in $\mathbb{R}^{C(X)}$. A similar fact is true for $P_c(X)$.

Proposition 2.4. *If X is realcompact, then $P_c(X)$ is homeomorphic to the closed convex hull of $e_X(X)$ in $\mathbb{R}^{C(X)}$.*

Proof. Obviously, $m_X(P_c(X))$ is a convex subset of $\mathbb{R}^{C(X)}$ containing the set $\operatorname{conv} e_X(X)$. It suffices to show that $m_X(P_c(X))$ coincides with the set $B = \overline{\operatorname{conv} e_X(X)}$. Suppose $\mu \in P_c(X)$. By Lemma 2.1(ii) and Proposition 2.2, for every $f \in C(X)$ the value $\mu(f)$ is determined by the restriction $f|s(\mu)$. So, there exists an element $\nu \in P(s(\mu))$ such that $\mu(f) = \nu(f|s(\mu))$, $f \in C(X)$ (see the proof of Proposition 2.2). Since the set $P_f(s(\mu))$ of all measures from $P(s(\mu))$ having finite supports is dense in $P(s(\mu))$ [17], there is a net $\{\nu_\alpha\}_{\alpha\in A} \subset P_f(s(\mu))$ converging to ν in $P(s(\mu))$. Each ν_α can be identified with the measure $\mu_\alpha \in P_c(X)$ defined by $\mu_\alpha(f) = \nu_\alpha(f|s(\mu))$, $f \in C(X)$. Moreover, the net $\{\mu_\alpha\}_{\alpha\in A}$ converges to μ in $P_c(X)$. Then $\{m_X(\mu_\alpha)\}_{\alpha\in A} \subset \operatorname{conv} e_X(X)$ and converges to $m_X(\mu)$ in $\mathbb{R}^{C(X)}$. So, $m_X(\mu) \in B$. In this way we obtained $m_X(P_c(X)) \subset B$.

On the other hand, since $m_X(M^+(X))$ is a closed and convex subset of $\mathbb{R}^{C(X)}$ containing $e_X(X)$, $B \subset m_X(M^+(X))$. So, the elements of B are of the form $m_X(\mu)$ with μ being a regular linear functional on $C(X)$. Since X is realcompact, according to [21, Theorem 18], any such a functional has a compact support in X. Therefore, $B \subset m_X(P_c(X))$. □

There exists a natural continuous map $j_X : P_c(X) \to P_c^*(X)$ assigning to each $\mu \in P_c(X)$ the measure $\nu = \mu|C^*(X)$. By Lemma 2.1 and Proposition 2.2, $s(\mu) = s^*(\nu)$ and $\mu(f)$ and $\nu(g)$ depend, respectively, on the restrictions $f|s(\mu)$ and $g|s^*(\nu)$ for all $f \in C(X)$ and $g \in C^*(X)$. This implies that j_X is one-to-one. Using again Lemma 2.1 and Proposition 2.2, one can show that j_X is surjective. According to next proposition, j_X is not always a homeomorphism.

A subset A of a space X is said to be *bounded* if $f(A) \subset \mathbb{R}$ is bounded for every $f \in C(X)$. This notion should be distinguished from the notion of a bounded set in a linear topological space.

Proposition 2.5. *For a given space X we have*:

(i) *The map j_X is a homeomorphism if and only if X is pseudocompact*;
(ii) *$P_c(X)$ is metrizable if and only if X is compact and metrizable.*

Proof. (i) Obviously, if X is pseudocompact, then $C(X) = C^*(X)$ and j_X is the identity on $P_c(X)$. Suppose X is not pseudocompact and choose $g \in C(X)$ and a discrete countable set $\{x(n)\colon n \geqslant 1\}$ in X such that $\{g(x(n))\colon n \geqslant 1\}$ is unbounded and discrete in $\mathbb{R}$. For every $n \geqslant 2$ define the measures $\mu_n \in P_c(X)$ and $\nu_n \in P_c^*(X)$ as follows: $\mu_1 = \delta_{x(1)}$, $\mu_n = (1 - 1/n)\delta_{x(1)} + \sum_{k=2}^{n+1}(1/n)^2\delta_{x(k)}$ and $\nu_1 = \delta^*_{x(1)}$, $\nu_n = (1 - 1/n)\delta^*_{x(1)} + \sum_{k=2}^{n+1}(1/n)^2\delta^*_{x(k)}$. Obviously, $j_X(\mu_n) = \nu_n$ for all $n \geqslant 1$ and $s(\mu_n) = s^*(\nu_n) = \{x(1), x(2), \dots, x(n+1)\}$, $n \geqslant 2$. So, $g(\bigcup_{n=1}^{\infty} s(\mu_n))$ is unbounded in $\mathbb{R}$. This, according to [35, Proposition 3.1] (see also [3]), means that the sequence $\{\mu_n\}_{n\geqslant1}$ is not compact. On the other hand, it is easily seen that $\{\nu_n\}_{n\geqslant2}$ converges in $P_c^*(X)$ to ν_1. Consequently, j_X is not a homeomorphism.

(ii) First we prove that $P_c(\mathbb{N})$ is not metrizable, where $\mathbb{N}$ is the set of the integers $n \geqslant 1$ with the discrete topology. For every $n \geqslant 1$ let $K(n) = P_c(\{1, 2, \dots, n\})$. Obviously, every $K(n)$ is homeomorphic to a simplex of dimension $n - 1$ and $K(n) \subset K(m)$ for $n \leqslant m$. Moreover, $P_c(\mathbb{N}) = \bigcup_{n\geqslant1} K(n)$.

Claim 2. *$P_c(\mathbb{N})$ is nowhere locally compact.*

Indeed, otherwise there would be $\mu \in P_c(\mathbb{N})$ and its open neighborhood $O(\mu)$ in $P_c(\mathbb{N})$ with $\overline{O(\mu)}$ being compact. Then, by [35, Proposition 3.1], $S = \bigcup\{s(\nu)\colon \nu \in O(\mu)\}$ is a bounded subset of $\mathbb{N}$. Hence, $S \subset \{1, 2, \dots, p\}$ for some $p \geqslant 1$. The last inclusion means that $O(\mu) \subset K(p)$, so $\dim O(\mu) \leqslant p - 1$. Therefore, $O(\mu)$ being open in $P_c(\mathbb{N})$ is also open in each $K(n)$, $n > p$. Since every open subset of $K(n)$ is of dimension $n - 1$, we obtain that $\dim O(\mu) > p - 1$, a contradiction.

Now, suppose $P_c(\mathbb{N})$ is metrizable and fix $\mu \in P_c(\mathbb{N})$. Since $P_c(\mathbb{N})$ is nowhere locally compact and $K(n)$, $n \geqslant 1$, are compact, $U(\mu) - K(n) \neq \emptyset$ for all $n \geqslant 1$ and all neighborhoods $U(\mu) \subset P_c(\mathbb{N})$ of μ. Using the last condition and the fact that μ has a countable local base (as a point in a metrizable space), we can construct a sequence $\{\mu_n\}_{n\geqslant1}$ converging to μ in $P_c(\mathbb{N})$ such that $\mu_n \notin K(n)$ for all n. Consequently, $s(\mu_n) \nsubseteq \{1, 2, \dots, n\}$, $n \geqslant 1$. To obtain a contradiction, we apply again [35, Proposition 3.1] to conclude that $s(\mu) \cup \bigcup_{n\geqslant1} s(\mu_n)$ is a bounded subset of $\mathbb{N}$ because $\{\mu, \mu_n\colon n \geqslant 1\}$ is a compact subset of $P_c(\mathbb{N})$. Therefore, $P_c(\mathbb{N})$ is not metrizable.

Let us complete the proof of (ii). If X is compact metrizable, then $P_c(X)$ is metrizable (see, for example [17]). Suppose $P_c(X)$ is metrizable. Then, by Proposition 2.3(i), X is also metrizable. If X is not compact, it should contain a C-embedded copy of $\mathbb{N}$ and, according to Proposition 2.3(ii), $P_c(X)$ should contain a copy of $P_c(\mathbb{N})$. So, $P_c(\mathbb{N})$ would be also metrizable, which is not possible. Therefore, X is compact and metrizable provided $P_c(X)$ is metrizable. □

Proposition 2.6. *If one of the spaces $P_c(X)$ and $P_c^*(X)$ is hereditarily Baire, then X is pseudocompact.*

Proof. We prove first that none of the spaces $P_c(\mathbb{N})$ and $P_c^*(\mathbb{N})$ has the Baire property. Indeed, this is true for $P_c(\mathbb{N})$ because it is the union of the compact sets $K(n)$, $n \geqslant 1$, and it is nowhere locally compact (see Claim 2 from Proposition 2.5). Suppose now that $P_c^*(\mathbb{N})$ is Baire. Since $P_c^*(\mathbb{N})$ is the union of the compact sets $K^*(n) = P_c^*(\{1, 2, \dots, n\})$, $n \geqslant 1$, there exists $m > 1$ such that $K^*(m)$ has a non-empty interior. Then $K(m) = P_c(\{1, 2, \dots, m\})$ has a non-empty interior in $P_c(\mathbb{N})$ because $K(m) = j_{\mathbb{N}}^{-1}(K^*(m))$. According to Claim 2, this is a contradiction.

If X is not pseudocompact, there exist a function $g \in C(X)$ and a discrete set $A = \{x_n\colon n \geqslant 1\}$ in X such that $g(x_n) \neq g(x_m)$ for $n \neq m$ and $g(A)$ is a discrete unbounded subset of $\mathbb{R}$. Since $g(A)$ is C-embedded in $\mathbb{R}$, it follows that

A is also C-embedded in X. So, A is a C-embedded copy of $\mathbb{N}$ in X. Then, by Proposition 2.3, $P_c(X)$ contains a closed copy of $P_c(\mathbb{N})$ and $P_c^*(X)$ contains a closed copy of $P_c^*(\mathbb{N})$. Since none of $P_c(\mathbb{N})$ and $P_c^*(\mathbb{N})$ has the Baire property, none of $P_c(X)$ and $P_c^*(X)$ can be hereditarily Baire. This completes the proof. □

Since every Čech-complete space is hereditarily Baire, we obtain the following corollary.

Corollary 2.7. *If one of the spaces $P_c(X)$ and $P_c^*(X)$ is Čech-complete, then X is pseudocompact.*

We say that an inverse system $S = \{X_\alpha, p_\beta^\alpha, A\}$ is *factorizing* [11] if for every $h \in C(X)$, where X is the limit space of S, there exist $\alpha \in A$ and $h_\alpha \in C(X_\alpha)$ with $h = h_\alpha \circ p_\alpha$. Here, $p_\alpha : X \to X_\alpha$ is the α-th limit projection. According to [9], P_c^* is a continuous functor, i.e. for every factorizing inverse system S the space $P_c^*(\lim S)$ is the limit of the inverse system $P_c^*(S) = \{P_c^*(X_\alpha), P_c^*(p_\beta^\alpha), A\}$. The same is true for the functor P_c.

Proposition 2.8. *P_c is a continuous functor.*

Proof. Let $S = \{X_\alpha, p_\beta^\alpha, A\}$ be a factorizing inverse system with a limit space X and let $\{\mu_\alpha\colon \alpha \in A\}$ be a thread of the system $P_c(S)$. For every $\alpha \in A$ we consider the measure $\nu_\alpha = j_{X_\alpha}(\mu_\alpha)$. Here, $j_{X_\alpha} : P_c(X_\alpha) \to P_c^*(X_\alpha)$ is the one-to-one surjection defined above. It is easily seen that $\{\nu_\alpha\colon \alpha \in A\}$ is a thread of the system $P_c^*(S)$, so it determines a unique measure $\nu \in P_c^*(X)$ (recall that P_c^* is a continuous functor). There exists a unique measure $\mu \in P_c(X)$ with $j_X(\mu) = \nu$. One can show that $P_c(p_\alpha)(\mu) = \mu_\alpha$ for all α. Hence, the set $P_c(X)$ coincides with the limit set of the system $P_c(S)$. It remains to show that for every $\mu^0 \in P_c(X)$ and its neighborhood U in $P_c(X)$ there exist $\alpha \in A$ and a neighborhood V of $\mu_\alpha^0 = P_c(p_\alpha)(\mu^0)$ in $P_c(X_\alpha)$ such that $P_c(p_\alpha)^{-1}(V) \subset U$. We can suppose that $U = \{\mu \in P_c(X)\colon |\mu(h_i) - \mu^0(h_i)| < \epsilon, i = 1, 2, \dots, k\}$ for some $\epsilon > 0$ and $h_i \in C(X)$, $i = 1, 2, \dots, k$. Since S is factorizing, we can find $\alpha \in A$ and functions $g_i \in C(X_\alpha)$ such that $h_i = g_i \circ p_\alpha$ for all $i = 1, \dots, k$. Then $V = \{\mu_\alpha \in P_c(X_\alpha)\colon |\mu_\alpha(g_i) - \mu_\alpha^0(g_i)| < \epsilon,\ i = 1, 2, \dots, k\}$ is the required neighborhood of μ_α^0. □

3. Milyutin maps and linear operators with compact supports

For every linear operator $u : C(X, E) \to C(Y, E)$, where E is a locally convex linear space, and $y \in Y$ there exists a linear map $T(y) : C(X, E) \to E$ defined by $T(y)(g) = u(g)(y)$, $g \in C(X, E)$. We say that u has *compact supports* (resp., u is *regular*) if each $T(y)$ has a compact support in X (resp., each $T(y)$ is regular). In a similar way we define a linear operator with compact supports if $u : C(X, E) \to C^*(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$ or $u : C^*(X, E) \to C(Y, E)$). Let us note that a linear map $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) is regular and has compact supports iff the formula

$$T(y)(g) = u(g)(y) \quad \text{with } g \in C(X, E) \ \big(\text{resp., } g \in C^*(X, E)\big) \tag{3}$$

produces a continuous map $T : Y \to P_c(X, E)$ (resp., $T : Y \to P_c^*(X, E)$). If $f : X \to Y$ is a surjective map, then a linear operator $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) is called *an averaging operator for f* if $u(\varphi \circ f) = \varphi$ for every $\varphi \in C(Y, E)$ (resp., $\varphi \in C^*(Y, E)$). It is easily seen that $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) is a regular averaging operator for f with compact supports if and only if the map $T : Y \to P_c(X, E)$ (resp., $T : Y \to P_c^*(X, E)$) defined by (3), has the following property: the support of every $T(y)$, $y \in Y$, is contained in $f^{-1}(y)$. Such a map T will be called *a map associated with f*. It is also clear that if $T : Y \to P_c(X, E)$ (resp., $T : Y \to P_c^*(X, E)$) is a map associated with f, then the equality (3) defines a regular averaging operator $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) for f with compact supports.

A surjective map $f : X \to Y$ is said to be *Milyutin* if f admits a regular averaging operator $u : C(X) \to C(Y)$ with compact supports, or equivalently, there exists a map $T : Y \to P_c(X)$ associated with f. A surjective map $f : X \to Y$ is called *weakly Milyutin* (resp., *strongly Milyutin*) if there exists a map $T : Y \to P_c^*(X)$ (resp., $T : P_c(Y) \to P_c(X)$) such that $s^*(T(y)) \subset f^{-1}(y)$ for all $y \in Y$ (resp., $s(T(\mu)) \subset f^{-1}(s(\mu))$ for all $\mu \in P_c(Y)$). Obviously, every strongly Milyutin map is Milyutin. Moreover, if $T : Y \to P_c(X)$ is a map associated with f, then the map $j_X \circ T : Y \to P_c^*(X)$ is witnessing that Milyutin maps are weakly Milyutin. One can also show that if $f : X \to Y$ is weakly Milyutin, then its Čech–Stone extension $\beta f : \beta X \to \beta Y$ is a Milyutin map.

We are going to establish some properties of (weakly) Milyutin maps.

Proposition 3.1. *Let $f : X \to Y$ be a weakly Milyutin map and E a complete locally convex space. Then f admits a regular averaging operator $u : C^*(X, E) \to C^*(Y, E)$ with compact supports.*

Proof. Let $T : Y \to P_c^*(X)$ be a map associated with f. For every $g \in C^*(X, E)$ let $B(g) = \overline{\operatorname{conv} g(X)}$ and consider the map $P_c^*(g) : P_c^*(X) \to P_c^*(B(g))$. Since $B(g)$ is a closed and bounded in E and E is complete, by [5, Theorem 3.4 and Proposition 3.10], there exists a continuous map $b : P_c^*(B(g)) \to B(g)$ assigning to each measure its barycenter. The composition $e(g) = b \circ P_c^*(g) : P_c^*(X) \to E$ is a continuous extension of g (we consider X as a subset of $P_c^*(X)$). Now, we define

$u : C^*(X, E) \to C^*(Y, E)$ by $u(g) = e(g) \circ T$. This a linear operator because $e(g)(\mu) = \int_X g\,d\mu$ for every $\mu \in P_c^*(X)$. Since $e(g)$ is a map from $P_c^*(X)$ into $B(g)$, the linear map $\Lambda(y) : C^*(X, E) \to E$, $\Lambda(y)(g) = u(g)(y)$, is regular for all $y \in Y$.

So, it remains to show that the support of each $\Lambda(y)$ is compact and it is contained in $f^{-1}(y)$. Because T is associated with f, $K(y) = s^*(T(y))$ is a compact subset of $f^{-1}(y)$, $y \in Y$. We are going to show that if $h|K(y) = g|K(y)$ with $h, g \in C^*(X, E)$, then $\Lambda(y)(h) = \Lambda(y)(g)$. That would imply the support of $\Lambda(y)$ is contained in $K(y) \subset f^{-1}(y)$, and hence it should be compact. To this end, observe that $T(y)$ can be considered as an element of $P(K(y))$ – the probability measures on $K(y)$. So, $T(y)$ is the limit of a net $\{\mu_\alpha\} \subset P(K(y))$ consisting of measures with finite supports. Each μ_α is of the form $\sum_{i=1}^{k(\alpha)} \lambda_i^\alpha \delta^*_{x_i^\alpha}$, where $x_i^\alpha \in K(y)$ and λ_i^α are positive reals with $\sum_{i=1}^{k(\alpha)} \lambda_i^\alpha = 1$. Then $\{e(g)(\mu_\alpha)\}$ converges to $e(g)(T(y))$ and $\{e(h)(\mu_\alpha)\}$ converges to $e(h)(T(y))$. On the other hand, $e(h)(\mu_\alpha) = \int_X h\,d\mu_\alpha = \sum_{i=1}^{k(\alpha)} \lambda_i^\alpha h(x_i^\alpha)$ and $e(g)(\mu_\alpha) = \sum_{i=1}^{k(\alpha)} \lambda_i^\alpha g(x_i^\alpha)$. Since $h|K(y) = g|K(y)$, $h(x_i^\alpha) = g(x_i^\alpha)$ for all α and i. Hence, $e(h)(T(y)) = e(g)(T(y))$ which means that $\Lambda(y)(h) = \Lambda(y)(g)$. Therefore, u is a regular averaging operator for f and has compact supports. □

Corollary 3.2. *Let X be a complete bounded convex subset of a locally convex space and $f : X \to Y$ be a weakly Milyutin map such that $f^{-1}(y)$ is convex for every $y \in Y$. Then there exists a map $g : Y \to X$ such that $g(y) \in f^{-1}(y)$ for all $y \in Y$.*

Proof. Let $T : Y \to P_c^*(X)$ be a map associated with f. By [5, Proposition 3.10], the barycenter $b(\mu)$ of each measure $\mu \in P_c^*(X)$ belongs to X and the map $b : P_c^*(X) \to X$ is continuous. Since the support of each $T(y)$, $y \in Y$, is compact subset of $f^{-1}(y)$ and $\overline{\operatorname{conv} s^*(T(y))} \subset f^{-1}(y)$ (recall that $f^{-1}(y)$ is convex), $b(T(y)) \in f^{-1}(y)$. So, the map $g = b \circ T$ is as required. □

Recall that a set-valued map $\Phi : X \to Y$ is lower semi-continuous (br., lsc) if for every open $U \subset Y$ the set $\Phi^{-1}(U) = \{x \in X\colon \Phi(x) \cap U \neq \emptyset\}$ is open in X.

Lemma 3.3. *For every space X and a linear space E the set-valued map $\Phi_X : P_c(X, E) \to X$ (resp., $\Phi_X^* : P_c^*(X, E) \to X$) defined by $\Phi_X(\mu) = s(\mu)$ (resp., $\Phi_X^*(\mu) = s^*(\mu)$) is lsc.*

Proof. A similar statement was established in [4, Lemma 1.2.7], so we omit the arguments. □

Proposition 3.4. *Let $f : X \to Y$ be a weakly Milyutin map. Then we have:*

(i) *$\beta f : \beta X \to \beta Y$ is a Milyutin map;*
(ii) *f is a Milyutin map provided f is perfect.*

Proof. Let $T : Y \to P_c^*(X)$ be a map associated with f. To prove (i), observe that $P_c^*(i) : P_c^*(X) \to P_c(\beta X)$ is an embedding, where $i : X \to \beta X$ is the standard embedding (see Proposition 2.3(iii)). Because $P_c(\beta X) = P(\beta X)$ is compact, we can extend T to a map $\tilde{T} : \beta Y \to P(\beta X)$. It suffices to show that $\tilde{T}$ is a map associated with βf. To this end, consider the lsc map $\Phi = \beta f \circ \Phi_{\beta X} \circ \tilde{T} : \beta Y \to \beta Y$. Since Φ is lsc and $\Phi(y) = y$ for all $y \in Y$, $\Phi(y) = y$ for any $y \in \beta Y$. This means that the support of any $\tilde{T}(y)$, $y \in \beta Y$, is contained in $(\beta f)^{-1}(y)$. So, βf is a Milyutin map.

The proof of (ii) follows from (i) and the following result of Choban [12, Proposition 1.1]: if βf admits a regular averaging operator and f is perfect, then f admits a regular averaging operator $u : C(X) \to C(Y)$ such that

$$\inf\{h(x)\colon x \in f^{-1}(y)\} \leqslant u(h)(y) \leqslant \sup\{h(x)\colon x \in f^{-1}(y)\}$$

for every $h \in C(X)$ and $y \in Y$. This implies that the support of each linear map $T(y) : C(X) \to \mathbb{R}$, $y \in Y$, defined by (3), is contained in $f^{-1}(y)$. Hence, $s(T(y))$ is compact because so is $f^{-1}(y)$ (recall that f is perfect). Therefore, f is a Milyutin map. □

Proposition 3.5. *Let $f : X \to Y$ be a Milyutin map. Then, in each of the following cases f is strongly Milyutin*: (i) *$f^{-1}(K)$ is compact for every compact set $K \subset Y$*; (ii) *every closed and bounded subset of X is compact.*

Proof. Let $u : C(X) \to C(Y)$, $u(h)(y) = g(y)(h)$, be a corresponding regular averaging operator with compact supports, where $g : Y \to P_c(X)$ is a map associated with f. We are going to extend g to a map $\tilde{g} : P_c(Y) \to P_c(X)$ such that $s(\tilde{g}(\mu)) \subset f^{-1}(s(\mu))$ for all $\mu \in P_c(Y)$. Let $\mu \in P_c(Y)$ and $K = s(\mu) \subset Y$. Then $g(K)$ is a compact subset of $P_c(X)$. Hence, by [35, Proposition 3.1], $H = \overline{\bigcup\{s(g(y))\colon y \in K\}}$ is a bounded and closed subset of X. Since $s(g(y)) \subset f^{-1}(y)$ for all $y \in Y$, $H \subset f^{-1}(K)$. So, in each of the cases (i) and (ii), H is compact. Define $\tilde{g}(\mu) : C(X) \to \mathbb{R}$ to be the linear functional $\tilde{g}(\mu)(h) = \mu(u(h))$, $h \in C(X)$. One can check that $\tilde{g}(\mu)(h) = 0$ provided $h(H) = 0$. This means that the support of $\tilde{g}(\mu)$ is a compact subset of H, so $\tilde{g}(\mu) \in P_c(X)$. Moreover, $\tilde{g}$, considered as a map from $P_c(Y)$ to $P_c(X)$ is continuous and satisfies the inclusions $s(\tilde{g}(\mu)) \subset f^{-1}(s(\mu))$, $\mu \in P_c(Y)$. Therefore, f is strongly Milyutin. □

A map $f : X \to Y$ is said to be 0-*invertible* [20] if for any space Z with $\dim Z = 0$ and any map $p : Z \to Y$ there exists a map $q : Z \to X$ such that $f \circ q = p$. Here, $\dim Z = 0$ means that $\dim \beta Z = 0$. We say that $f : X \to Y$ has *a metrizable kernel* if there exist a metrizable space M and an embedding $X \subset Y \times M$ such that $\pi_Y|X = f$, where $\pi_Y : Y \times M \to Y$ is the projection.

Next theorem is a generalization of [13, Theorem 3.4] and [20, Corollary 1].

Theorem 3.6. *Let $f : X \to Y$ be a surjection with a metrizable kernel and Y a paracompact space. Then the following conditions are equivalent*:

(i) *f is (weakly) Milyutin*;
(ii) *The set-valued map $f^{-1} : Y \to X$ admits a lsc compact-valued selection*;
(iii) *f is 0-invertible.*

Proof. (i) ⇒ (ii) Let f be weakly Milyutin and $T : Y \to P_c^*(X)$ is a map associated with f. By Lemma 3.3, the map $\Phi_X^* : P_c^*(X) \to X$ is lsc, so is the map $\Phi_X^* \circ T$. Moreover, $\Phi_X^*(T(y)) = s^*(T(y)) \subset f^{-1}(y)$ for all $y \in Y$. Hence, $\Phi_X^* \circ T$ is a compact-valued selection of f^{-1}.

(ii) ⇒ (iii) Suppose M is a metrizable space such that $X \subset Y \times M$ and $\pi_Y|X = f$. Suppose also that f^{-1} admits a compact-valued lsc selection $\Phi : Y \to X$. To show that f is 0-invertible, take a map $p : Z \to Y$ with $\dim Z = 0$, and let $Z_1 = (\beta p)^{-1}(Y)$. Then Z_1 is paracompact (as a perfect preimage of Y) and $\dim Z_1 = 0$ because $\beta Z_1 = \beta Z$ is 0-dimensional. The set-valued map $\pi_M \circ \Phi \circ p_1 : Z_1 \to M$ is lsc and compact-valued, where $\pi_M : Y \times M \to M$ is the projection and $p_1 = (\beta p)|Z_1$. According to [23], $\pi_M \circ \Phi \circ p_1$ admits a (single-valued) continuous selection $q_1 : Z_1 \to M$. Finally, the map $q : Z \to X$, $q(z) = (p(z), q_1(z))$ is the required lifting of p, i.e. $f \circ q = p$.

(iii) ⇒ (i) By [28], there exists a perfect weakly Milyutin map $p : Z \to Y$ with Z being a 0-dimensional paracompact. Then, by Proposition 3.4(ii), p is a Milyutin map. Since f is 0-invertible, there exists a map $g : Z \to X$ with $f \circ g = p$. If $T : Y \to P_c(Z)$ is a map associated with p, then $\tilde{T} = P_c(g) \circ T : Y \to P_c(X)$ is a map associated with f because $s(\tilde{T}(y)) \subset g(p^{-1}(y)) \subset f^{-1}(y)$ for all $y \in Y$. Hence, f is a Milyutin map. □

Corollary 3.7. *Let $f : X \to Y$ be a surjective map such that either X and Y are metrizable or f is perfect. Then the following are equivalent*: (i) *f is weakly Milyutin*; (ii) *f is Milyutin*; (iii) *f is strongly Milyutin.*

Proof. If X and Y are metrizable, this follows from Proposition 3.5 and Theorem 3.6. In case f is perfect, we apply Propositions 3.4 and 3.5. □

A space Z is called *a $k_{\mathbb{R}}$-space* if every function on Z is continuous provided it is continuous on every compact subset of Z.

Theorem 3.8. *The product f of any family $\{f_\alpha : X_\alpha \to Y_\alpha,\ \alpha \in A\}$ of weakly Milyutin maps is also weakly Milyutin. If, in addition, $Y = \prod\{Y_\alpha\colon\ \alpha \in A\}$ is a $k_{\mathbb{R}}$-space and for every $\alpha \in A$ the closed and bounded subsets of X_α are compact, then f is Milyutin provided each f_α is Milyutin.*

Proof. Let $T_\alpha : Y_\alpha \to P_c^*(X_\alpha)$ be a map associated with f_α for each α. Then, by Proposition 3.4, βf_α is a Milyutin map and $\beta T_\alpha : \beta Y_\alpha \to P(\beta X_\alpha)$ is associated with βf_α. So, $u_\alpha : C(\beta X_\alpha) \to C(\beta Y_\alpha)$, $u_\alpha(h)(y) = \beta T_\alpha(y)(h)$, $y \in \beta Y_\alpha$ and $h \in C(\beta X_\alpha)$, is a regular averaging operator for βf_α. Let $X = \prod\{X_\alpha\colon\ \alpha \in A\}$, $\tilde{X} = \prod\{\beta X_\alpha\colon\ \alpha \in A\}$, $\tilde{Y} = \prod\{\beta Y_\alpha\colon\ \alpha \in A\}$ and $\tilde{f} = \prod\{\beta f_\alpha\colon\ \alpha \in A\}$. According to [26], there exists a regular averaging operator $u : C(\tilde{X}) \to C(\tilde{Y})$ for $\tilde{f}$ such that $u(h \circ p_\alpha) = u_\alpha(h) \circ q_\alpha$, $\alpha \in A$, $h \in C(\beta X_\alpha)$, where $p_\alpha : \tilde{X} \to \beta X_\alpha$ and $q_\alpha : \tilde{Y} \to \beta Y_\alpha$ are the projections. This implies that, if $\tilde{T} : \tilde{Y} \to P(\tilde{X})$ is the map associated to $\tilde{f}$ and generated by u, we have $s(\tilde{T}(y)) \subset \prod\{s(T_\alpha(q_\alpha(y)))\colon\ \alpha \in A\}$, $y \in Y$. Hence, $s(\tilde{T}(y)) \subset f^{-1}(y)$ for every $y \in Y$. So, $\tilde{T}$ maps Y into the subspace H of $P(\tilde{X})$ consisting of all measures $\mu \in P(\tilde{X})$ with $s(\mu) \subset X$. Now, let $\pi : \beta X \to \tilde{X}$ be the natural map and $P(\pi) : P(\beta X) \to P(\tilde{X})$. Then, $\theta = P(\pi)|P_c^*(X) : P_c^*(X) \to H$ is a homeomorphism (for more general result see [9, Proposition 1]). Therefore, $T = \theta^{-1} \circ (\tilde{T}|Y) : Y \to P_c^*(X)$ is a map associated with f. Thus, f is weakly Milyutin.

Suppose now that Y is a $k_{\mathbb{R}}$-space, f_α are Milyutin maps and the closed and bounded subsets of each X_α are compact. We already proved that there exists a regular averaging operator $u : C^*(X) \to C^*(Y)$ for f and a corresponding to u map $T : Y \to P_c^*(X)$ associated with f such that $s^*(T(y)) \subset \prod\{s(T_\alpha(q_\alpha(y)))\colon\ \alpha \in A\} \subset f^{-1}(y)$ for every $y \in Y$. Here, each $T_\alpha : Y_\alpha \to P_c(X_\alpha)$ is a map associated with f_α (recall that f_α are Milyutin maps). For any $h \in C(X)$ and $n \geqslant 1$ define $h_n \in C^*(X)$ by $h_n(x) = h(x)$ if $|h(x)| \leqslant n$, $h_n(x) = n$ if $h(x) \geqslant n$ and $h_n(x) = -n$ if $h(x) \leqslant -n$. Since for every $y \in Y$ the support $s^*(T(y)) \subset X$ is compact, $h|s^*(T(y)) = h_n|s^*(T(y))$ with $n \geqslant n_0$ for some n_0. Hence, the formula $v(h)(y) = \lim u(h_n)(y)$, $y \in Y$, defines a function on Y. Let us show that $v(h)$ is continuous. Since Y is a $k_{\mathbb{R}}$-space, it suffices to prove that $v(h)$ is continuous on every compact set $K \subset Y$. Then each of the sets $T_\alpha(K_\alpha) \subset P_c(X_\alpha)$ is compact, where $K_\alpha = q_\alpha(K)$. By [35, Proposition 3.1], $Z_\alpha = \overline{\bigcup\{s(\mu)\colon\ \mu \in T_\alpha(K_\alpha)\}}$ is bounded in X_α and, hence compact (recall that all closed and bounded subsets of X_α are compact). Let Z be the closure in X of the set $\bigcup\{s^*(\mu)\colon\ \mu \in T(K)\}$. Since $Z \subset \prod\{Z_\alpha\colon\ \alpha \in A\}$, Z is also compact. So, there exists m such that $h|Z = h_n|Z$ for all $n \geqslant m$. This implies that $v(h)|K = u(h_m)|K$. Hence, $v(h)$ is continuous on K. Since for every $y \in Y$ the support of $T(y)$ is compact and each $u(h)(y)$, $h \in C^*(X)$, depends on $h|s^*(T(y))$, $v : C(X) \to C(Y)$ is linear and the support of $T'(y) \in P_c(X)$ is contained in $s^*(T(y)) \subset f^{-1}(y)$, where $T' : Y \to P_c(X)$ is defined by $T'(y)(h) = v(h)(y)$, $h \in C(X)$, $y \in Y$. Moreover, it follows from the definition of v that it is regular and $v(\phi \circ f) = \phi$ for every $\phi \in C(Y)$. Therefore, v is a regular averaging operator for f with compact supports. □

Corollary 3.9. *A product of perfect Milyutin maps is also Milyutin.*

Proof. Since any product of perfect maps is perfect, the proof follows from Corollary 3.7 and Theorem 3.8. □

Corollary 3.10. *Let $Y = \prod\{Y_\alpha\colon \alpha \in A\}$ be a product of metrizable spaces. Then there exists a* 0*-dimensional product X of metrizable spaces space and a* 0*-invertible perfect Milyutin map $f : X \to Y$.*

Proof. By [12, Theorem 1.2.1], for every $\alpha \in A$ there exist a 0-dimensional metrizable space X_α and a perfect Milyutin map $f_\alpha : X_\alpha \to Y_\alpha$. Then, by Corollary 3.9, $f = \prod\{f_\alpha\colon \alpha \in A\}$ is a perfect Milyutin map from $X = \prod\{X_\alpha\colon \alpha \in A\}$ onto Y. It is easily seen that f is 0-invertible because each f_α is 0-invertible (see Theorem 3.6). Moreover, since $\dim X_\alpha = 0$ for each α, $\dim X = 0$. □

Recall that X is a p-paracompact space [2] if it admits a perfect map onto a metrizable space.

Corollary 3.11. *For every p-paracompact space Y there exist a* 0*-dimensional p-paracompact space X and a perfect* 0*-invertible Milyutin map $f : X \to Y$.*

Proof. Since Y is p-paracompact, it can be considered as a closed subset of $M \times \mathbb{I}^\tau$, where M is metrizable and $\tau \geqslant \aleph_0$. There exist perfect Milyutin maps $\phi : \mathfrak{C} \to \mathbb{I}$ and $g : M_0 \to M$ with $\mathfrak{C}$ being the Cantor set [26] and M_0 a 0-dimensional metrizable space. [12, Theorem 1.2.1]. Then the product map $\Phi = g \times \phi^\tau : M_0 \times \mathfrak{C}^\tau$ is a perfect 0-invertible Milyutin map (see Corollary 3.10), and let $T : M \times \mathbb{I}^\tau \to P_c(M_0 \times \mathfrak{C}^\tau)$ be a map associated with Φ. Define $X = \Phi^{-1}(Y)$ and $f = \Phi|X$. Since X is closed in $M_0 \times \mathfrak{C}^\tau$, it is a 0-dimensional p-paracompact. Since Φ is 0-invertible (as a product of 0-invertible maps, see Theorem 3.6), so is f. To show that f is Milyutin, observe that X is C-embedded in $M_0 \times \mathfrak{C}^\tau$. So, $P_c(X)$ is embedded in $P_c(M_0 \times \mathfrak{C}^\tau)$ such that $T(y) \in P_c(X)$ for all $y \in Y$. This means that $T|Y$ is a map associated with f. Hence, f is Milyutin. □

Now, we provide a specific class of Milyutin maps. Suppose $B \subset Z$ and $g : B \to D$. We say that g is *a Z-normal map* provided for every $h \in C(D)$ the function $h \circ g$ can be continuously extended to a function on Z. A map $f : X \to Y$ is called 0*-soft* [10] if for any 0-dimensional space Z, any two subspaces $Z_0 \subset Z_1 \subset Z$, and any Z-normal maps $g_0 : Z_0 \to X$ and $g_1 : Z_1 \to Y$ with $f \circ g_0 = g_1|Z_0$, there exists a Z-normal map $g : Z_1 \to X$ such that $f \circ g = g_1$.

Proposition 3.12. *Every* 0*-soft map is Milyutin.*

Proof. Let $f : X \to Y$ be 0-soft. Consider Y as a C-embedded subset of $\mathbb{R}^{C(Y)}$ and let $\varphi : Z \to \mathbb{R}^{C(Y)}$ be a perfect Milyutin map with $\dim Z = 0$ (see Corollary 3.10). Since Y is C-embedded in $\mathbb{R}^{C(Y)}$, $g_1 = \varphi|Z_1 : Z_1 \to Y$ is a Z-normal map, where $Z_1 = \varphi^{-1}(Y)$. Because f is 0-soft, there exists a Z-normal map $g : Z_1 \to X$ with $f \circ g = g_1$. Now, for every $h \in C(X)$ choose an extension $e(h) \in C(Z)$ of $h \circ g$ (such $e(h)$ exist since g is Z-normal). Define $v : C(X) \to C(Y)$ by $v(h) = u(e(h))|Y$, where $u : C(Z) \to C(\mathbb{R}^{C(Y)})$ is a regular averaging operator for φ having compact supports. The map v is linear because for every $y \in Y$ $u(e(h))(y)$ depends on the restriction $e(h)|\varphi^{-1}(y)$. By the same reason v has compact supports. Moreover, v is a regular averaging operator for f. Hence, f is Milyutin. □

4. *AE*(0)-spaces and regular extension operators with compact supports

Let X be a subspace of Y. A linear operator $u : C(X, E) \to C(Y, E)$ is said to be *an extension operator* provided each $u(f)$, $f \in C(X, E)$ is an extension of f. One can show that such an extension operator u is regular and has compact supports if and only if there exists a map $T : Y \to P_c(X, E)$ such that $T(x) = \delta_x$ for every $x \in X$. Sometimes a map $T : Y \to P_c(X, E)$ satisfying the last condition will be called *a P_c-valued retraction*. The connection between u and T is given by the formula $T(y)(f) = u(f)(y)$, $f \in C(X, E)$, $y \in Y$.

Pelczynski [26] introduced the class of Dugundji spaces: a compactum X is a *Dugundji space* if for every embedding of X in another compact space Y there exists an extension regular operator $u : C(X) \to C(Y)$ (note that u has compact supports because X is compact). Later Haydon [19] proved that a compact space X is a Dugundji space if and only if it is an absolute extensor for 0-dimensional compact spaces (br., $X \in AE(0)$). The notion of $X \in AE(0)$ was extended by Chigogidze [10] in the class of all Tychonoff spaces as follows: a *space X is an $AE(0)$* if for every 0-dimensional space Z and its subspace $Z_0 \subset Z$, every Z-normal map $g : Z_0 \to X$ can be extended to the whole of Z.

We show that an analogue of Haydon's result remains true and for the extended class of *AE*(0)-spaces.

Theorem 4.1. *For any space X the following conditions are equivalent*:

(i) *X is an $AE(0)$-space*;
(ii) *For every C-embedding of X in a space Y there exists a regular extension operator $u : C(X) \to C(Y)$ with compact supports*;
(iii) *For every C-embedding of X in a space Y there exists a regular extension operator $u : C^*(X) \to C^*(Y)$ with compact supports.*

Proof. (i) $\Rightarrow$ (ii) Suppose X is C-embedded in Y and take a set A such that Y is C-embedded in $\mathbb{R}^A$. It suffices to show there exists a regular extension operator $u : C(X) \to C(\mathbb{R}^A)$ with compact supports, or equivalently, we can find a map $T : \mathbb{R}^A \to P_c(X)$ with $T(x) = \delta_x$ for all $x \in X$. By Corollary 3.10, there exist a 0-dimensional space Z and a Milyutin map $f : Z \to \mathbb{R}^A$. This means that the map $g : \mathbb{R}^A \to P_c(Z)$ associated with f is an embedding. Since X is C-embedded in $\mathbb{R}^A$, the restriction $f|f^{-1}(X)$ is a Z-normal map. So, there exists a map $q : Z \to X$ extending $f|f^{-1}(X)$ (recall that $X \in AE(0)$). Then $T = P_c(q) \circ g : \mathbb{R}^A \to P_c(X)$ has the required property that $T(x) = \delta_x$ for all $x \in X$.

(ii) $\Rightarrow$ (iii) Let X be C-embedded in Y and $u : C(X) \to C(Y)$ a regular extension operator with compact supports. Then $u(f) \in C^*(Y)$ for all $f \in C^*(X)$ because u is regular. Hence, $u|C^*(X) : C^*(X) \to C^*(Y)$ is a regular extension operator with compact supports.

(iii) $\Rightarrow$ (i) Suppose X is C-embedded in $\mathbb{R}^A$ for some A and $u : C^*(X) \to C^*(\mathbb{R}^A)$ is a regular extension operator with compact supports. So, there exists a map $T : \mathbb{R}^A \to P_c^*(X)$ with $T(x) = \delta_x$, $x \in X$. Assume that A is the set of all ordinals $\{\lambda\colon \lambda < \omega(\tau)\}$, where $\omega(\tau)$ is the first ordinal of cardinality τ.

For any sets $B \subset D \subset A$ we use the following notations: $\pi_B : \mathbb{R}^A \to \mathbb{R}^B$ and $\pi^D_B : \mathbb{R}^D \to \mathbb{R}^B$ are the natural projections, $X(B) = \pi_B(X)$, $p_B = \pi_B|X$ and $p^D_B = \pi^D_B|X(D)$. A set $B \subset A$ is called *T-admissible* if for any $x \in X$ and $y \in \mathbb{R}^A$ the equality $\pi_B(x) = \pi_B(y)$ implies $P_c^*(p_B)(\delta_x) = P_c^*(p_B)(T(y))$. Let us note that if B is T-admissible, then there exists a map

$$T_B : \mathbb{R}^B \to P_c^*\big(X(B)\big) \quad \text{such that} \quad T_B(z) = \delta_z \quad \text{for all } z \in X(B). \tag{4}$$

Indeed, take an embedding $i : \mathbb{R}^B \to \mathbb{R}^A$ such that $\pi_B \circ i$ is the identity on $\mathbb{R}^B$, and define $T_B = P_c^*(p_B) \circ T \circ i$.

Claim 3. *For every countable set $B \subset A$ there exists a countable T-admissible set $D \subset A$ containing B.*

We construct by induction an increasing sequence $\{D(n)\}_{n\geqslant 1}$ of countable subsets of A such that $D \subset D(1)$ and for all $n \geqslant 1$, $x \in X$ and $y \in \mathbb{R}^A$ we have

$$P_c^*(p_{D(n)})(\delta_x) = P_c^*(p_{D(n)})\big(T(y)\big) \quad \text{provided } \pi_{D(n+1)}(x) = \pi_{D(n+1)}(y). \tag{5}$$

Suppose we have already constructed $D(1), \dots, D(n)$. Since $D(n)$ is countable, the topological weight of $X(D(n))$ is $\aleph_0$. So is the weight of $P_c^*(X(D(n)))$ [9]. Then the map $P_c^*(p_{D(n)}) \circ T : \mathbb{R}^A \to P_c^*(X(D(n)))$ depends on countable many coordinates (see, for example [27]). This means that there exists a countable set $D(n+1)$ satisfying (5). We can assume that $D(n+1)$ contains $D(n)$, which completes the induction. Obviously, the set $D = \bigcup_{n\geqslant 1} D(n)$ is countable. Let us show it is T-admissible. Suppose $\pi_D(x) = \pi_D(y)$ for some $x \in X$ and $y \in \mathbb{R}^A$. Hence, for every $n \geqslant 1$ we have $\pi_{D(n+1)}(x) = \pi_{D(n+1)}(y)$ and, by (5), $P_c^*(p_{D(n)})(\delta_x) = P_c^*(p_{D(n)})(T(y))$. This means that the support of each measure $P_c^*(p_{D(n)})(T(y))$ is the point $p_{D(n)}(x)$. The last relation implies that the support of $P_c^*(p_D)(T(y))$ is the point $p_D(x)$. Therefore, $P_c^*(p_D)(T(y)) = P_c^*(p_D)(\delta_x)$ and D is T-admissible.

Claim 4. *Any union of T-admissible sets is T-admissible.*

Suppose B is the union of T-admissible sets $B(s)$, $s \in S$, and $\pi_B(x) = \pi_B(y)$ with $x \in X$ and $y \in \mathbb{R}^A$. Then $\pi_{B(s)}(x) = \pi_{B(s)}(y)$ for every $s \in S$. Hence, $P_c^*(p_{B(s)})(T(y)) = P_c^*(p_{B(s)})(\delta_x)$, $s \in S$. So, the support of each $P_c^*(p_{B(s)})(T(y))$ is the point $p_{B(s)}(x)$. Consequently, the support of $P_c^*(p_B)(T(y))$ is the point $p_B(x)$ because $p_B(x) = \bigcap\{(p^B_{B(s)})^{-1}(p_{B(s)}(x))\colon s \in S\}$. This means that B is T-admissible.

Claim 5. *Let $B \subset A$ be T-admissible. Then we have*:

(a) *$X(B)$ is a closed subset of $\mathbb{R}^B$*;
(b) *$p_B(V)$ is functionally open in $X(B)$ for any functionally open subset V of X.*

Since B is T-admissible, according to (4) there exists a map $T_B : \mathbb{R}^B \to P_c^*(X(B))$ such that $T_B(z) = \delta_z$ for all $z \in X(B)$. To prove condition (a), suppose $\{z_\alpha\colon \alpha \in \Lambda\}$ is a net in $X(B)$ converging to some $z \in \mathbb{R}^B$. Then $\{T_B(z_\alpha)\}$ converges to $T_B(z)$. But $T_B(z_\alpha) = \delta_{z_\alpha} \in i^*_{X(B)}(X(B))$ for every α and, since $i^*_{X(B)}(X(B))$ is a closed subset of $P_c^*(X(B))$ (see Proposition 2.3(i)), $T_B(z) \in i^*_{X(B)}(X(B))$. Hence, $T_B(z) = \delta_y$ for some $y \in X(B)$. Using that $i^*_{X(B)}$ embeds $X(B)$ in $P_c^*(X(B))$, we obtain that $\{z_\alpha\}$ converges to y, so $y = z \in X(B)$.

To prove (b), let V be a functionally open subset of X and $g : X \to [0,1]$ a continuous function with $V = g^{-1}((0,1])$. Then $u(g) \in C^*(\mathbb{R}^A)$ with $0 \leqslant u(g)(y) \leqslant 1$ for all $y \in \mathbb{R}^A$ and let $W = u(g)^{-1}((0,1])$. Since $\pi_B(W)$ is functionally open in $\mathbb{R}^B$

(see, for example [34]), $\pi_B(W) \cap X(B)$ is functionally open in $X(B)$. So, it suffices to show that $p_B(V) = \pi_B(W) \cap X(B)$. Because $u(g)$ extends g, we have $V \subset W$. So, $p_B(V) \subset \pi_B(W) \cap X(B)$. To prove the other inclusion, let $z \in \pi_B(W) \cap X(B)$. Choose $x \in X$ and $y \in W$ with $\pi_B(x) = \pi_B(y)$. Then $P_c^*(p_B)(T(y)) = P_c^*(p_B)(\delta_x) = \delta_z$ (recall that B is T-admissible). Hence, $s^*(T(y)) \subset p_B^{-1}(z)$. Since $y \in W$, $T(y)(g) = u(g)(y) \in (0, 1]$. This implies that $s^*(T(y)) \cap V \neq \emptyset$ (otherwise $T(y)(g) = 0$ because $g(X - V) = 0$, see Proposition 2.1(ii)). Therefore, $z \in p_B(V)$, i.e. $\pi_B(W) \cap X(B) \subset p_B(V)$. The proof of Claim 5 is completed.

Let us continue the proof of (iii) ⇒ (i). Since A is the set of all ordinals $\lambda < \omega(\tau)$, according to Claim 3, for every λ there exists a countable T-admissible set $B(\lambda) \subset A$ containing λ. Let $A(\lambda) = \bigcup\{B(\eta)\colon \eta < \lambda\}$ if λ is a limit ordinal, and $A(\lambda) = \bigcup\{B(\eta)\colon \eta \leqslant \lambda\}$ otherwise. By Claim 4, every $A(\lambda)$ is T-admissible. We are going to use the following simplified notations:

$$X_\lambda = X\big(A(\lambda)\big), \qquad p_\lambda = p_{A(\lambda)} : X \to X_\lambda \quad \text{and} \quad p_\lambda^\eta : X\eta \to X_\lambda \quad \text{provided } \lambda < \eta.$$

Since A is the union of all $A(\lambda)$ and each X_λ is closed in $\mathbb{R}^{A(\lambda)}$ (see Claim 5(a)), we obtain a continuous inverse system $S = \{X_\lambda, p_\lambda^\eta,\ \lambda < \eta < \omega(\tau)\}$ whose limit space is X. Recall that S is continuous if for every limit ordinal γ the space X_γ is the limit of the inverse system $\{X_\lambda, p_\lambda^\eta,\ \lambda < \eta < \gamma\}$. Because of the continuity of S, $X \in AE(0)$ provided $X_1 \in AE(0)$ and each short projection $p_\lambda^{\lambda+1}$ is 0-soft. The space X_1 being a closed subset of $\mathbb{R}^{A(1)}$ is a Polish space, so an $AE(0)$ [10]. Hence, it remains to show that all $p_\lambda^{\lambda+1}$ are 0-soft.

We fix $\lambda < \omega(\tau)$ and let $E(\lambda) = A(\lambda) \cap (B(\lambda) \cup B(\lambda + 1))$. Since $E(\lambda)$ is countable, there exists a sequence $\{\beta_n\} \subset A(\lambda)$ such that $\beta_n \leqslant \lambda$ for each n and $E(\lambda) \subset C(\lambda) \subset A(\lambda)$, where $C(\lambda) = \bigcup\{B(\beta_n)\colon n \geqslant 1\}$. By Claim 4, the sets $C(\lambda)$ and $D(\lambda) = B(\lambda) \cup B(\lambda + 1) \cup C(\lambda)$ are countable and T-admissible. Consider the following diagram:

$$\begin{array}{ccc} X_{\lambda+1} & \xrightarrow{\ p_\lambda^{\lambda+1}\ } & X_\lambda \\ {\scriptstyle p_{D(\lambda)}^{A(\lambda+1)}}\big\downarrow & & \big\downarrow{\scriptstyle p_{C(\lambda)}^{A(\lambda)}} \\ X(D(\lambda)) & \xrightarrow{\ p_{C(\lambda)}^{D(\lambda)}\ } & X(C(\lambda)). \end{array}$$

We are going to prove first that the diagram is a cartesian square. This means that the map $g : X_{\lambda+1} \to Z$, $g(x) = (p_{D(\lambda)}^{A(\lambda+1)}(x), p_\lambda^{\lambda+1}(x))$, is a homeomorphism. Here $Z = \{(x_1, x_2) \in X(D(\lambda)) \times X_\lambda\colon p_{C(\lambda)}^{D(\lambda)}(x_1) = p_{C(\lambda)}^{A(\lambda)}(x_2)\}$ is the fibered product of $X(D(\lambda))$ and X_λ with respect to the maps $p_{C(\lambda)}^{D(\lambda)}$ and $p_{C(\lambda)}^{A(\lambda)}$. Let $z = (x(1), x(2)) \in Z$. Since $(D(\lambda) - C(\lambda)) \cap (A(\lambda) - C(\lambda)) = \emptyset$ and $A(\lambda + 1) = (D(\lambda) - C(\lambda)) \cup (A(\lambda) - C(\lambda)) \cup C(\lambda)$, there exists exactly one point $x \in \mathbb{R}^{A(\lambda+1)}$ such that $\pi_{D(\lambda)}^{A(\lambda+1)}(x) = x(1)$ and $\pi_{A(\lambda)}^{A(\lambda+1)}(x) = x(2)$. Choose $y \in \mathbb{R}^A$ with $\pi_{A(\lambda+1)}(y) = x$. Since $D(\lambda)$ and $A(\lambda)$ are T-admissible, $P_c^*(p_{D(\lambda)})(T(y)) = \delta_{x(1)}$ and $P_c^*(p_{A(\lambda)})(T(y)) = \delta_{x(2)}$. Consequently, $p_{D(\lambda)}^{A(\lambda+1)}(H) = x(1)$ and $p_{A(\lambda)}^{A(\lambda+1)}(H) = x(2)$, where H is the support of the measure $P_c^*(p_{A(\lambda+1)})(T(y))$. Hence, $H = \{x\}$ is the unique point of $X_{\lambda+1}$ with $g(x) = z$. Thus, g is a surjective and one-to-one map between $X_{\lambda+1}$ and Z. To prove g is a homeomorphism, it remains to show that g^{-1} is continuous. The above arguments yield that $x = g^{-1}(z)$ depends continuously from $z \in Z$. Indeed, since $D(\lambda) \cap A(\lambda) = C(\lambda)$, we have

$$x(1) = (a, b) \in \mathbb{R}^{D(\lambda)-C(\lambda)} \times \mathbb{R}^{C(\lambda)} \quad \text{and} \quad x(2) = (b, c) \in \mathbb{R}^{C(\lambda)} \times \mathbb{R}^{A(\lambda)-C(\lambda)},$$

where $z = (x(1), x(2)) \in Z$. Hence, $g^{-1}(z) = (a, b, c)$ is a continuous function of z.

Since $D(\lambda)$ and $C(\lambda)$ are countable and T-admissible sets, both $X(D(\lambda))$ and $X(C(\lambda))$ are Polish spaces and $p_{C(\lambda)}^{D(\lambda)}$ is functionally open (see Claim 5(b)). Hence, $p_{C(\lambda)}^{D(\lambda)}$ is 0-soft [10]. This yields that $p_\lambda^{\lambda+1}$ is also 0-soft because the above diagram is a cartesian square. □

Next proposition provides a characterization of $AE(0)$-spaces in terms of extension of vector-valued functions. This result was inspired by [7].

Theorem 4.2. *A space $X \in AE(0)$ if and only if for any complete locally convex space E and any C-embedding of X in a space Y there exists a regular extension operator: $C^*(X, E) \to C^*(Y, E)$ with compact supports.*

Proof. Suppose $X \in AE(0)$ and X is C-embedded in a space Y. Then by Theorem 4.1(iii), there exists a regular extension operator $v : C^*(X) \to C^*(Y)$ with compact supports. This is equivalent to the existence of a P_c^*-valued retraction $T : Y \to P_c^*(X)$. We can extend each $f \in C^*(X, E)$ to a continuous bounded map $e(f) : P_c^*(X) \to E$. Indeed, let $B(f) = \overline{\operatorname{conv} f(X)}$ and consider the map $P_c^*(f) : P_c^*(X) \to P_c^*(B(f))$. Obviously, $B(f)$ is a bounded convex closed subset of E, so it is complete. Then, by [5, Theorem 3.4 and Proposition 3.10], there exists a continuous map $b : P_c^*(B(f)) \to B(f)$ assigning to each measure $\nu \in P_c^*(B(f))$ its barycenter $b(\nu)$. The composition $e(f) = b \circ P_c^*(f) : P_c^*(X) \to B(f)$ is a bounded continuous extension of f. We also have

$$e(f)(\mu) = \int_X f \, d\mu \quad \text{for every } \mu \in P_c^*(X). \tag{6}$$

Finally, we define $u : C^*(X, E) \to C^*(Y, E)$ by $u(f) = e(f) \circ T$, $f \in C^*(X, E)$. The linearity of u follows from (6). Moreover, for every $y \in Y$ the linear map $\Lambda(y) : C^*(X, E) \to E$, $\Lambda(y)(f) = u(f)(y)$, is regular because $\Lambda(y)(f) \in \overline{\operatorname{conv} f(X)}$. Using the arguments from the proof of Proposition 3.1 (the final part), we can show that each $\Lambda(y)$, $y \in Y$, has a compact support which is contained in $K(y) = s^*(T(y)) \subset X$. Therefore, u is a regular extension operator with compact supports.

The other implication follows from Theorem 4.1. Indeed, since $\mathbb{R}$ is complete, there exists a regular extension operator $u : C^*(X) \to C^*(Y)$ provided X is C-embedded in Y. Hence, by Theorem 4.1(iii), $X \in AE(0)$. □

Recall that a space X is an absolute retract [10] if for every C-embedding of X in a space Y there exists a retraction from Y onto X.

Corollary 4.3. *Let X be a convex bounded and complete subset of a locally convex topological space. Then X is an absolute retract provided $X \in AE(0)$.*

Proof. Suppose X is C-embedded in a space Y. According to [5, Theorem 3.4 and Proposition 3.10], the barycenter of each $\mu \in P_c(X)$ belongs to X and the map $b : P_c(X) \to X$ is continuous. Since $X \in AE(0)$, by Theorem 4.1, there exists a P_c-valued retraction $T : Y \to P_c(X)$. Then $r = b \circ T : Y \to X$ is a retraction. □

Lemma 4.4. *Let $X \subset Y$ and $u : C(X) \to C(Y)$ be a regular extension operator with compact supports. Suppose every closed bounded subset of X is compact. Then there exists a map $T_c : P_c(Y) \to P_c(X)$ (resp., $T_c^* : P_c^*(Y) \to P_c^*(X)$) such that $P_c(i) \circ T_c$ (resp., $P_c^*(i) \circ T_c^*$) is a retraction, where $i : X \to Y$ is the embedding of X into Y.*

Proof. For every $\mu \in P_c(Y)$ define $T_c(\mu) : C(X) \to \mathbb{R}$ by $T_c(\mu)(f) = \mu(u(f))$, $f \in C(X)$. Obviously, each $T_c(\mu)$ is linear. Let us show that $T_c(\mu) \in P_c(X)$ for all $\mu \in P_c(Y)$. Since u has compact supports, the map $T : Y \to P_c(X)$ generated by u is continuous. Hence, $T(s(\mu))$ is a compact subset of $P_c(X)$ (recall that $s(\mu) \subset Y$ is compact). Then by [3] (see also [35, Proposition 3.1]), $H(\mu) = \overline{\bigcup \{s(T(y)) : \ y \in s(\mu)\}}$ is closed and bounded in X, and hence compact. Let us show that the support of $T_c(\mu)$ is compact. That will be done if we prove that $s(T_c(\mu)) \subset H(\mu)$. To this end, let $f(H(\mu)) = 0$ for some $f \in C(X)$. Consequently, $T(y)(f) = 0$ for all $y \in s(\mu)$. So, $u(f)(s(\mu)) = 0$. The last equality means that $T_c(\mu)(f) = 0$. Hence, each $T_c(\mu)$ has a compact support and T_c is a map from $P_c(Y)$ to $P_c(X)$. It is easily seen that $P_c(i)(T_c(\mu)) = \mu$ for all $\mu \in P_c(i)(P_c(X))$. Therefore, $P_c(i) \circ T_c$ is a retraction from $P_c(Y)$ onto $P_c(i)(P_c(X))$.

Now, we consider the linear operators $T_c^*(\nu) : C^*(X) \to \mathbb{R}$, $T_c^*(\nu)(h) = \nu(u(h))$ with $\nu \in P_c^*(Y)$ and $h \in C^*(X)$. Observed that $u(h) \in C^*(Y)$ for $h \in C^*(X)$ because u is a regular operator, so the above definition is correct. To show that T_c^* is a map from $P_c^*(Y)$ to $P_c^*(X)$, for every $\nu \in P_c^*(Y)$ take the unique $\mu \in P_c(Y)$ with $j_Y(\mu) = \nu$. Then $s(\mu) = s^*(\nu)$ according to Proposition 2.1. Hence, $T_c^*(\nu)(h) = 0$ provided $h \in C^*(X)$ with $h|s(T_c(\mu)) = 0$. So, the support of $T_c^*(\nu)$ is contained in $s(T_c(\mu))$. This means that T_c^* maps $P_c^*(Y)$ into $P_c^*(X)$. Moreover, one can show that $P_c^*(i) \circ T_c^*$ is a retraction. □

Ditor and Haydon [14] proved that if X is a compact space, then $P(X)$ is an absolute retract if and only if X is a Dugundji space of weight $\leqslant \aleph_1$. A similar result concerning the space of all σ-additive probability measures was established by Banakh, Chigogidze, and Fedorchuk [6]. Next theorem shows that the same is true when $P_c(X)$ or $P_c^*(X)$ is an AR.

Theorem 4.5. *For a space X the following are equivalent*:

(i) *$P_c(X)$ (resp., $P_c^*(X)$) is an absolute retract*;
(ii) *$P_c(X)$ (resp., $P_c^*(X)$) is an $AE(0)$*;
(iii) *X is a Dugundji space of weight $\leqslant \aleph_1$.*

Proof. (i) ⇒ (ii) This implication is trivial because every AR is an $AE(0)$.

(ii) ⇒ (iii) It suffices to show that X is compact. Indeed, then both $P_c(X)$ and $P_c^*(X)$ are $AE(0)$ and coincide with $P(X)$. So, by Corollary 4.3, $P(X)$ is an AR. Applying the mentioned above result of Ditor–Haydon, we obtain that X is a Dugundji space of weight $\leqslant \aleph_1$.

Suppose X is not compact. Since $P_c(X)$ (resp., $P_c^*(X)$) is an $AE(0)$-space, it is realcompact. Hence, so is X as a closed subset of $P_c(X)$ (resp., $P_c^*(X)$). Consequently, X is not pseudocompact (otherwise it would be compact), and there exists a closed C-embedded subset Y of X homeomorphic to $\mathbb{N}$ (see the proof of Proposition 2.6). Since Y is an $AE(0)$, according to Theorem 4.1, there exists a regular extension operator $u : C(Y) \to C(X)$ with compact supports. Then, by Lemma 4.4, $P_c(Y)$ (resp., $P_c^*(Y)$) is homeomorphic to a retract of $P_c(X)$ (resp., $P_c^*(X)$). Hence, one of the spaces $P_c(Y)$ and $P_c^*(Y)$ is an $AE(0)$ (as a retract of an $AE(0)$-space). Suppose $P_c^*(Y) \in AE(0)$. Since $P_c^*(Y)$ is second countable, this implies $P_c^*(Y)$ is Čech-complete. Hence, by Corollary 2.7, Y is pseudocompact, a contradiction. If $P_c(Y) \in AE(0)$, then $P_c(Y)$ is metrizable according to a result of Chigogidze [10] stating that every $AE(0)$-space whose points are G_δ-sets is metrizable (the points of $P_c(Y)$ are G_δ because $j_Y : P_c(Y) \to P_c^*(Y)$ is an one-to-one surjection and $P_c^*(Y)$ is metrizable). But by Proposition 2.5(ii), $P_c(Y)$ is metrizable only if Y is compact and metrizable. So, we have again a contradiction.

(iii) ⇒ (i) This implication follows from the stated above result of Ditor and Haydon [14]. □

5. Properties preserved by Milyutin maps

In this section we show that some topological properties are preserved under Milyutin maps. Let $\mathfrak{F}$ be a family of closed subsets of X. We say that X *is collectionwise normal with respect to* $\mathfrak{F}$ if for every discrete family $\{F_\alpha\colon \alpha \in A\} \subset \mathfrak{F}$ there exists a discrete family $\{V_\alpha\colon \alpha \in A\}$ of open in X sets with $F_\alpha \subset V_\alpha$ for each $\alpha \in A$. When X is collectionwise normal with respect to the family of all closed subsets, it is called collectionwise normal.

Theorem 5.1. *Every weakly Milyutin map preserves paracompactness and collectionwise normality.*

Proof. Let $f : X \to Y$ be a weakly Milyutin map and $u : C^*(X) \to C^*(Y)$ a regular averaging operator for f with compact supports.

Suppose X is collectionwise normal, and let $\{F_\alpha\colon \alpha \in A\}$ be a discrete family of closed sets in Y. Then $\{f^{-1}(F_\alpha)\colon \alpha \in A\}$ is a discrete collection of closed sets in X. So, there exists a discrete family $\{V_\alpha\colon \alpha \in A\}$ of open sets in X with $f^{-1}(F_\alpha) \subset V_\alpha$, $\alpha \in A$. Let $V_0 = X - \bigcup\{f^{-1}(F_\alpha)\colon \alpha \in A\}$ and $\gamma = \{V_\alpha\colon \alpha \in A\} \cup \{V_0\}$. Since γ is a locally finite open cover of X and X is normal (as collectionwise normal), there exists a partition of unity $\xi = \{h_\alpha\colon \alpha \in A\} \cup \{h_0\}$ on X subordinated to γ such that $h_\alpha(f^{-1}(F_\alpha)) = 1$ for every α. Observe that $h_{\alpha(1)}(x) + h_{\alpha(2)}(x) \leqslant 1$ for any $\alpha(1) \neq \alpha(2)$ and any $x \in X$. So, $u(h_{\alpha(1)})(y) + u(h_{\alpha(2)})(y) \leqslant 1$ for all $y \in Y$. This yields that $\{u(h_\alpha)^{-1}((1/2, 1])\colon \alpha \in A\}$ is a disjoint open family in Y. Moreover, $F_\alpha \subset u(h_\alpha)^{-1}((1/2, 1])$ for every α. Therefore, Y is collectionwise normal (see [16, Theorem 5.1.17]).

Let X be paracompact and ω an open cover of Y. So, there exists a locally finite open cover γ of X which an index-refinement of $f^{-1}(\omega)$. Let ξ be a partition of unity on X subordinated to γ. It is easily seen that $u(\xi)$ is a partition of unity on Y subordinated to ω. Hence, by [24], Y is paracompact. □

Corollary 5.2. *Let $f : X \to Y$ be a weakly Milyutin map and X a* (*completely*) *metrizable space. Then Y is also* (*completely*) *metrizable.*

Proof. Let $T : Y \to P_c^*(X)$ be a map associated with f. Then $\phi = \Phi_X^* \circ T : Y \to X$ is a lsc compact-valued map (see Lemma 3.3 for the map Φ_X^*) such that $\phi(y) \subset f^{-1}(y)$ for every $y \in Y$. Since Y is paracompact (by Theorem 4.1), we can apply Michael's selection theorem [25] to find an upper semi-continuous (br., usc) compact-valued selection $\psi : Y \to X$ for ϕ (recall that ψ is usc provided the set $\{y \in Y\colon \psi(y) \cap F \neq \emptyset\}$ is closed in Y for every closed $F \subset X$). Then $f|X_1 : X_1 \to Y$ is a perfect surjection, where $X_1 = \bigcup\{\psi(y)\colon y \in Y\}$. Hence, Y is metrizable as a perfect image of a metrizable space.

If X is completely metrizable, then so is Y. Indeed, by [1, Theorem 1.2], there exists a closed subset $X_0 \subset X$ such that $f|X_0 : X_0 \to X$ is an open surjection. Then Y is complete (as a metric space being an open image of a complete metric space). □

Proposition 5.3. *Let $f : X \to Y$ be a weakly Milyutin map with X being a product of metrizable spaces. Then we have*:

(i) *The closure of any family of G_δ-sets in Y is a zero-set in Y*;
(ii) *Y is collectionwise normal with respect to the family of all closed G_δ-sets in Y.*

Proof. Let $X = \prod\{X_\gamma\colon \gamma \in \Gamma\}$, where each X_γ is metrizable. Suppose $u : C^*(X) \to C^*(Y)$ is a regular averaging operator for f with compact supports.

(i) Let G be a union of G_δ-sets in Y. Then so is $f^{-1}(G)$ in X and, by [22, Corollary], there exists $h \in C^*(X)$ with $h^{-1}(0) = \overline{f^{-1}(G)}$. Since $h(T(y)) = 0$ for each $y \in G$, $u(h)(G) = 0$. On the other hand, $\inf\{h(x)\colon x \in T(y)\} > 0$ for $y \notin \overline{G}$. Hence, $u(h)(y) > 0$ for any $y \notin \overline{G}$. Consequently, $u(h)^{-1}(0) = \overline{G}$.

(ii) Let $\{F_\alpha\colon \alpha \in A\}$ be a discrete family of closed G_δ-sets in Y. Then so is the family $\{H_\alpha = f^{-1}(F_\alpha)\colon \alpha \in A\}$ in X. Moreover, by (i), each F_α is a zero-set in Y, hence H_α is a zero-set in X.

We can assume that Γ is uncountable (otherwise Y is metrizable and the proof is trivial). Consider the Σ-product $\Sigma(a)$ of all X_γ with a base-point $a \in X$. Since $\Sigma(a)$ is G_δ-dense in X (i.e., every G_δ-subset of X meets $\Sigma(a)$), $\Sigma(a)$ is C-embedded in X [32] and

$$H_\alpha = \overline{H_\alpha \cap \Sigma(a)} \quad \text{for any } \alpha. \tag{7}$$

Because $\Sigma(a)$ is collectionwise normal [18], there exists a discrete family $\{W_\alpha\colon \alpha \in A\}$ of open subsets of $\Sigma(a)$ such that $H_\alpha \cap \Sigma(a) \subset W_\alpha$, $\alpha \in A$. Let $W_0 = \Sigma(a) - \bigcup\{H_\alpha \cap \Sigma(a)\colon \alpha \in A\}$. Choose a partition of unity $\{h_\alpha\colon \alpha \in A\} \cup \{h_0\}$ in $\Sigma(a)$ subordinated to the locally finite cover $\{W_\alpha\colon \alpha \in A\} \cup \{W_0\}$ of $\Sigma(a)$ such that $h_\alpha(H_\alpha \cap \Sigma(a)) = 1$ for each α. Since $\Sigma(a)$ is

C-embedded in X, each h_α can be extended to a function g_α on X. Because of (7), $g_\alpha(H_\alpha)=1$, $\alpha\in A$. The density of $\Sigma(a)$ in X implies that $g_{\alpha(1)}(x)+g_{\alpha(2)}(x)\leqslant 1$ for any $\alpha(1)\neq\alpha(2)$ and any $x\in X$. As in the proof of Theorem 5.1, this implies that $F_\alpha\subset U_\alpha=u(g_\alpha)^{-1}((1/2,1])$ and the family $\{U_\alpha\colon\ \alpha\in A\}$ is disjoint. Then, as in the proof of [16, Theorem 5.1.17], there exists a discrete family $\{V_\alpha\colon\ \alpha\in A\}$ of open subsets of Y with $F_\alpha\subset V_\alpha$, $\alpha\in A$. □

A space X is called *κ-metrizable* [29] if there exists a *κ-metric on* X, i.e., a non-negative real-valued function d on $X\times\mathcal{RC}(X)$, where $\mathcal{RC}(X)$ denotes the family of all regularly closed subset of X (i.e., closed sets $F\subset X$ with $F=\overline{int_X(F)}$) satisfying the following conditions:

(K1) $d(x,F)=0$ iff $x\in F$ for every $x\in X$ and $F\in\mathcal{RC}(X)$;

(K2) $F_1\subset F_2$ implies $d(x,F_2)\leqslant d(x,F_1)$ for every $x\in X$;

(K3) $d(x,F)$ is continuous with respect to x for every $F\in\mathcal{RC}(X)$;

(K4) $d(x,\overline{\bigcup\{F_\alpha\colon\ \alpha\in A\}})=\inf\{d(x,F_\alpha)\colon\ \alpha\in A\}$ for every $x\in X$ and every increasing linearly ordered by inclusion family $\{F_\alpha\}_{\alpha\in A}\subset\mathcal{RC}(X)$.

If $\mathcal{K}(X)$ is a family of closed subsets of X, then a function $d: X\times\mathcal{K}(X)\to\mathcal{R}$ satisfying conditions (K1)–(K3) with $\mathcal{RC}(X)$ replaced by $\mathcal{K}(X)$ is called *a monotone continuous annihilator* of the family $\mathcal{K}(X)$ [15]. When $\mathcal{K}(X)$ consists of all zero sets in X, then any monotone continuous annihilator is said to be a *δ-metric on* X [15]. The well-known notion of stratifiability [8] can be express as follows: X is stratifiable iff there exists a monotone continuous annihilator on X for the family of all closed subsets of X.

A space X is perfectly κ-normal [30] provided every $F\in\mathcal{RC}(X)$ is a zero-set in X.

Theorem 5.4. *Every weakly Milyutin map $f: X\to Y$ preserves the following properties: stratifiability, δ-metrizability, and perfectly κ-normality. If, in addition, $cl_X(f^{-1}(U))=f^{-1}(cl_Y(U))$ for every open $U\subset Y$, then f preserves κ-metrizability.*

Proof. We consider only the case when f satisfies the additional condition which is denoted by (s) (the proof of the other cases is similar). Let $u: C^*(X)\to C^*(Y)$ be a regular averaging operator for f having compact supports, and $d(x,F)$ be a κ-metric on X. We may assume that $d(x,F)\leqslant 1$ for any $x\in X$ and $F\in\mathcal{RC}(X)$, see [29]. Let $F_G=cl_X(f^{-1}(int_Y(G)))$ for each $G\in\mathcal{RC}(Y)$, and define $h_G(x)=d(x,F_G)$. Consider the function $\rho: Y\times\mathcal{RC}(Y)\to\mathbb{R}$, $\rho(y,G)=u(h_G)(y)$. We are going to check that ρ is a κ-metric on Y.

Suppose $G(1),G(2)\in\mathcal{RC}(Y)$ and $G(1)\subset G(2)$. Then $F_{G(1)}\subset F_{G(2)}$, so $h_{G(2)}\leqslant h_{G(1)}$. Consequently, $\rho(y,G(2))\leqslant\rho(y,G(1))$ for any $y\in Y$. On the other hand, obviously, $\rho(y,G)$ is continuous with respect to y for every $G\in\mathcal{RC}(Y)$. Hence, ρ satisfies conditions (K2) and (K3).

Suppose $G\in\mathcal{RC}(Y)$. Then $s^*(T(y))\subset f^{-1}(y)\subset F_G$ for every $y\in int_Y(G)$, where $T: Y\to P_c^*(X)$ is the associated map to f generated by u. Consequently, $h_G|s^*(T(y))=0$ which implies $u(h_G)(y)=0$, $y\in int_Y(G)$. On the other hand, if $y\notin G$, then $s^*(T(y))\cap F_G=\emptyset$ and $h_G(x)>0$ for all $x\in s^*(T(y))$. Since $u(h_G)(y)\geqslant\inf\{h_G(x)\colon\ x\in s^*(T(y))\}$ (recall that u is an averaging operator for f), $u(h_G)(y)>0$. Hence, $u(h_G)(y)=\rho(y,G)=0$ iff $y\in G$, so ρ satisfies condition (K1).

To check condition (K4), suppose $\{G(\alpha)\colon\ \alpha\in A\}\subset\mathcal{RC}(Y)$ is an increasing linearly ordered by inclusion family and $G=cl_Y(\bigcup\{G(\alpha)\colon\ \alpha\in A\})$. Using that f satisfies condition (ac), we have $F_G=cl_X(\bigcup\{F_{G(\alpha)}\colon\ \alpha\in A\})$. Since $\{F_{G(\alpha)}\colon\ \alpha\in A\}$ is also increasing and linearly ordered by inclusion, according to condition (K4), $h_G(x)=\inf\{h_{G(\alpha)}(x)\colon\ \alpha\in A\}$ for every $x\in X$. Let $y\in Y$ and $\epsilon>0$. Then for every $x\in X$ there exists $\alpha_x\in A$ such that $h_{G(\alpha_x)}(x)<h_G(x)+\epsilon$. Choose a neighborhood $V(x)$ of x in X such that $h_{G(\alpha_x)}(z)<h_G(z)+\epsilon$ for all $z\in V(x)$. Since $s^*(T(y))$ is compact, it can be covered by finitely many $V(x(i))$, $i=1,\dots,n$, with $x(i)\in s^*(T(y))$. Let $\beta=\max\{\alpha_{x(i)}\colon\ i\leqslant n\}$. Then $h_{G(\beta)}(x)<h_G(x)+\epsilon$ for all $x\in s^*(T(y))$. The last equality yields $\rho(y,G(\beta))\leqslant\rho(y,G)+\epsilon$ because $u(h_{G(\beta)})(y)$ and $u(h_G)(y)$ depend only on the restrictions $h_{G(\beta)}|s^*(T(y))$ and $h_G|s^*(T(y))$, respectively. Thus, $\inf\{\rho(y,G(\alpha))\colon\ \alpha\in A\}\leqslant\rho(y,G)$. The inequality $\rho(y,G)\leqslant\inf\{\rho(y,G(\alpha))\colon\ \alpha\in A\}$ is obvious because G contains each $G(\alpha)$, so ρ satisfies condition (K4). Therefore, Y is κ-metrizable. □

Next corollary provides a positive answer to a question of Shchepin [31].

Corollary 5.5. *Every $AE(0)$-space is κ-metrizable.*

Proof. Let X be an $AE(0)$-space of weight τ. By [10, Theorem 4], there exists a surjective 0-soft map $f:\mathbb{N}^\tau\to X$. Since $\mathbb{N}^\tau\in AE(0)$ (as a product of $AE(0)$-spaces) and every 0-soft map between $AE(0)$-spaces is functionally open [10, Theorem 1.15], f satisfies condition (s) from the previous theorem. On the other hand, $\mathbb{N}^\tau$ is κ-metrizable as a product of metrizable spaces [29, Theorem 15]. Hence, the proof follows from Proposition 3.12 and Theorem 5.4. □

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