ON THE SPACES WHICH HAVE A τ -LATTICE OF OPEN MAPPINGS

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A criterion is obtained for $\mathscr{C}(\tau)$-embeddability ($\mathscr{C}$-embeddability) of a given space into a space with a τ-lattice of open mappings. As a corollary it is shown that several classes of topological spaces have the same absolute (neighbourhood) retracts.

1. Notations and terminology. Completely regular spaces and continuous mappings are considered only. The terminology from the Engelking's book [4] is adopted except for the notations which are explicitly defined here.

In a topological space X, a subset which is an intersection of $\tau\geq\aleph_0$ open sets is called a G_δ^τ-set and a union of an arbitrary number of G_δ^τ-sets is called a $G_{\delta\Sigma}^\tau$-set (see [8]). Let f and g be two mappings defined on the same space X. Then we write $f<g$ iff there exists a mapping $h:f(X)\rightarrow g(X)$ with $g=h\circ f$.

Let τ be an infinite cardinal number. Following E. Shchepin [9] we say that a family L of mappings, defined on X, is a τ-lattice of X, if it satisfies the following conditions:

1. If $\{h_s:s\in S\}\subset$ L and for every finite set $S'\subset S$ the diagonal mapping $\Delta_{s\in S'} h_s$ belongs to L, then $\Delta_{s\in S} h_s\in$ L.

2. For every mapping f, defined on X, there exist mappings $h\in$ L and $g:h(X)\rightarrow f(X)$ such $f=g\circ h$ and $w(h(X))\leq w(f(X))$.

An $\aleph_0$-lattice is called simply a lattice. When X is a compact space these concepts coincide with the concepts of τ-lattice and lattice, introduced by Shchepin in [9], where he proved that every $\varkappa$-metrizable compact space possesses a lattice of open mappings. There are non-compact spaces which possess a lattice of open mappings. For example, if $X=\Pi\{X_\alpha:\alpha\in A\}$ and $w(X_\alpha)\leq\tau$ for every $\alpha\in A$, then the collection of all projections of X is a τ-lattice.

A subset M of a given space X is called $\mathscr{C}(\tau)$-embedded in X, if every mapping f defined on M with $w(f(M))\leq\tau$ has a continuous extension $f:X\rightarrow f(M)$. It is obvious that if M is $\mathscr{C}(\tau)$-embedded in X, then it is $\mathscr{C}$-embedded in X, but the converse is not true even for $\tau=\aleph_0$. For example, the set N of all natural numbers is $\mathscr{C}$-embedded in R and N is not $\mathscr{C}(\aleph_0)$-embedded in R.

The space X is said to be τ-pseudocompact if X is $\mathscr{C}(\tau)$)-embedded in βX, i. e. if the existence of a mapping from X onto some Y with $w(Y)\leq\tau$ implies that Y is a compact space. Obviously, the space X is $\aleph_0$-pseudocompact iff X is a pseudocompact space.

Finally, we recall that the space X is an absolute (neighborhood) retract for a given class $\mathscr{P}$ of topological spaces, briefly $A(N)R(\mathscr{P})$, if $X\in\mathscr{P}$ and for

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every closed embedding of X into $Y \in \mathscr{P}$ there exists a retraction $r: Y \to X$ (a neighbourhood U of X in Y and a retraction $r: U \to X$).

2. Theorems. We need the following two lemmas.

Lemma 1 [11]. *Let the space X possess a τ-lattice L of open mappings and let $\mathscr{A}$ be a family of G_δ^λ-subsets of X. Then there exists a subfamily $\mathscr{B}$ of $\mathscr{A}$ and a mapping $h \in$ L, such that* Card $(\mathscr{B}) \leq \tau\lambda$, $wh(X) \leq \tau\lambda$, $(h^{-1}h(\overline{U\mathscr{B}}) = \overline{U\mathscr{B}} = \overline{U\mathscr{A}}$ *and* $h^{-1}h(\bar{B}) = \bar{B}$ *for every* $B \in \mathscr{B}$.

Lemma 2 [11]. *Let the space X possess a τ-latice L of open mappings and let F be a $G_{\delta\Sigma}^\lambda$-subset of X. If U and V are disjoint open subsets of $\bar{F}$, then there is such a mapping $h \in L$ that $h^{-1}h(\bar{F}) = \bar{F}$, $h(U) \cap h(V) = \emptyset$ and $wh(X) \leq \tau.\lambda$.*

Theorem 1. *Let the space X possess a lattice of open mappings and let F be a closed $G_{\delta\Sigma}^{\aleph_0}$-subset of X. If βY is a closed continuous image of F then Y is a pseudocompact space.*

Proof. Let $f: F \to \beta Y$ is a closed mapping from F onto βY. Let us suppose that there exists an unbounded function $g: Y \to R$. If $\tilde{g}; \beta Y \to \beta R$ is the natural extension of g we put $\tilde{f} = \tilde{g} \circ f$. For every $r \in R$ the set $\tilde{f}^{-1}(r)$ is a closed $G_{\delta\Sigma}^{\aleph_0}$-sumbset of X. Thus, by Lemma 1, there exists a mapping $h_r \in$ L such that $h_r^{-1} h_r(\tilde{f}^{-1}(r)) = \tilde{f}^{-1}(r)$ and $wh_r(X) \leq \aleph_0$. Therefore, $\tilde{f}^{-1}(r)$ is a closed $G_\delta^{\aleph_0}$-subset of X. If we use again the Lemma 1 for the family $\mathscr{A} = \{\tilde{f}^{-1}(r): r \in R\}$ we get a countable subfamily $\mathscr{B} = \{\tilde{f}^{-1}(r): r \in R'\}$ and a mapping $h \in$ L such that $h^{-1}h(\tilde{f}^{-1}(r)) = \tilde{f}^{-1}(r)$ for every $r \in R'$, $h^{-1}h(\overline{U\mathscr{B}}) = \overline{U\mathscr{B}} = \overline{U\mathscr{A}}$ and $wh(X) \leq \aleph_0$. Let us define a multivalued map $\Phi: h(\overline{U\mathscr{B}} \to \beta R$ by the formula $\Phi(z) = \tilde{f}(h^{-1}(z))$. Since h is open and $\Phi(z)$ is an one-point set for every $z \in h(U\mathscr{B})$, the mapping Φ is singlevalued and continuous. Besides we have that $\Phi(h(\overline{U\mathscr{B}}) = \beta R$, i. e. $w(\beta R) \leq \aleph_0$. Consequently, the space Y is pseudocompact.

The special case of this theorem, when βY is a dyadic compact, was proved by R. Engelking and A. Pelczynski in [6].

Proposition 1. *Let the space X possess a τ-lattice L of open mappings and let the closure $\bar{A}$ of the set $A < X$ be a $G_{\delta\Sigma}^\lambda$-set in X. Then A is $\mathscr{C}(\mu)$-embedded in $\bar{A}$ iff $\varphi(A)$ is $\mathscr{C}(\mu)$-embedded in $\varphi(\bar{A})$ for each $\varphi \in$ L with $\varphi^{-1}\varphi(\bar{A}) = \bar{A}$ and $w\varphi(X) = \tau\lambda\mu$.*

Proof. Let $\varphi(A)$ be $\mathscr{C}(\mu)$-embedded in $\varphi(\bar{A})$ for each $\varphi \in$ L, for which $\varphi^{-1}\varphi(\bar{A}) = \bar{A}$ and $w\varphi(X) \leq \tau\lambda\mu$. Let $f: A \to Y$ be a mapping from A onto a space Y with $w(Y) \leq \mu$. Let $\mathscr{B}$ be a base of Y and Card $\mathscr{C} \leq \mu$. A pair $\sigma_s = (W_s^1, W_s^2)$, where $W_s^i \in \mathscr{B}$, $i = 1, 2$ is called disjoint if $W_s^1 \cap W_s^2 = \emptyset$. Let $\sigma = \{\sigma_s : s \in S\}$ be the set of all disjoint pairs. Obviously, card $\sigma \leq \mu$. We put $U_s^i = f^{-1}(W_s^i)$, $i = 1, 2$. If $\tilde{U}_s^i$ are the maximal open subsets of $\bar{A}$ with $U_s^i = \tilde{U}_s^i \cap A$, $i = 1, 2$, then $\tilde{U}_s^1 \cap \tilde{U}_s^2 = \emptyset$. By Lemma 2, there us a mapping $\varphi \in$ L such that $\varphi^{-1}\varphi(\bar{A}) = \bar{A}$, $\varphi(\tilde{U}_s^1) \cap \varphi(U_s^2) = \emptyset$ for every $s \in S$ and $w\varphi(X) \leq \tau\lambda\mu$. Then $f(x_1) = f(x_2)$ if $\varphi(x_1) = (x_2)$ and $x_1, x_2 \in A$. Let is define a mapping $f_1: \varphi(A) \to Y$ by the formula $f_1(\varphi(x)) = f(\varphi^{-1}(\varphi(x)) \cap A)$ for $x \in A$. Obviously, this definition is correct and $f_1(\varphi(A)) = Y$. It is easily seen that f_1 is continuous, so there exists a continuous extension $\tilde{f}_1:\varphi(\bar{A})\to Y$ of f_1. Then the mapping $\tilde{f}_1\circ\varphi$ is a continuous extension of f.

Let A be $\mathscr{C}(\mu)$-embedded in $\bar{A}$ and $f:\varphi(A)\to Y$ is a mapping from $\varphi(A)$ onto Y with $w(Y)\le\mu$, where $\varphi\in L$, $\varphi^{-1}\varphi(\bar{A})=\bar{A}$ and $w\varphi(X)\le\tau\lambda\mu$. Let $f_1:\bar{A}\to Y$ be a continuous extension of $f_0\varphi$. Then the map $\tilde{f}:\varphi(\bar{A})\to Y$, defined by the formula $\tilde{f}(\varphi(x))=f_1(\varphi^{-1}\varphi(x))$, is continuous, since $\varphi^{-1}\varphi(\bar{A})=\bar{A}$ and φ is open. This completes the proof.

Remark 1. In the same manner we can get that the set A of Proposition 1 is $\mathscr{C}$-embeded ($\mathscr{C}^*$-embedded, respectively) in $\bar{A}$ iff $\varphi(A)$ is $\mathscr{C}$-embedded ($\mathscr{C}^*$-embedded, respectively) in $\varphi(\bar{A})$ for every $\varphi\in L$ satisfying the conditions $\varphi^{-1}\varphi(\bar{A})=(\bar{A})$ and $w\varphi(X)=\tau\lambda$.

Theorem 2. *Let the compact space X possess a τ-lattice* L *of open mappings and let the closure $\bar{A}$ be a $G^{\tau}_{\delta\Sigma}$-set in X. Then the space A is τ-pseudocompact iff $\varphi(A)=\varphi(\bar{A})$ for each $\varphi\in L$ with $w\varphi(X)\le\tau$.*

Proof. Obviously, if A is τ-pseudocompact then $\varphi(A)=\varphi(\bar{A})$ for $\varphi\in L$ with $w\varphi(X)\le\tau$.

Conversely, let $\varphi(A)=\varphi(\bar{A})$ for each $\varphi\in L$ with $w\varphi(X)\le\tau$ and let $f:A\to Y$ be a mapping from A onto a space Y with $w(Y)\le\tau$. By Proposition 1 A is $\mathscr{C}(\tau)$-embedded in $\bar{A}$, so there exists a continuous extension $\tilde{f}:\bar{A}\to Y$ of f, i.e. Y is a compact space. Hence, A is τ-pseudocompact and the theorem is proved.

Remark 2. (i). The conclusion of Theorem 2 can be expressed in a different way: A is τ-pseudocompact iff there exists no non-empty G^{τ}_{δ}-subset of $\bar{A}$ contained in $\bar{A}\setminus A$.

(ii) The space X is τ-pseudocompact iff the Stone-Chech remainder $\beta X\setminus X$ does not contain a non-empty G^{τ}_{δ}-subset of βX.

3. Corollary 1. *Let the compact space X possess a τ-lattice of open mappings and lef F be a closed $G^{\tau}_{\delta\Sigma}$-subset of X. If $f:F\to Y$ is an open mapping from F onto a space Y then Y is the Stone-Chech compactification of any of its dense and τ-pseudocompact subset.*

Proof. Let M be a dense and τ-pseudcompact subset of Y. By Remark 2 (ii), the set $Y\setminus M$ does not contain non-empty G^{τ}_{δ}-subset of Y. Therefore $F\setminus f^{-1}(M)$ does not contain non-empty G^{τ}_{δ}-subset of F and, by Remark 2(i), $f^{-1}(M)$ is τ-pseudocompact. From Proposition 1 it follows that F is the Stone-Chech compactification of $f^{-1}(M)$. Let $h:M\to[0,1]$ be a real function and let $\tilde{f}:F\to[0,1]$ be the continuous extension of h_0f. Then the map $\tilde{h}:Y\to[0,1]$, defined by the formula $\tilde{h}(y)=\sup\{\tilde{f}(x):x\in f^{-1}(y)\}$, is a continuous extension of h (see [1, pp. 358, problem 128]). Hence $\beta M=Y$.

The special case of Corollary 1, when $X=Y=F$, $f=\mathrm{id}$ and X is a Cartesian product of metrizable compact spaces, was proved by R. Engelking and B. Efimov in [5].

Corollary 2 [3]. *If G is a topological group, then βG is a topological group iff G is pseudocompact.*

Proof. Let G be a pseudocompact topological group. By Theorem 1 of [3], there exists such a compact group $\tilde{G}$ that G is embedded in $\tilde{G}$ as a dense subgroup. Since every compact topological group is $\varkappa$-metrizable [9], i. e. it possesses a lattice of open mappings, by Corollary 1 $\tilde{G}=\beta G$. If βG is a topological group it follows, by Theorem 1, that G is pseudocompact.

Corollary 3. *Let $\{X_\alpha;\ \alpha\in A\}$ be a family of compact spaces and let M_α be a dense τ-pseudocompact subset of X_α for every $\alpha\in A$. If the space $X=\Pi\{X_\alpha:\ \alpha\in A\}$ possesses a τ-lattice of open mappings the space $M=\Pi\{M_\alpha:\ \alpha\in A\}$ is τ-pseudocompact.*

Proof. By Corollary 2 it suffices to show that every non-empty G_δ^τ-subset of X intersects M. Let $P=\cap\{U_\beta:\beta<\tau\}\neq\emptyset$, where for each $\beta<\tau$ the set U_β is open in X. We can consider the sets U_β as elements of the standard base of X, i. e. $U_\beta=\Pi\{U_{\beta,\alpha}:\alpha\in A\}$. Then for every $\alpha\in A$ the set $W_\alpha=\cap\{U_{\beta\alpha}:\beta<\tau\}$ is a non-empty G_δ^τ-subset of X_α and by Remark 2(ii), $W_\alpha\cap M_\alpha\neq\emptyset$. If $x_\alpha\in W_\alpha\cap M_\alpha$, the point $x=(x_\alpha)_{\alpha\in A}$ belongs to $M\cap P$.

We shall say that the space X is locally pseudocompact (locally $\varkappa$-metrizable) if for every point x of X and for every its neighbourhood U there exists such a closed pseudocompact ($\varkappa$-metrizable) neighbourhood V with $x\in V\subset U$.

Corollary 4. *For any space X the following conditions are equivalent:*

a) *$X\in ANR$ (completely regular spaces);*
b) *$X\in ANR$ (locally compact spaces);*
c) *$X\in ANR$ (locally pseudocompact spaces);*
d) *$X\in ANR$ (locally $\varkappa$-metrizable spaces).*

For proving Corollary 4 we need the following two assertions:

A) *Let $\beta X\subset I^\tau$, where $\tau>\lambda=w(\beta X)$ and let $Y=I^\tau\setminus(\beta X\setminus X)$. If for some neighbourhood U of X in Y there exists a retraction $r:U\to X$, then X is locally compact.*

Proof. Let $x_0\in X$. Since $x_0\notin\overline{Y\setminus U}^{I^\tau}$ there is such an open neighbourhood V of x_0 in I^τ that $\bar{V}^{I^\tau}\cap\overline{Y\setminus U}^{I^\tau}=\emptyset$. Hence $M=\bar{V}^{I^\tau}\setminus(\beta X\setminus\lambda)=U\cap(\bar{V}^{I^\tau})$. Suppose there exists a non-empty G_δ^λ-subset P of $\bar{V}^{I^\tau}$ which is contained in $\beta X\setminus X$. Since $\bar{V}^{I^\tau}$ is a $G_\delta^{\aleph_0}$-subset of I^τ (see [9]), then P is a G_δ^λ-subset of I^τ, i. e. $w(P)=\tau$. This is a contradiction, because $w(P)\leq w(\beta X)=\lambda$. Therefore M is, by Remark 2(i), a λ-pseudocompact space, i. e. $r(M)$ is a compact space; besides $r(M)$ is a neighbourhood of the point x_0 in X. Consequently, X is a locally compact space.

B) *The space Y defined in* A), *is locally pseudocompact and locally $\varkappa$-metrizable.*

Proof. From the choice of τ it follows that Y is a dense subset of I^τ E. Shchepin has proved in [9], that $\varkappa$-metrizability is hereditary with respect to dense subsets and to closed domains. Hence, Y is locally $\varkappa$-metrizable. Let $y_0\in Y$ and let W be an open neighbourhood of y_0 in Y. Then there exists such an open subset W_1 of I^τ that $W_1\cap Y=W$. We have $\bar{W}^Y=\bar{W}_1^{I^\tau}\cap Y$, i. e. $\bar{W}^Y=\bar{W}_1^{I^\tau}\setminus(\beta X\setminus X)$. Since $w(\beta X\setminus X)\leq\lambda<\tau$ it follows from Theorem 2 that $\bar{W}^Y$ is a pseudocompact space. Therefore, Y is a locally pseudocompact space.

Proof of the Corollary 4. The implications a) $\to$ b) and c) $\to$ b) follow from A) and B) respectively.

d) $\to$ b). If $X \in$ ANR (locally $\varkappa$-metrizable spaces), by A) and B), X is a locally compact space. Since each locally compact space is a closed subset of $I^{\tau} \setminus \{a\}$ for some $\tau > \aleph_0$ and $a \in I^{\tau}$, which is locally $\varkappa$-metrizable, $X \in$ ANR (locally compact spaces).

b) $\to$ a). Let X be a closed subset of Y. Then X is a closed subset of the locally compact space $\beta Y \setminus (\overline{X}^{\beta Y} \setminus X)$ and now it is easy to end the proof.

b) $\to$ c). It follows from the implication b) $\to$ a).

b) $\to$ d). Let $X \in ANR$ (locally compact spaces). In view of the implication b) $\to$ a) it suffices to show that X is locally $\varkappa$-metrizable. Let the one-point compactification ωX of X is embedded in I^{τ} for some $\tau > \aleph_0$. Then there exists a retraction $r: U \to X$, where U is an open subset of $I^{\tau} \setminus \{\omega\}$. If $x_0 \in X$ and Ox_0 its neighbourhood in X, there exists such an open neighbourhood V of x_0 in I^{τ} that $\overline{V}^{I^{\tau}} \subset U$, $r(\overline{V}^{I^{\tau}}) \subset Ox_0$ and $\overline{V}^{I^{\tau}}$ top I^{τ}. Let $\tilde{r}$ be the restriction of r on $\overline{V}^{I^{\tau}}$ and $W = \tilde{r}^{-1}(V \cap X)$ Obviously, W is on open subset of $\overline{V}^{I^{\tau}}$ and $\tilde{r}(\overline{W}^{I^{\tau}}) = \overline{V \cap X}^{I^{\tau}} \subset X$ so the set $\overline{V \cap X}^{I^{\tau}}$ is a retract of $\overline{W}^{I^{\tau}}$. Since $\varkappa$-metrizability is hereditary with respect to closed domains and any retract of a $\varkappa$-metrizable compact space is one (see [9; 10]), the set $\overline{V \cap X}^{I^{\tau}}$ is a $\varkappa$-metrizable neighbourhood of x_0 in X and $\overline{V \cap X}^{I^{\tau}} \subset Ox_0$. Hence X is locally $\varkappa$-metrizable.

Corollary 5. *For every space X the following conditions are equivalent:*

a) *$X \in AR$ (completely regular spaces);*
b) *$X \in AR$ (compact spaces);*
c) *$X \in AR$ (locally compact spaces);*
d) *$X \in AR$ (pseudocompact spaces);*
e) *$X \in AR$ ($\varkappa$-metrizable spaces).*

The proof of Corollary 5 is similar to the proof of Corollary 4, so it is omitted.

Let us note, that the implication a) $\to$ b) in Corollary 5 answers the question of V. Belnov [2] about the existence of servant retracts in the class of all Hausdorff topological Abelian groups in the negative.

Yu. Lisitza called my attention to the fact that the equivalence of the conditions a) and b) in the last two corollaries is proved by O. Hanner in [7], although it is not formulated explicitly.

The next propositions are dimensional analogs (here the covering dimension is meant) of Corollary 4 and Corollary 5, respectively.

Corollary 6. *For every space X the following conditions are equivalent:*

a) *$X \in ANR$ (n-dimensional completely regular space);*
b) *$X \in ANR$ (n-dimensional locally compact spaces);*
c) *$X \in ANR$ (n-dimensional locally pseudocompact spaces).*

Corollary 7. *The following conditions are equivalent for space X:*

a) *$X \in AR$ (n-dimensional completely regular spaces);*
b) *$X \in AR$ (n-dimensional compact spaces);*
c) *$X \in AR$ (n-dimensional locally compact spaces).*

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Added in proof. The equivalence of the conditions a) and b) in both Corollary 4 and Corollary 5 was proved by E. Michael in [12]. Our proof is diferent from Michael's one.

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