

KRASINKIEWICZ SPACES AND PARAMETRIC KRASINKIEWICZ MAPS

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ABSTRACT. We say that a metrizable space M is a Krasinkiewicz space if any map from a metrizable compactum X into M can be approximated by Krasinkiewicz maps (a map $g: X \rightarrow M$ is Krasinkiewicz provided every continuum in X is either contained in a fiber of g or contains a component of a fiber of g). In this paper we establish the following property of Krasinkiewicz spaces: Let $f: X \rightarrow Y$ be a perfect map between metrizable spaces and M a Krasinkiewicz complete ANR -space. If Y is a countable union of closed finite-dimensional subsets, then the function space $C(X, M)$ with the source limitation topology contains a dense G_δ -subset of maps g such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are Krasinkiewicz maps. The same conclusion remains true if M is homeomorphic to a closed convex subset of a Banach space and Y is a C -space.

1. INTRODUCTION

All spaces in the paper are assumed to be metrizable and all maps continuous. Unless stated otherwise, any function space $C(X, M)$ is endowed with the *source limitation topology*. This topology, known also as the *fine topology*, was introduced by Whitney [14] and has a base at a given $f \in C(X, M)$ consisting of the sets

$$B_\varrho(f, \varepsilon) = \{g \in C(X, M) : \varrho(g, f) < \varepsilon\},$$

where ϱ is a fixed compatible metric on M and $\varepsilon: X \rightarrow (0, 1]$ runs over continuous functions into $(0, 1]$. The symbol $\varrho(f, g) < \varepsilon$ means that $\varrho(f(x), g(x)) < \varepsilon(x)$ for

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all $x \in X$. The source limitation topology doesn't depend on the metric ρ [5] and has the Baire property provided M is completely metrizable [9]. Obviously, this topology coincides with the uniform convergence topology when X is compact.

Based on paper [4], we introduce the following notion. A space M is said to be a *Krasinkiewicz space* if for any compactum X the function space $C(X, M)$ contains a dense subset of Krasinkiewicz maps. Recall that a map $g: X \rightarrow M$, where X is compact, is said to be Krasinkiewicz [6] if every continuum in X is either contained in a fiber of g or contains a component of a fiber of g . Krasinkiewicz [4] proved that every 1-manifold is a Krasinkiewicz space (for the unit interval \mathbb{I} this was established by Levin-Lewis [6]). The first author, generalizing the Krasinkiewicz result, proved in [7] that all compact polyhedra, as well as all 1-dimensional Peano continua and manifolds modeled on a Menger cube are Krasinkiewicz spaces.

The main results in this paper is the following theorem:

Theorem 1.1. *Let M be a Krasinkiewicz complete ANR-space and $f: X \rightarrow Y$ a perfect map with Y being a strongly countable-dimensional space. Then the function space $C(X, M)$ contains a dense G_δ -set of maps g such that all restrictions $g|_{f^{-1}(y)}$, $y \in Y$, are Krasinkiewicz maps. Moreover, if in addition M is a closed convex subset of a Banach space, then the same conclusion remains true provided Y is a C -space.*

Recall that X is a C -space if for any sequence $\{\nu_n\}_{n=1}^\infty$ of open covers of X there exists a sequence $\{\gamma_n\}_{n=1}^\infty$ of disjoint open families in X such that each γ_n refines ν_n and $\cup_{n=1}^\infty \gamma_n$ is a cover of X . Every strongly countable-dimensional space (i.e. a space which is a union of countably many closed finite-dimensional subsets), as well as every countable-dimensional space (a countable union of 0-dimensional subsets) is a C -space [2] and there exists a compact C -space which is not countable-dimensional.

In what follows, by a *polyhedron* we mean the underlying space of a simplicial complex equipped with the metric topology. A compactum is called a *Bing space* if each of its subcontinua is hereditarily indecomposable. According to Corollary 3.2, each polyhedron is a Krasinkiewicz space. Moreover, it follows from [11] that for any polyhedron P without isolated points and a compactum X the space $C(X, P)$ contains a dense set of *Bing maps* (maps g such that all fibers $g^{-1}(y)$, $y \in P$, are Bing spaces). Therefore, Theorem 1.1 and [13, Theorem 1.1] imply the following corollary:

Corollary 1.2. *Let P be a complete polyhedron without isolated points and $f: X \rightarrow Y$ a perfect map. Then the function space $C(X, P)$ contains a dense G_δ -set of maps g such that all restrictions $g|_{f^{-1}(y)}$, $y \in Y$, are both Bing and Krasinkiewicz*

maps in each of the following cases: (i) Y is strongly countable-dimensional; (ii) Y is a C -space and P is a closed convex subset of a Banach space.

Most part of the paper is devoted to the proof of Theorem 1.1, given in Section 2. In Section 3 we provide some properties of Krasinkiewicz spaces. For example, we show that a complete ANR is a Krasinkiewicz space if and only if it has an open cover of Krasinkiewicz subspaces. In particular, all n -manifolds, $n \geq 1$, are Krasinkiewicz spaces.

2. PROOF OF THEOREM 1.1

We fixed a metric d on X and for every $A \subset X$ and $\delta > 0$ let $B(A, \delta) = \{x \in X : d(x, A) < \delta\}$. If $y \in Y$ and $m, n \geq 1$, then $\mathcal{K}(m, n, y)$ denotes the set of all maps $g \in C(X, M)$ satisfying the following condition:

- For each subcontinuum $L \subset f^{-1}(y)$ with $\text{diam}g(L) \geq 1/n$ there exists $x \in L$ such that $C(x, g|f^{-1}(y)) \subset B(L, 1/m)$. Here, $g|f^{-1}(y)$ is the restriction of g over $f^{-1}(y)$ and $C(x, g|f^{-1}(y))$ denotes the component of the fiber $g^{-1}(g(x)) \cap f^{-1}(y)$ of $g|f^{-1}(y)$ containing x .

For $H \subset Y$ let $\mathcal{K}(m, n, H)$ be the intersection of all $\mathcal{K}(m, n, y)$, $y \in H$. We also denote by $\mathcal{K}(H)$ the set of all maps $g \in C(X, M)$ such that $g|f^{-1}(y) : f^{-1}(y) \rightarrow M$ is a Krasinkiewicz map for each $y \in H$.

Proposition 2.1. $\mathcal{K}(H) = \bigcap_{m, n \in \mathbb{N}} K(m, n, H)$.

PROOF. Obviously $\mathcal{K}(H) \subset \bigcap_{m, n \in \mathbb{N}} K(m, n, H)$. So, we need to prove the inclusion $\bigcap_{m, n \in \mathbb{N}} K(m, n, H) \subset \mathcal{K}(H)$. Let $g \in \bigcap_{m, n \in \mathbb{N}} K(m, n, H)$, $y \in H$ and $L \subset f^{-1}(y)$ be a subcontinuum such that $\text{diam}g(L) > 0$. We are going to prove that there exists a subcontinuum $L_2 \subset L_1 = L$ such that $\text{diam}g(L_2) > 0$ and $C(x, g|f^{-1}(y)) \subset B(L_1, 1/2)$ for each $x \in L_2$. Since $\text{diam}g(L_1) > 0$, there exists $n_1 \in \mathbb{N}$ such that $\text{diam}g(L_1) \geq 1/n_1$. Since $g \in K(2, n_1, y)$, there exists a point $x \in L_1$ such that $C(x, g|f^{-1}(y)) \subset B(L_1, 1/2)$. Let $E \subset L_1$ be the set of all such points. It is easy to see that:

- (#) every $x \in E$ has a neighborhood U_x in L_1 with $C(z, g|f^{-1}(y)) \subset B(L_1, 1/2)$ for all $z \in U_x$.

Let $x_0 \in E$ and \mathcal{D} be the family of all subcontinua D of L_1 such that $x_0 \in D$ and $C(d, g|f^{-1}(y)) \subset B(L_1, 1/2)$ for each $d \in D$. Since $\{x_0\} \in \mathcal{D}$, $\mathcal{D} \neq \emptyset$.

Claim. There exists $D^* \in \mathcal{D}$ such that $\text{diam}g(D^*) > 0$.

Assume that $g(D)$ is a singleton for each $D \in \mathcal{D}$. Then $\text{cl}(\bigcup \mathcal{D}) \in \mathcal{D}$. In fact, if $d, d' \in \text{cl}(\bigcup \mathcal{D})$ then $C(d, g|f^{-1}(y)) = C(d', g|f^{-1}(y))$ (note that $g(\text{cl}(\bigcup \mathcal{D}))$ is

a singleton). Hence $C(d, g|f^{-1}(y)) \subset B(L_1, 1/2)$ for each $d \in \text{cl}(\bigcup \mathcal{D})$, and this implies $\text{cl}(\bigcup \mathcal{D}) \in \mathcal{D}$. Then $\text{cl}(\bigcup \mathcal{D})$ is a maximal element of \mathcal{D} . If $\text{cl}(\bigcup \mathcal{D}) \neq L_1$, then by (#) there exists a proper subcontinuum $D' \subset L_1$ such that D' contains $\text{cl}(\bigcup \mathcal{D})$ as a proper subcontinuum of D' and $C(d, g|f^{-1}(y)) \subset B(L_1, 1/2)$ for each $d \in D'$. But this contradicts the fact that $\text{cl}(\bigcup \mathcal{D})$ is a maximal element of \mathcal{D} . So $\text{cl}(\bigcup \mathcal{D}) = L_1$. But this is a contradiction because $\text{diam}g(L_1) > 0$ and $g(\text{cl}(\bigcup \mathcal{D}))$ is a singleton. So there exists $D^* \in \mathcal{D}$ such that $\text{diam}g(D^*) > 0$. This completes the proof of claim.

Let $L_2 = D^*$. Then L_2 has the required property. By induction, we can find a decreasing sequence $\{L_k\}_{k=1}^\infty$ of subcontinua of L such that for any $k \in \mathbb{N}$ we have

- (*) $\text{diam}g(L_k) > 0$;
- (**) $C(x, g|f^{-1}(y)) \subset B(L_k, 1/(k+1))$ for each $x \in L_{k+1}$.

It is easy to see that $C(x, g|f^{-1}(y)) \subset L$ for each $x \in \bigcap_{k=1}^\infty L_k$. This implies $g \in \mathcal{K}(H)$, which completes the proof. \square

Obviously, if $Y = \bigcup_{m=1}^\infty Y_m$, $\mathcal{K}(Y) = \bigcap_{i,m=1}^\infty \mathcal{K}(Y_m)$. Therefore, according to Proposition 2.1, it suffices to show that $\mathcal{K}(m, n, H)$ is open and dense in $C(X, M)$ with respect to the source limitation topology for $m, n \geq 1$ and any closed $H \subset Y$ in the following cases: (i) H is finite-dimensional and M a Krasinkiewicz ANR-space; (ii) H is a C -space and M a Krasinkiewicz space homeomorphic to a closed convex subset of a Banach space.

In both of the above two cases we follow the scheme from the proof of [13, Theorem 1.1]. In particular, we need the following lemma established in [13, Lemma 2.1].

Lemma 2.2. [13] *Every complete ANR-space M' admits a complete metric ϱ generating its topology satisfying the following condition: If Z is a paracompact space, $A \subset Z$ a closed set and $\varphi: Z \rightarrow M'$ a map, then for every function $\alpha: Z \rightarrow (0, 1]$ and every map $g: A \rightarrow M'$ with $\varrho(g(z), \varphi(z)) < \alpha(z)/8$ for all $z \in A$, there exists a map $\bar{g}: Z \rightarrow M'$ extending g such that $\varrho(\bar{g}(z), \varphi(z)) < \alpha(z)$ for all $z \in Z$.*

2.1. Proof that $\mathcal{K}(m, n, H)$ is open in $C(X, M)$ for any $m, n \geq 1$ and any closed $H \subset Y$. In this subsection we prove that all sets $\mathcal{K}(m, n, H)$ are open in $C(X, M)$, where (M, ϱ) is a complete metric (not necessarily an ANR or a Krasinkiewicz) space.

Lemma 2.3. *Let $g \in \mathcal{K}(m, n, y)$ for some $y \in Y$ and $m, n \geq 1$. Then there exists a neighborhood V_y of y in Y and $\delta_y > 0$ such that $y' \in V_y$ and $\varrho(g_1(x), g(x)) < \delta_y$ for all $x \in f^{-1}(y')$ yields $g_1 \in \mathcal{K}(m, n, y')$.*

PROOF. Indeed, otherwise we can find a local base $\{V_k\}_{k \in \mathbb{N}}$ of neighborhoods of y in Y , points $y_k \in V_k$ and maps $g_k \in C(X, M)$ such that $\varrho(g_k(x), g(x)) < 1/k$ for all $x \in f^{-1}(y_k)$ but g_k does not belong to $\mathcal{K}(m, n, y_k)$. Consequently, for every k there exists a continuum $F_k \subset f^{-1}(y_k)$ such that $\text{diam}g_k(F_k) \geq 1/n$ and $C(x, g_k|f^{-1}(y_k))$ is not a subset of $B(F_k, 1/m)$ for any $x \in F_k$. Then all F_k are contained in the compact set $P = f^{-1}(\{y_k\}_{k \in \mathbb{N}} \cup \{y\})$. We may assume that $\{F_k\}_{k \in \mathbb{N}}$ converges to a continuum F . It follows that $F \subset f^{-1}(y)$ and $\text{diam}g(F) \geq 1/n$. Since $g \in \mathcal{K}(m, n, y)$ there exists $t \in F$ such that $C(t, g|f^{-1}(y)) \subset B(F, 1/m)$. Since $\lim F_k = F$, for each k there exists $t_k \in F_k$ with $\lim t_k = t$. We may assume that $\{C(t_k, g_k|f^{-1}(y_k))\}_{k \in \mathbb{N}}$ converges to a continuum C . Note that $C \subset C(t, g|f^{-1}(y))$. Since $C(t_k, g_k|f^{-1}(y_k)) \setminus B(F_k, 1/m) \neq \emptyset$, it is easy to see that C is not contained in $B(F, 1/m)$. This is a contradiction. \square

Now, we are in a position to show that the sets $\mathcal{K}(m, n, H)$ are open in $C(X, M)$.

Proposition 2.4. *For any closed $H \subset Y$ and any $m, n \geq 1$, the set $\mathcal{K}(m, n, H)$ is open in $C(X, M)$ with respect to the source limitation topology.*

PROOF. Let $g_0 \in \mathcal{K}(m, n, H)$. Then, by Lemma 2.3, for every $y \in H$ there exist a neighborhood V_y and a positive $\delta_y \leq 1$ such that $g \in \mathcal{K}(m, n, y')$ provided $g|f^{-1}(y')$ is δ_y -closed to $g_0|f^{-1}(y')$. The family $\{V_y \cap H : y \in H\}$ can be supposed to be locally finite in H . Consider the set-valued lower semi-continuous map $\psi : H \rightarrow (0, 1]$, $\psi(y) = \bigcup\{(0, \delta_z] : y \in V_z\}$. By [10, Theorem 6.2, p.116], ψ admits a continuous selection $\beta : H \rightarrow (0, 1]$. Let $\bar{\beta} : Y \rightarrow (0, 1]$ be a continuous extension of β and $\alpha = \bar{\beta} \circ f$. It remains only to show that if $g \in C(X, M)$ with $\varrho(g_0(x), g(x)) < \alpha(x)$ for all $x \in X$, then $g \in \mathcal{K}(m, n, y)$ for all $y \in H$. So, we take such a g and fix $y \in H$. Then there exists $z \in H$ with $y \in V_z$ and $\alpha(x) \leq \delta_z$ for all $x \in f^{-1}(y)$. Hence, $\varrho(g(x), g_0(x)) < \delta_z$ for each $x \in f^{-1}(y)$. According to the choice of V_z , $g \in \mathcal{K}(m, n, y)$. This completes the proof. \square

2.2. $\mathcal{K}(m, n, H)$ is dense in $C(X, M)$ for finite-dimensional H . In this subsection we show that $\mathcal{K}(m, n, H)$ is dense in $C(X, M)$ with respect to the source limitation topology provided $H \subset Y$ is a closed finite-dimensional subset and M a Krasinkiewicz complete ANR-space. We need to show that $B_\varrho(g, \varepsilon) = \{g' \in C(X, M) : \varrho(g, g') < \varepsilon\}$ meets $\mathcal{K}(m, n, H)$ for every $g \in C(X, M)$ and every continuous function $\varepsilon : X \rightarrow (0, 1]$, where ϱ is a complete metric on M satisfying the

hypotheses of Lemma 2.2. To this end, fix $g_0 \in C(X, M)$ and $\varepsilon \in C(X, (0, 1/64])$. Consider the set-valued map $\Phi_\varepsilon: Y \rightarrow C(X, M)$, $\Phi_\varepsilon(y) = \mathcal{K}(m, n, y) \cap B_\varrho(g_0, \varepsilon)$, where $C(X, M)$ carries the compact open topology.

Lemma 2.5. *The map Φ_ε satisfies the following conditions:*

- (1) $\Phi_\varepsilon(y) \neq \emptyset$ for all $y \in Y$;
- (2) If $\Phi_\varepsilon(y_0)$ contains a compact set K for some $y_0 \in Y$, then there exists a neighborhood $V(y_0)$ of y_0 such that $K \subset \Phi_\varepsilon(y)$ for every $y \in V(y_0)$.

PROOF. For any $y \in Y$ let $\eta_y = \min\{\varepsilon(x) : x \in f^{-1}(y)\}$. Since M is a Krasinkiewicz space, there exists a Krasinkiewicz map $h \in C(f^{-1}(y), M)$ such that $\varrho(h(x), g_0(x)) < \eta_y/8$ for all $x \in f^{-1}(y)$. Then, by Lemma 2.2, h can be extended to a map $g \in C(X, M)$ with $\varrho(g(x), g_0(x)) < \varepsilon(x)$ for all $x \in X$. Obviously, $g \in B_\varrho(g_0, \varepsilon)$. Because h is a Krasinkiewicz map on $f^{-1}(y)$, $g \in \mathcal{K}(m, n, y)$. So, $g \in \Phi_\varepsilon(y)$.

To prove the second item, suppose there exists a sequence $\{y_j\}_{j \geq 1}$ converging to y_0 in Y such that $K \setminus \Phi_\varepsilon(y_j) \neq \emptyset$. Let $g_j \in K \setminus \Phi_\varepsilon(y_j)$, $j \geq 1$, and $P = f^{-1}(\{y_0\} \cup \{y_j\}_{j \geq 1})$. The restriction map $\pi_P: C(X, M) \rightarrow C(P, M)$ is continuous when both $C(X, M)$ and $C(P, M)$ are equipped with the compact open topology. Moreover, the compact open topology on $C(P, M)$ coincides with the uniform convergence. Hence, there exists a subsequence $\{g_{j_k}\}$ of $\{g_j\}$ such that $\pi_P(g_{j_k})$ converges to $\pi_P(g)$ in $C(P, M)$ for some $g \in K$. Since $g \in \mathcal{K}(m, n, y_0)$, we can apply Lemma 2.3 to find a neighborhood V of y_0 in Y and a positive $\delta > 0$ such that $y' \in V$ and $\varrho(g(x), g'(x)) < \delta$ for all $x \in f^{-1}(y')$ implies $g' \in \mathcal{K}(m, n, y')$. Now, choose j_k with $y_{j_k} \in V$ and $\varrho(g(x), g_{j_k}(x)) < \delta$ for any $x \in f^{-1}(y_{j_k})$. Then $g_{j_k} \in \mathcal{K}(m, n, y_{j_k})$. So, $g_{j_k} \in \Phi_\varepsilon(y_{j_k})$ which contradicts the choice of the functions g_j . \square

Lemma 2.6. *Every $\Phi_\varepsilon(y)$ has the following property: If $\hat{v}: \mathbb{S}^k \rightarrow \Phi_\varepsilon(y)$ is continuous, where $k \geq 0$ and \mathbb{S}^k is the k -sphere, then \hat{v} can be extended to a continuous map $\hat{u}: \mathbb{B}^{k+1} \rightarrow \Phi_{64\varepsilon}(y)$.*

PROOF. Let us mention the following property of the function space $C(X, M)$ with the compact open topology: For any metrizable space Z a map $\hat{w}: Z \rightarrow C(X, M)$ is continuous if and only if the map $w: Z \times X \rightarrow M$, $w(z, x) = \hat{w}(z)(x)$, is continuous. Hence, every map $\hat{v}: \mathbb{S}^k \rightarrow \Phi_\varepsilon(y)$ generates a continuous map $v: \mathbb{S}^k \times X \rightarrow M$ defined by $v(z, x) = \hat{v}(z)(x)$ such that $\varrho(v(z, x), g_0(x)) < \varepsilon(x)$ for all $(z, x) \in \mathbb{S}^k \times X$.

Let $\pi_y: C(X, M) \rightarrow C(f^{-1}(y), M)$ be the restriction map. It is easily seen that π_y is continuous and open when both $C(X, M)$ and $C(f^{-1}(y), M)$ are equipped

with the source limitation or the compact open topology. Since $f^{-1}(y)$ is compact, the source limitation, the compact open and the uniform convergence topologies on $C(f^{-1}(y), M)$ coincide. Therefore, $\pi_y(\mathcal{K}(m, n, y))$ is open in $C(f^{-1}(y), M)$ and contains the compact set $\pi_y(\hat{v}(\mathbb{S}^k))$. Consequently, the distance (in the space $C(f^{-1}(y), M)$) between $\pi_y(\hat{v}(\mathbb{S}^k))$ and $C(f^{-1}(y), M) \setminus \pi_y(\mathcal{K}(m, n, y))$ is positive. Denote this distance by δ_1 .

Obviously $\delta_2 = \inf\{\varepsilon(x) - \varrho(v(z, x), g_0(x)) : (z, x) \in \mathbb{S}^k \times f^{-1}(y)\}$ is positive. According to Lemma 2.2, there exists a continuous extension $v_1: \mathbb{B}^{k+1} \times f^{-1}(y) \rightarrow M$ of the map $v|(\mathbb{S}^k \times f^{-1}(y))$ with $\varrho(v_1(z, x), g_0(x)) < 8\varepsilon(x)$ for all $(z, x) \in \mathbb{B}^{k+1} \times f^{-1}(y)$. Let $\delta_3 = \inf\{8\varepsilon(x) - \varrho(v_1(z, x), g_0(x)) : (z, x) \in \mathbb{B}^{k+1} \times f^{-1}(y)\}$. Since M is a Krasinkiewicz space, there exists a Krasinkiewicz map $v_2: \mathbb{B}^{k+1} \times f^{-1}(y) \rightarrow M$ such that $\varrho(v_2(z, x), v_1(z, x)) < \delta/8$ for all $(z, x) \in \mathbb{B}^{k+1} \times f^{-1}(y)$, where $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Therefore, we have a map $\hat{v}_2: \mathbb{B}^{k+1} \rightarrow C(f^{-1}(y), M)$. The choice of δ_3 implies

$$(1) \quad \varrho(v_2(z, x), g_0(x)) < 8\varepsilon(x)$$

for all $(z, x) \in \mathbb{B}^{k+1} \times f^{-1}(y)$. Moreover, v_2 being a Krasinkiewicz map yields that all maps $\hat{v}_2(z): f^{-1}(y) \rightarrow M, z \in \mathbb{B}^{k+1}$, are also Krasinkiewicz. On the other hand, by Lemma 2.2 and (1), every $\hat{v}_2(z)$ can be extended to a map from X into M . Therefore,

$$(2) \quad \hat{v}_2(\mathbb{B}^{k+1}) \subset \pi_y(\mathcal{K}(m, n, y))$$

Representing the ball \mathbb{B}^{k+1} as a cone with a base \mathbb{S}^k and a vertex z_0 , we can consider v_2 as a homotopy from $\mathbb{S}^k \times f^{-1}(y) \times [0, 1]$ into M between the maps $v_2|(\mathbb{S}^k \times f^{-1}(y) \times \{0\})$ and $v_2|(\{z_0\} \times f^{-1}(y))$. Observe also that $\varrho(v_2(z, x, 0), v(z, x)) < \delta/8$ for any $(z, x) \in \mathbb{S}^k \times f^{-1}(y)$. Hence, the map $\varphi: \mathbb{S}^k \times f^{-1}(y) \times \{0, 1\} \rightarrow M$,

$$\varphi(z, x, t) = \begin{cases} v(z, x) & \text{if } t = 0; \\ v_2(z, x, 0) & \text{if } t = 1. \end{cases}$$

is $(\delta/8)$ -close to v . Consequently, by Lemma 2.2, φ admits a continuous extension $v_3: \mathbb{S}^k \times f^{-1}(y) \times [0, 1] \rightarrow M$ such that $\varrho(v_3(z, x, t), v(z, x)) < \delta$ for every $(z, x, t) \in \mathbb{S}^k \times f^{-1}(y) \times [0, 1]$. Since $\delta < \min\{\delta_1, \delta_2\}$, for any $(z, x, t) \in \mathbb{S}^k \times f^{-1}(y) \times [0, 1]$ we have

$$(3) \quad \varrho(v_3(z, x, t), v(z, x)) < \delta_1,$$

and

$$(4) \quad \varrho(v_3(z, x, t), g_0(x)) < \varepsilon(x).$$

Therefore, v_3 is a homotopy connecting the maps v and $v_2|(\mathbb{S}^k \times f^{-1}(y) \times \{0\})$, while v_2 is a homotopy connecting the maps $v_2|(\mathbb{S}^k \times f^{-1}(y) \times \{0\})$ and $v_2|(\{z_0\} \times f^{-1}(y))$. Combining these two homotopies, we obtain a map $u_1: \mathbb{S}^k \times f^{-1}(y) \times [0, 1] \rightarrow M$ such that $u_1(z, x, 0) = v(z, x)$, $u_1(z, x, 1) = v_2(z_0, x)$ and $\varrho(u_1(z, x, t), g_0(x)) < 8\varepsilon(x)$ for all $(z, x, t) \in \mathbb{S}^k \times f^{-1}(y) \times [0, 1]$. Obviously, u_1 can also be considered as a map from $\mathbb{B}^{k+1} \times f^{-1}(y)$ into M such that $u_1|(\mathbb{S}^k \times f^{-1}(y)) = v$ and $\varrho(u_1(z, x), g_0(x)) < 8\varepsilon(x)$, $(z, x) \in \mathbb{B}^{k+1} \times f^{-1}(y)$. Now consider the map $u_2: (\mathbb{B}^{k+1} \times f^{-1}(y)) \cup (\mathbb{S}^k \times X) \rightarrow M$ with $u_2|((\mathbb{B}^{k+1} \times f^{-1}(y))) = u_1$ and $u_2|(\mathbb{S}^k \times X) = v$. Finally, using Lemma 2.2, we extend u_2 to a map $u: \mathbb{B}^{k+1} \times X \rightarrow M$ such that

$$(5) \quad \varrho(u(z, x), g_0(x)) < 64\varepsilon(x)$$

for any $(z, x) \in \mathbb{B}^{k+1} \times X$. Then $\hat{u}: \mathbb{B}^{k+1} \rightarrow C(X, M)$ extends the map \hat{v} . Moreover, (2), (3) and the choice of δ_1 implies that $\hat{u}(\mathbb{B}^{k+1}) \subset \mathcal{K}(m, n, y)$. On the other hand, (5) yields $\hat{u}(\mathbb{B}^{k+1}) \subset B_\varrho(g_0, 64\varepsilon)$. Hence, $\hat{u}(\mathbb{B}^{k+1}) \subset \Phi_{64\varepsilon}(y)$. \square

Next proposition completes the proof of Theorem 1.1 in the case Y is strongly countable-dimensional.

Proposition 2.7. *Let $H \subset Y$ be a closed finite-dimensional set. Then $\mathcal{K}(m, n, H)$, $m, n \geq 1$, are dense sets in $C(X, M)$ with respect to the source limitation topology.*

PROOF. Let $\dim H \leq k$. Define the set-valued maps $\Phi_j: H \rightarrow C(X, M)$, $j = 0, \dots, k$, $\Phi_j(y) = \Phi_{\varepsilon/8^{2(k-j)+1}}(y)$. Obviously, $\Phi_0(y) \subset \Phi_1(y) \subset \dots \subset \Phi_k(y) = \Phi_{\varepsilon/8}(y)$. According to Lemma 2.6, every map from \mathbb{S}^k into $\Phi_j(y)$ can be extended to a map from \mathbb{B}^{k+1} into $\Phi_{j+1}(y)$, where $j = 0, 1, \dots, k-1$ and $y \in H$. Moreover, by Lemma 2.5, any $\Phi_j(y)$ has the following property: if $K \subset \Phi_j(y)$ is compact, then there exists a neighborhood V_y of y in Y such that $K \subset \Phi_j(z)$ for all $z \in V_y \cap H$. So, we may apply [3, Theorem 3.1] to find a continuous selection $\theta: H \rightarrow C(X, M)$ of Φ_k . Hence, $\theta(y) \in \Phi_{\varepsilon/8}(y)$ for all $y \in H$. Now, consider the map $g: f^{-1}(H) \rightarrow M$, $g(x) = \theta(f(x))(x)$. Using that $C(X, M)$ carries the compact open topology, one can show that g is continuous. Moreover, $\varrho(g(x), g_0(x)) < \varepsilon(x)/8$ for all $x \in f^{-1}(H)$. Then, by Lemma 2.2, g can be extended to a continuous map $\bar{g}: X \rightarrow M$ with $\varrho(\bar{g}(x), g_0(x)) < \varepsilon(x)$, $x \in X$. It follows from the definition of g that $\bar{g}|f^{-1}(y) = \theta(y)|f^{-1}(y)$ for every $y \in H$. Since $\theta(y) \in \mathcal{K}(m, n, y)$ for all $y \in H$, $\bar{g} \in \mathcal{K}(m, n, H)$. Hence, $B_\varrho(g_0, \varepsilon) \cap \mathcal{K}(m, n, H) \neq \emptyset$. \square

2.3. $\mathcal{K}(m, n, H)$ is dense in $C(X, M)$ for H being a C -space. We now turn to the proof of Theorem 1.1 in the case Y is a C -space and M a Krasinkiewicz space homeomorphic to a closed convex subset M' of a given Banach space E .

Suppose $M = M'$ and let ϱ be the metric on M inherited from the norm of E and $\Psi_\varepsilon: Y \rightarrow C(X, M)$ be the set-valued map $\Psi_\varepsilon(y) = \overline{B}_\varrho(g_0, \varepsilon) \cap \mathcal{K}(m, n, y)$, where $C(X, M)$ is equipped again with the compact open topology and

$$\overline{B}_\varrho(g_0, \varepsilon) = \{g \in C(X, M) : \varrho(g_0(x), g(x)) \leq \varepsilon(x) \text{ for all } x \in X\}.$$

Lemma 2.8. $\Psi_\varepsilon(y)$ is k -connected for all $k \geq 0$ (i.e. every map $\hat{v}: \mathbb{S}^k \rightarrow \Psi_\varepsilon(y)$ can be extended to a map $\hat{u}: \mathbb{B}^{k+1} \rightarrow \Psi_\varepsilon(y)$).

PROOF. All function spaces in this proof are equipped with the compact open topology. Let $\pi_y: C(X, M) \rightarrow C(f^{-1}(y), M)$ be the restriction map and $P(y) = \overline{B}_\varrho(g_0, \varepsilon, y) \setminus \pi_y(\mathcal{K}(m, n, y))$, where $\overline{B}_\varrho(g_0, \varepsilon, y)$ is the set

$$\{g \in C(f^{-1}(y), M) : \varrho(g_0(x), g(x)) \leq \varepsilon(x) \text{ for all } x \in f^{-1}(y)\}.$$

Since $\pi_y(\mathcal{K}(m, n, y))$ is open in $C(f^{-1}(y), M)$, $P(y) \subset \overline{B}_\varrho(g_0, \varepsilon, y)$ is closed.

We are going to show that $P(y)$ is a Z -set in $\overline{B}_\varrho(g_0, \varepsilon, y)$, i.e., every map $\hat{w}: K \rightarrow \overline{B}_\varrho(g_0, \varepsilon, y)$, where K is compact, can be approximated by a map $\hat{w}_1: K \rightarrow \overline{B}_\varrho(g_0, \varepsilon, y) \setminus P(y) = \overline{B}_\varrho(g_0, \varepsilon, y) \cap \pi_y(\mathcal{K}(m, n, y))$. To this end, fix $\delta > 0$ and let $w: K \times f^{-1}(y) \rightarrow M$ be the map generated by \hat{w} . So, $\varrho(w(z, x), g_0(x)) \leq \varepsilon(x)$ for all $(z, x) \in K \times f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there exists $\lambda \in (0, 1)$ such that $\lambda \max\{\varepsilon(x) : x \in f^{-1}(y)\} < \delta/2$. Define the map $w_1: K \times f^{-1}(y) \rightarrow M$ by $w_1(z, x) = (1 - \lambda)w(z, x) + \lambda g_0(x)$. Then, for all $(z, x) \in K \times f^{-1}(y)$ we have

$$\varrho(w_1(z, x), w(z, x)) \leq \lambda \varepsilon(x) < \delta/2$$

and

$$\varrho(w_1(z, x), g_0(x)) \leq (1 - \lambda)\varepsilon(x) < \varepsilon(x).$$

Since M is a Krasinkiewicz space, there exists a Krasinkiewicz map $w_2: K \times f^{-1}(y) \rightarrow M$ which is δ_1 -close to w_1 , where $\delta_1 = \min\{\lambda \varepsilon(x) : x \in f^{-1}(y)\}$. Hence, for every $(z, x) \in K \times f^{-1}(y)$ we have

$$\varrho(w_2(z, x), g_0(x)) \leq \varepsilon(x) \text{ and } \varrho(w_2(z, x), w(z, x)) < \delta.$$

The last two inequalities imply that the map $\hat{w}_2: K \rightarrow C(f^{-1}(y), M)$ is δ -close to \hat{w} and $\hat{w}_2(K) \subset \overline{B}_\varrho(g_0, \varepsilon, y)$. Moreover, every $\hat{w}_2(z)$, $z \in K$, being a map from $f^{-1}(y)$ into M , can be extended to a map from X to M because M is a closed convex subset of E . Since w_2 is a Krasinkiewicz map, so are the maps $\hat{w}_2(z)$, $z \in K$. Hence, $\hat{w}_2(K) \subset \pi_y(\mathcal{K}(m, n, y))$. So, $P(y)$ is a Z -set in $\overline{B}_\varrho(g_0, \varepsilon, y)$.

Let us complete the proof of the lemma. For every map $\hat{v}: \mathbb{S}^k \rightarrow \Psi_\varepsilon(y)$ the composition $\pi_y \circ \hat{v}$ is a map from \mathbb{S}^k into $\overline{B}_\varrho(g_0, \varepsilon, y) \cap \pi_y(\mathcal{K}(m, n, y))$. Since $P(y)$ is a Z -set in the convex set $\overline{B}_\varrho(g_0, \varepsilon, y)$, by [12, Proposition 6.3], there exists a

map $\hat{v}_1: \mathbb{B}^{k+1} \rightarrow \overline{B}_\rho(g_0, \varepsilon, y) \cap \pi_y(\mathcal{K}(m, n, y))$ extending $\pi_y \circ \hat{v}$. Consider the map $v_2: A \rightarrow M$, where $A = (\mathbb{B}^{k+1} \times f^{-1}(y)) \cup (\mathbb{S}^k \times X)$, defined by $v_2|_{(\mathbb{B}^{k+1} \times f^{-1}(y))} = v_1$ and $v_2|_{(\mathbb{S}^k \times X)} = v$. Next, take a selection $u: \mathbb{B}^{k+1} \times X \rightarrow M$ for the set-valued map $\phi: \mathbb{B}^{k+1} \times X \rightarrow M$, $\phi(z, x) = v_2(z, x)$ if $(z, x) \in A$ and $\phi(z, x) = \text{cl}(B_\rho(g_0(x), \varepsilon(x)))$ if $(z, x) \notin A$. Such u exists by Michael's [8] convex-valued selection theorem. Obviously u extends v_2 and $\rho(u(z, x), g_0(x)) \leq \varepsilon(x)$ for every $(z, x) \in \mathbb{B}^{k+1} \times X$. Finally, observe that \hat{u} is the required extension of \hat{v} . \square

We can finish the proof of Theorem 1.1.

Proposition 2.9. *Suppose $H \subset Y$ is a closed C -space and M a closed convex subset of a Banach space E . Then the sets $\mathcal{K}(m, n, H)$, $m, n \geq 1$, are dense in $C(X, M)$ with respect to the source limitation topology.*

PROOF. Consider the set-valued map $\Psi_\varepsilon: H \rightarrow C(X, M)$. It follows from the proof of Lemma 2.5 that if $K \subset \Psi_\varepsilon(y_0)$ for some compactum K and $y_0 \in H$, then y_0 admits a neighborhood $V \subset H$ with $K \subset \Psi_\varepsilon(y)$ for all $y \in V$. Moreover, according to Lemma 2.8, every image $\Psi_\varepsilon(y)$ is aspherical, i.e., any map from \mathbb{S}^k into $\Psi_\varepsilon(y)$, $k \geq 0$, can be extended to a map from \mathbb{B}^{k+1} to $\Psi_\varepsilon(y)$. Then, by the Uspenskij selection theorem [12, Theorem 1.3], Ψ_ε admits a continuous selection $\theta: H \rightarrow C(X, M)$. Repeating the arguments from the proof of Proposition 2.7, we obtain a map $g: f^{-1}(H) \rightarrow M$ such that $\rho(g(x), g_0(x)) \leq \varepsilon(x)$ for every $x \in f^{-1}(H)$ and $g|_{f^{-1}(y)} = \theta(y)|_{f^{-1}(y)}$, $y \in H$. Applying once more the Michael [8] convex-valued selection theorem for the set-valued map $\vartheta: X \rightarrow M$, $\vartheta(x) = g(x)$ if $x \in f^{-1}(H)$ and $\vartheta(x) = \overline{B}_\rho(g_0(x), \varepsilon(x))$ if $x \notin f^{-1}(H)$, we obtain a selection \bar{g} for ϑ . Obviously, \bar{g} extends g and $\bar{g} \in \overline{B}_\rho(g_0, \varepsilon)$. Since $\theta(y) \in \mathcal{K}(m, n, y)$ for all $y \in H$, we have $\bar{g} \in \overline{B}_\rho(g_0, \varepsilon) \cap \mathcal{K}(m, n, H)$. Hence, $\mathcal{K}(m, n, H)$ is dense in $C(X, M)$. \square

3. SOME PROPERTIES OF KRASINKIEWICZ SPACES

In this section we investigate the class of Krasinkiewicz spaces and we show that several well-known classes of spaces are subclasses of Krasinkiewicz spaces. Let us start with the following proposition whose proof is straightforward.

Proposition 3.1. *For every space M we have:*

- (1) *If M is a Krasinkiewicz space, then so is any open subset of M ;*
- (2) *If every compact set in M is contained in a Krasinkiewicz subset of M , then M is also a Krasinkiewicz space.*

Corollary 3.2. *Every polyhedron is a Krasinkiewicz space.*

PROOF. Apply Proposition 3.1(2) and the fact that each compact polyhedron is a Krasinkiewicz space [7]. □

Next proposition is an analogue of [11, Theorem 4.2].

Proposition 3.3. *Suppose M is completely metrizable and for every $\varepsilon > 0$ there exist a Krasinkiewicz space Z_ε and maps $r: M \rightarrow Z_\varepsilon$ and $\phi: Z_\varepsilon \rightarrow M$ such that ϕ is light and $\phi \circ r$ is ε -close to the identity on M . Then M is a Krasinkiewicz space.*

PROOF. Let $g \in C(X, M)$ and $\varepsilon > 0$, where X is compact. Then there exists a Krasinkiewicz space $Z_{\varepsilon/2}$ and two maps $r: M \rightarrow Z_{\varepsilon/2}$, $\phi: Z_{\varepsilon/2} \rightarrow M$ such that ϕ is light and $\phi \circ r$ is $\varepsilon/2$ -close to the identity on M . Take $\delta > 0$ and a neighborhood U of $r(g(X))$ in $Z_{\varepsilon/2}$ such that $dist(\phi(z_1), \phi(z_2)) < \varepsilon/2$ provided $z_1, z_2 \in U$ and $dist(z_1, z_2) < \delta$. Next, choose a Krasinkiewicz map $h: X \rightarrow Z_{\varepsilon/2}$ which is δ -close to $r \circ g$ and $h(X) \subset U$. Finally, $g' = \phi \circ h$ is ε -close to g and, since ϕ is light, g' is a Krasinkiewicz map (see [7, Proposition 3.1]). □

Proposition 3.3 is of special interest when all Z_ε are subsets of M and the maps r are retractions (in such a case we say that M admits small retractions to Krasinkiewicz spaces). Since every compact Menger manifold (a manifold modeled on the Menger cube μ^n for some $n \geq 1$), as well as every 1-dimensional Peano continuum, admits small retractions to compact polyhedra, it was observed in [7, Theorem 3.2-3.3] that any such a space is Krasinkiewicz. Moreover, every Nöbeling manifold also admits small retractions to polyhedra, see [1]. So, by Proposition 3.3, we have:

Corollary 3.4. *Each of the following are Krasinkiewicz spaces: 1-dimensional Peano continua, Menger manifolds and Nöbeling manifolds.*

Proposition 3.5. *A product of finitely many Krasinkiewicz spaces is a Krasinkiewicz space.*

PROOF. We need to prove the proposition for a product of two Krasinkiewicz spaces M_1 and M_2 . In this case, the proof is reduced to show that if X is a metric compactum and $g_i: X \rightarrow M_i$, $i = 1, 2$, are Krasinkiewicz maps, then the product map $g = g_1 \Delta g_2: X \rightarrow M_1 \times M_2$ is also a Krasinkiewicz map. And that easily follows. □

Some more examples of Krasinkiewicz spaces are provided by next theorem.

Theorem 3.6. *A complete ANR-space M is a Krasinkiewicz space if and only if it has an open cover of Krasinkiewicz spaces.*

PROOF. It suffices to show that M is Krasinkiewicz if each $y \in M$ has a neighborhood U_y in M which is a Krasinkiewicz space. We fix a compactum X and choose $\varepsilon_y > 0$, $y \in M$, with $B(y, 3\varepsilon_y) \subset U_y$. Let H_y be the set of all maps $g : X \rightarrow M$ satisfying next condition:

- (a) If $L \subset X$ is a subcontinuum such that $\text{diam } g(L) > 0$ and $g(L) \subset \text{cl}(B(y, \varepsilon_y))$, then there exists $x \in L$ with $C(x, g) \subset L$.

Now, for each $m, n \in \mathbb{N}$ consider the set $H_{m,n,y} \subset C(X, M)$ of all maps g such that:

- (b) If $L \subset X$ is continuum with $\text{diam } g(L) \geq 1/n$ and $g(L) \subset \text{cl}(B(y, \varepsilon_y))$, then $C(x, g) \subset B(L, 1/m)$ for some $x \in L$.

Claim 1. $H_y = \bigcap_{m,n \in \mathbb{N}} H_{m,n,y}$.

The proof of this claim is similar to the proof of Proposition 2.1, so it is omitted.

Claim 2. Every $H_{m,n,y}$ is open in $C(X, M)$.

Let $f \in \text{cl}(C(X, M) \setminus H_{m,n,y})$. Then there exists a sequence of maps $\{f_i\}_{i=1}^\infty \subset C(X, M) \setminus H_{m,n,y}$ with $\lim f_i = f$. For each $i = 1, 2, \dots$, there exists a subcontinuum $L_i \subset X$ such that $\text{diam } f_i(L_i) \geq 1/n$, $f_i(L_i) \subset \text{cl}(B(y, \varepsilon_y))$ and $C(x, f_i)$ is not contained in $B(L_i, 1/m)$ for each $x \in L_i$. We may assume that L_i converges to a subcontinuum $L \subset X$. It is easy to see that $\text{diam } f(L) \geq 1/n$ and $f(L) \subset \text{cl}(B(y, \varepsilon_y))$. Let $x \in L$ be arbitrary. Then x is the limit of a sequence $\{x_i\}_{i=1}^\infty \subset X$ such that $x_i \in L_i$ for each $i = 1, 2, \dots$. We may assume that $C(x_i, f_i)$ converges to a subcontinuum $C \subset X$. Since each $C(x_i, f_i)$ is not contained in $B(L_i, 1/m)$, C is not contained in $B(L, 1/m)$. Moreover, $x \in C \subset C(x, f)$. So, $f \in C(X, M) \setminus H_{m,n,y}$. This completes the proof of Claim 2.

Claim 3. Every H_y is dense in $C(X, M)$.

Let $f \in C(X, M)$ and $\varepsilon > 0$ with $\varepsilon < \varepsilon_y$. Since M is an ANR, there is a $\delta > 0$ such that each map $g : A \rightarrow M$, where $A \subset X$ is closed, has a continuous extension $\hat{g} : X \rightarrow M$ which is ε -close to f provided g is δ -close to $f|_A$. Since U_y is a Krasinkiewicz space and $f^{-1}(\text{cl}(B(y, 2\varepsilon_y)))$ is compact, there exists a Krasinkiewicz map $k : f^{-1}(\text{cl}(B(y, 2\varepsilon_y))) \rightarrow U_y$ such that k is δ -close to $f|_{f^{-1}(\text{cl}(B(y, 2\varepsilon_y)))}$. Then there exists a continuous extension $\hat{k} : X \rightarrow Y$ of k such that \hat{k} is ε -close to f . We are going to show that $\hat{k} \in H_y$. Indeed, let L be a subcontinuum of X such that $\text{diam } \hat{k}(L) > 0$ and $\hat{k}(L) \subset \text{cl}(B(y, \varepsilon_y))$. Then $L \subset f^{-1}(\text{cl}(B(y, 2\varepsilon_y)))$. Since $k : f^{-1}(\text{cl}(B(y, 2\varepsilon_y))) \rightarrow U_y$ is a Krasinkiewicz map,

there exists $x \in L$ such that $C(x, k) \subset L$. Note that $C(x, k) = C(x, \hat{k})$ because $\hat{k}^{-1}(z) = k^{-1}(z)$ for each $z \in \text{cl}(B(y, \varepsilon_y))$. This completes the proof of Claim 3.

Now, we can complete the proof of Theorem 3.6. Let $f \in C(X, M)$ and $\varepsilon > 0$. Since $f(X)$ is compact, there exist finitely many points $y_1, y_2, \dots, y_N \in f(X)$ such that $f(X) \subset \bigcup_{i=1}^N B(y_i, 2^{-1}\varepsilon_{y_i})$. Let $\delta_0 = \min\{\varepsilon, 2^{-1}\varepsilon_{y_1}, 2^{-1}\varepsilon_{y_2}, \dots, 2^{-1}\varepsilon_{y_N}\}$. By previous claims, $\bigcap_{i=1}^N H_{y_i}$ is a dense G_{δ} -subset of $C(X, M)$. So, we can find a map $g_0 \in \bigcap_{i=1}^N H_{y_i}$ δ_0 -close to f . It suffices to show that g_0 is a Krasinkiewicz map. To this end, let T be a subcontinuum of X with $\text{diam}g_0(T) > 0$. Note that $g_0(T) \subset \bigcup_{i=1}^N B(y_i, \varepsilon_{y_i})$. Hence, there exists a subcontinuum $T' \subset T$ and $j \in \{1, 2, \dots, N\}$ such that $\text{diam}g_0(T') > 0$ and $g_0(T') \subset \text{cl}(B(y_j, \varepsilon_{y_j}))$. Since $g_0 \in H_{y_j}$, there exists a point $x_0 \in T'$ such that $C(x_0, g_0) \subset T' \subset T$. This completes the proof. \square

Our final proposition provides spaces which are not Krasinkiewicz. It implies, for example, that hereditarily indecomposable continua can not be Krasinkiewicz spaces.

Proposition 3.7. *Let Y be a non-degenerate continuum such that some open subset of Y contains no arc. Then the projection $p : Y \times \mathbb{I} \rightarrow Y$ can not be approximated by Krasinkiewicz maps.*

PROOF. Let U be an open subset of Y such that U contains no arc. Choose a non-degenerate continuum $L \subset U$ and let $\delta = \text{diam}L$ and $\varepsilon = \min\{\delta/2, \text{dist}(L, X \setminus U)\}$. We claim that every map $q : Y \times \mathbb{I} \rightarrow Y$ which is ε -close to p can not be Krasinkiewicz. Indeed, suppose there exists such a Krasinkiewicz map q_0 and let $t \in \mathbb{I}$. Then $q_0(L \times \{t\})$ is not a singleton, so there exists $y \in q_0(L \times \{t\})$ and a component C of $q_0^{-1}(y)$ such that $C \subset L \times \{t\}$. Take any point $z \in p(C)$. Then $q_0(\{z\} \times \mathbb{I})$ is not a singleton. So $q_0(\{z\} \times \mathbb{I})$ contains an arc. On the other hand, $q_0(\{z\} \times \mathbb{I}) \subset U$. This is a contradiction. \square

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