ISSN 0001-4346, Mathematical Notes, 2011, Vol. 89, No. 3, pp. 319–327. © Pleiades Publishing, Ltd., 2011.
Published in Russian in Matematicheskie Zametki, 2011, Vol. 89, No. 3, pp. 331–341.

Extenders and $\varkappa$-Metrizable Compacta*

V. Valov**

Nipissing University, North Bay, ON, Canada

Received August 6, 2009

Abstract—A characterization of $\varkappa$-metrizable compacta in terms of extension of functions and upper semicontinuous compact-valued retractions to superextensions is established.

DOI: `10.1134/S0001434611030023`

Keywords: *Dugundji spaces, $\varkappa$-metrizable spaces, spaces with closed binary normal subbases, superextensions, probability space.*

1. INTRODUCTION

All topological spaces considered in this paper are assumed to be compact and Hausdorff and all single-valued maps, continuous.

Our main result is a characterization of *$\varkappa$-metrizable* compacta (see Theorem 1.1 below) similar to the original definition of *Dugundji spaces* given by Pełczyński in [1] (recall that the class of $\varkappa$-metrizable spaces was introduced by Shchepin [2] and contains all Dugundji spaces). This characterization is inspired by Shapiro's theorem [3, Theorem 3] that a compact space X is $\varkappa$-metrizable if and only if, for any embedding of X in a compact space Y, there exists a *monotone extender*

$$u\colon C_+(X) \to C_+(Y),$$

i.e., $u(f)|X = f$ for each $f \in C_+(X)$ and $f \le g$ implies $u(f) \le u(g)$. Here by $C(X)$ and $C_+(X)$ we denote, respectively, all continuous and continuous nonnegative functions on X. Following [3], we say that a map $u\colon C(X) \to C(Y)$ is (i) *monotone* (ii) *homogeneous*, or (iii) *weakly additive* if, for any $f, g \in C(X)$ and every real number k,

(i) $f \le g$ implies $u(f) \le u(g)$,

(ii) $u(k \cdot f) = k \cdot u(f)$, or

(iii) $u(f + k) = u(f) + k$,

respectively. If a map u satisfies conditions (i), (ii), and (iii), then $u(1) = 1$, $u(0) = 0$, and $u(g) \ge 0$ for any $f \in C(X)$ and $g \in C_+(X)$.

We say that a map $\varphi\colon C(X) \to \mathbb{R}$ is monotone, homogeneous, or weakly additive if φ satisfies condition (i), (ii), or (iii) with $\mathbb{R}$ instead of $C(Y)$. We also introduce and investigate a covariant functor S which takes each X to the space $S(X)$ of all monotone homogeneous weakly additive maps $\varphi\colon C(X) \to \mathbb{R}$ with the topology of pointwise convergence. It is easy to see that the map defined by $x \to \delta_x$ for $x \in X$, where $\delta_x(f) = f(x)$ for all $f \in C(X)$ and $x \in X$, is an embedding of X into $S(X)$. It turns out that $\varkappa$-metrizable compacta are precisely S-injective compact spaces in the sense of Shchepin [4]. The functor S, which is a subfunctor of the functor O introduced by Radul in [5], has remarkable properties. For example, S is open and weakly normal (see Sec. 2).

The characterization of $\varkappa$-metrizable compacta mentioned above is as follows.

---

[*] The text was submitted by the author in English.

[**] E-mail: `veskov@nipissingu.ca`

Theorem 1.1. *For a compact space X, the following conditions are equivalent*:

- i) *X is $\varkappa$-metrizable*;
- ii) *for any embedding of X in a compact space Y, there exists a monotone homogeneous weakly additive extender $u\colon C(X) \to C(Y)$*;
- iii) *for any embedding of X in a compact space Y, there exists a continuous map $r\colon Y \to S(X)$ such that $r(x) = \delta_x$ for all $x \in X$*;
- iv) *for any embedding of X in a compact space Y, there exists an upper semicontinuous compact-valued map $r\colon Y \to \lambda X$ such that $r(x) = \eta_x$ for all $x \in X$.*

Theorem 1.1 remains valid if S is replaced by O; see Theorem 4.2.

Supercompacta and superextensions were introduced by de Groot [6]. A space is said to be *supercompact* if it has a binary subbase for closed subsets. A family $\mathscr{S}$ of closed subsets of X is *binary* if any linked subfamily of $\mathscr{S}$ has nonempty intersection (we say that a system of subsets of X is *linked* if any two elements of this system intersect). Of special interest to us are supercompacta with a binary normal subbase. A subbase $\mathscr{S}$ is said to be *normal* if, for any $S_0, S_1 \in \mathscr{S}$ with $S_0 \cap S_1 = \varnothing$, there exist $T_0, T_1 \in \mathscr{S}$ such that

$$S_0 \cap T_1 = \varnothing = T_0 \cap S_1 \qquad \text{and} \qquad T_0 \cup T_1 = X.$$

A space X possessing a binary normal subbase $\mathscr{S}$ is said to be *normally supercompact* [7] and denoted by $(X, \mathscr{S})$.

The *superextension* λX of X consists of all maximal linked systems of closed sets in X. The family of sets

$$U^+ = \{\eta \in \lambda X : F \subset U \text{ for some } F \in \eta\},$$

where U ranges over all open subsets of X, is a subbase for the topology of λX. It is well known that λX is normally supercompact. Let η_x, where $x \in X$, be a maximal linked system of all closed sets containing x in X. The map $x \to \eta_x$ is an embedding of X into λX. The book [7] by van Mill contains more detailed information about normally supercompact spaces and superextensions; see also the book [8] by Fedorchuk and Filippov.

This paper is organized as follows. Section 2 is devoted to the functor S. In Sec. 3, we investigate a relationship between $\varkappa$-metrizable compacta and spaces with a closed binary subbase. As corollaries, we obtain Ivanov's results from [9] and [10] concerning superextensions of $\varkappa$-metrizable compacta, as well as results of Moiseev [11] on closed hyperspaces of inclusions. The proof of Theorem 1.1 is given in the final Section 4.

2. MONOTONE HOMOGENEOUS WEAKLY ADDITIVE FUNCTIONALS

The space $S(X)$ can be represented as a subset of the product space $\mathbb{R}^{C(X)}$ by identifying each $\varphi \in S(X)$ with $(\varphi(f))_{f \in C(X)} \in \mathbb{R}^{C(X)}$. Note that, according to [5, Lemma 1], every monotone weakly additive functional is nonexpanding. Thus, all $\mu \in S(X)$ are continuous maps on $C(X)$.

Proposition 2.1. *Let X be a compact space. Then $S(X)$ is a compact convex subset of $\mathbb{R}^{C(X)}$ containing the space $P(X)$ of all regular probability measures on X.*

Proof. It can be proved by a standard argument that $S(X)$ is a convex compact subset of $\mathbb{R}^{C(X)}$. Since $P(X)$ is the space of all monotone linear maps $\mu\colon C(X) \to \mathbb{R}$ with $\mu(1) = 1$ endowed with the topology of pointwise convergence, it follows that $P(X)$ is a subspace of $S(X)$. $\square$

Obviously, for any map $f\colon X \to Y$ between compacta, the formula

$$(S(f)(\mu))(h) = \mu(h \circ f), \qquad \text{where} \quad \mu \in S(X), \quad h \in C(Y),$$

defines a map $S(f)\colon S(X) \to S(Y)$. Moreover, $S(g \circ f) = S(g) \circ S(f)$. Thus, S is a covariant functor in the category COMP of compacta and (continuous) maps.

We say that a covariant functor F in the category COMP is *weakly normal* if it satisfies the following conditions:

(a) F preserves injective maps (i.e., F is monomorphic);

(b) F preserves surjective maps (i.e., F is epimorphic);

(c) F preserves intersections (i.e., $F\bigl(\bigcap_{\alpha\in A} X_\alpha\bigr) = \bigcap_{\alpha\in A} F(X_\alpha)$ for any family of closed subsets X_α of a given compact space X);

(d) F is continuous (i.e., F preserves the limits of inverse systems);

(e) F preserves the weight of infinite compacta;

(f) F preserves points and the empty set.

The above properties were considered in [4] by Shchepin, who introduced the important notion of a *normal functor*, that is, a weakly normal functor preserving preimages of sets. We have already mentioned that the functor S is a subfunctor of the functor O of order-preserving functionals introduced by Radul in [5]. Since O is weakly normal [5], it follows that S is injective and preserves weight, points, and the empty set.

Theorem 2.2. *The functor S is weakly normal but not normal.*

Proof. We follow the proof of a similar assertion for O given in [5]. First, let us show that S is epimorphic. Suppose that $f\colon X \to Y$ is a surjective map between compacta and $\nu \in S(Y)$. Let A denote the subspace of $C(X)$ consisting of all compositions $h \circ f$ for $h \in C(Y)$. We have $0_X \in A$, and if $\varphi \in A$, then A contains $\varphi + k$ and $k\varphi$ for any constant $k \in \mathbb{R}$ (such a subspace A is called a *weakly additive homogeneous subspace* of $C(X)$). Consider the monotone weakly additive homogeneous functional $\mu'\colon A \to \mathbb{R}$ defined by $\mu'(f \circ h) = \nu(h)$. If μ' can be extended to a functional $\mu \in S(X)$, then $S(f)(\mu) = \nu$. The proof that S is epimorphic is completed by applying the following statement.

Statement 1. *Let $\mu'\colon A \to \mathbb{R}$ be a monotone weakly additive homogeneous functional. Then μ' can be extended to a monotone weakly additive homogeneous functional $\mu\colon C(X) \to \mathbb{R}$.*

Following the proof of [5, Lemma 2], consider all pairs (B, μ_B), where $B \supset A$, B is a weakly additive homogeneous subspace of $C(X)$, and μ_B is a monotone weakly additive homogeneous functional on B. We define a partial order on these pairs by setting $(B, \mu_B) \le (C, \mu_C)$ if and only if $B \subset C$ and μ_C extends μ_B. By Zorn's lemma, there exists a maximal pair (B_0, μ_0). If $B_0 \ne C(X)$, then we take any $\varphi_0 \in C(X)\backslash B_0$ and define B_0^+ (B_0^-) to be the set of all $\varphi \in B_0$ such that $\varphi \ge \varphi_0$ (respectively, $\varphi \le \varphi_0$). Since μ_0 is monotone, there exists a $p \in \mathbb{R}$ for which $\mu_0(B_0^-) \le p \le \mu_0(B_0^+)$. The set B_0 is disjoint from $\{k\varphi + c : k, c \in \mathbb{R}\}$, and

$$D = B_0 \cup \{k\varphi + c : k, c \in \mathbb{R}\}$$

is a weakly additive homogeneous subspace of $C(X)$. Consider the functional $\mu\colon D \to \mathbb{R}$ defined by

$$\mu|B_0 = \mu_0 \qquad \text{and} \qquad \mu(k\varphi + c) = kp + c$$

for all constants k and c. It is easy to see that μ is a monotone weakly additive homogeneous functional on D extending μ_0, which contradicts the maximality of (B_0, μ_0). Hence $B_0 = C(X)$.

Therefore, S is epimorphic. This fact, together with the continuity of the functor O (see [5]), implies the continuity of S. The argument used in the proof of [5, Lemma 5 and Proposition 5] implies that S preserves intersections. Thus, S is weakly normal. In [5], an example showing that O does not preserve preimages is given; the same example, without any modification, shows that S does not preserve preimages either. □

Since S is monomorphic, it follows that the set

$$\operatorname{supp}(\mu) = \bigcap\{H \subset X : H \text{ is closed and } \mu \in S(H)\}$$

is well defined for any $\mu \in S(X)$; this set is called the *support* of μ. Moreover, S is continuous and epimorphic; thus, Proposition 3.5 in [4] implies the following assertion.

Corollary 2.3. *The set $S_\omega(X) = \{\mu \in S(X) : \operatorname{supp}(\mu)$ is finite$\}$ is dense in $S(X)$ for any compact space X.*

Theorem 2.4. *A surjective map $f\colon X \to Y$ is open if and only if so is the map $S(f)\colon S(X) \to S(Y)$.*

Proof. We say that a subset $A \subset S(X)$ is S-convex if A contains all $\mu \in S(X)$ for which

$$\inf A \le \mu \le \sup A.$$

Here $\inf A$ is the functional $\inf A\colon C(X) \to \mathbb{R}$ defined by

$$(\inf A)(h) = \inf\{\nu(h)\colon \nu \in S(X)\};$$

the definition of $\sup A$ is similar. Note that the functionals $\inf A$ and $\sup A$ are monotone and weakly additive but not homogeneous. Hence they are not elements of $S(X)$.

The proof of the theorem is completed by using the notion of S-continuity and following the same line of reasoning as in the proof of Theorem 1 in [12]. □

Recall that $\varkappa$-metrizable compacta are precisely compacta which can be represented as limits of inverse sigma-systems with open bonding maps [4]; therefore, Theorem 2.4 has the following corollary.

Corollary 2.5. *A compact space X is $\varkappa$-metrizable if and only if so is $S(X)$.*

Proposition 2.6. *For each compact space X, there exists an embedding $i\colon \lambda X \to S(X)$ such that $i(\eta_x) = \delta_x$ for any $x \in X$. Moreover, if X is disconnected, then $i(\lambda X) \ne S(X)$.*

Proof. The same argument as in the proof of Theorem 3 in [3] shows that

$$\max_{F\in\eta}\min_{x\in F} f(x) = \min_{F\in\eta}\max_{x\in F} f(x) \tag{1}$$

for any $\eta \in \lambda X$ and $f \in C(X)$, and $\varphi_\eta(f) = \max_{F\in\eta}\min_{x\in F} f(x)$ defines a monotone homogeneous weakly additive map $\varphi_\eta\colon C(X) \to \mathbb{R}$. Therefore, $\varphi_\eta \in S(X)$ for any $\eta \in \lambda X$. Moreover,

$$\varphi_{\eta_x}(f) = \max_{\{F:x\in F\}}\min_{y\in F} f(y) = f(x) \qquad \text{for any} \quad x \in X.$$

Thus, the map $i\colon \lambda X \to S(X)$ defined by $i(\eta) = \varphi_\eta$ satisfies the condition $i(\eta_x) = \delta_x$ for $x \in X$. To prove the first part of the proposition, we must show that i is injective and continuous.

Take $\eta \in \lambda X$ and let W be a neighborhood of φ_η in $S(X)$. We can assume that W consists of all $\varphi \in S(X)$ such that $|\varphi_\eta(f_i) - \varphi(f_i)| < \epsilon_i$ for some functions $f_i \in C(X)$ and numbers $\epsilon_i > 0$, where $i = 1, \dots, k$. We set

$$U_i = \{x \in X : f_i(x) > \varphi_\eta(f_i) - \epsilon_i\}, \qquad V_i = \{x \in X : f_i(x) < \varphi_\eta(f_i) + \epsilon_i\}.$$

By virtue of (1), for any $i \le k$, there exist $F_i, H_i \in \eta$ such that $F_i \subset U_i$ and $H_i \subset V_i$. The intersection $G = \bigcap_{i=1}^{i=k}(U_i^+ \cap V_i^+)$ is a neighborhood of η in λX, and $i(\xi) \in W$ for all $\xi \in G$. Thus, the map i is continuous.

Let η and ξ be two different elements of λX. Then there exist $F_0 \in \eta$ and $H_0 \in \xi$ for which $F_0 \cap H_0 = \varnothing$. Let $f \in C_+(X)$ be a function with the properties $f \le 1$, $f(F_0) = 1$, and $f(H_0) = 0$. Then

$$\varphi_\eta(f) = \max_{F\in\eta}\min_{x\in F} f(x) = 1 \qquad \text{and} \qquad \varphi_\xi(f) = \min_{H\in\xi}\max_{x\in H} f(x) = 0.$$

This implies the injectivity of i.

The second part of the proposition follows directly from the fact that $S(X)$ is always connected (as a convex set in $\mathbb{R}^{C(X)}$), whereas λX is disconnected for disconnected X. □

3. $\varkappa$-METRIZABLE COMPACT SPACES AND SUPEREXTENSIONS

Recall that a compact space X is said to be openly generated [4] if X is the limit space of an inverse sigma-system with open bonding maps. Shchepin [2] proved that any $\varkappa$-metrizable compact space is openly generated. On the other hand, any openly generated compact space is $\varkappa$-metrizable (see [4]); this follows from Ivanov's theorem that λX is a Dugundji space for any openly generated X [9]. We apply Ivanov's result to prove Theorem 1.1 in the next section. For this reason, in this section, we give an independent proof of a more general assertion (Proposition 3.2). A part of this proof is Lemma 3.1, which was first proved by Shirokov in [13]. Shirokov's proof of Lemma 3.1 is based on Ivanov's result mentioned above, while our proof is based on Shchepin's results [14] on inverse systems with open bonding maps.

We say that a subspace X in a space Y is *regularly embedded* in Y if there exists a map e from the topology of X to the topology of Y which satisfies the conditions $e(\varnothing)=\varnothing$, $e(U)\cap X=U$, and $e(U)\cap e(V)=\varnothing$ provided that $U\cap V=\varnothing$.

Lemma 3.1. *Any embedding of an openly generated compact space X in a space Y is regular.*

Proof. It suffices to prove the existence of a regular embedding of X in the Tychonoff cube $\mathbb{I}^A$ for some A. Such an embedding exists if X is metrizable (see [15, Sec. 21. XI, Theorem 2]. Suppose that the required assertion holds for any openly generated compact space of weight $<\tau$ and consider an openly generated compact space X of weight τ. Such a space X is the limit of a well-ordered inverse system

$$S=\{X_\alpha,p^\alpha_\beta,\beta<\alpha<\omega(\tau)\}$$

in which all bonding maps $p_\alpha\colon X\to X_\alpha$ are open surjections and all X_α are openly generated compacta of weight $<\tau$ (see [4]). Here $\omega(\tau)$ denotes the first ordinal of cardinality τ. By assumption, for any $\alpha<\omega(\tau)$, there exists a regular embedding of X_α into the cube $\mathbb{I}^{A(\alpha)}$. According to [14], X has a base $\mathscr{B}$ consisting of open sets $U\subset X$ with finite rank $d(U)$, where

$$d(U)=\{\alpha: p_{\alpha+1}(U)\neq(p^{\alpha+1}_\alpha)^{-1}(p_\alpha(U))\}.$$

For any $U\in\mathscr{B}$, we set

$$\Gamma(U)=\{\alpha_0,\alpha,\alpha+1:\alpha\in d(U)\},$$

where $\alpha_0\in A$ is fixed. Obviously, X is a subset of $\prod\{X_\alpha:\alpha<\omega(\tau)\}$. For any $U\in\mathscr{B}$, consider the open set $\gamma_1(U)\subset\prod\{X_\alpha:\alpha<\omega(\tau)\}$ defined by

$$\gamma_1(U)=\prod\{p_\alpha(U):\alpha\in\Gamma(U)\}\times\prod\{X_\alpha:\alpha\notin\Gamma(U)\}.$$

Let

$$\gamma(W)=\bigcup\{\gamma_1(U):U\in\mathscr{B}\text{ and }\overline{U}\subset W\},\qquad W\in\mathscr{T}_X.$$

Statement 2. *The following assertions hold*:

i) $\gamma(W_1)\cap\gamma(W_2)=\varnothing$ *whenever* W_1 *and* W_2 *are disjoint open sets in* X;

ii) $\gamma(W)\cap X=W$ *for* $W\in\mathscr{T}_X$.

Proof. Suppose that $W_1\cap W_2=\varnothing$. To prove assertion (i), it suffices to show that $\gamma_1(U_1)\cap\gamma_1(U_2)=\varnothing$ for any pair of sets $U_1,U_2\in\mathscr{B}$ such that $\overline{U_i}\subset W_i$ for $i=1,2$. Indeed, take such a pair U_1,U_2 and let $\beta=\max\{\Gamma(U_1)\cap\Gamma(U_2)\}$. Then β is either α_0 or $\max\{d(U_1)\cap d(U_2)\}+1$. In both cases, we have $d(U_1)\cap d(U_2)\cap[\beta,\omega(\tau))=\varnothing$. It follows from [14, Sec. 5, Lemma 3] that $p_\beta(U_1)\cap p_\beta(U_2)=\varnothing$. The relation $\beta\in\Gamma(U_1)\cap\Gamma(U_2)$ implies $\gamma_1(U_1)\cap\gamma_1(U_2)=\varnothing$.

Obviously, $W\subset\gamma(W)\cap X$ for all $W\in\mathscr{T}_X$. Thus, condition (ii) will follow as soon as we prove that $\gamma(W)\cap X\subset W$. Take $x\in\gamma(W)\cap X$ and $U\in\mathscr{B}$ for which $x\in\gamma_1(U)$ and $\overline{U}\subset W$. Let

$$\beta(U)=\max d(U)+1.$$

Then $p_\alpha(x) \in p_\alpha(U)$ for all $\alpha \leq \beta(U)$. Since $\alpha \notin d(U)$ for all $\alpha \geq \beta(U)$, it follows that

$$(p^\alpha_{\beta(U)})^{-1}(p_{\beta(U)}(x)) \subset p_\alpha(U) \qquad \text{for} \quad \alpha > \beta(U).$$

Therefore, $p_\alpha(x) \in p_\alpha(U)$ for all α, whence $x \in \overline{U}$. Thus, $x \in W$, which proves the required assertion. □

We want to show that the set $\prod\{X_\alpha : \alpha < \omega(\tau)\}$ is regularly embedded in $\prod\{\mathbb{I}^{A(\alpha)} : \alpha < \omega(\tau)\}$. The regular embedding of X_α in $\mathbb{I}^{A(\alpha)}$ implies the existence of a regular operator $e_\alpha\colon \mathscr{T}_{X_\alpha} \to \mathscr{T}_{\mathbb{I}^{A(\alpha)}}$ for each α. Let $\mathscr{B}_1$ be a base for the topology of $\prod\{X_\alpha : \alpha < \omega(\tau)\}$ consisting of sets of the form

$$V = \prod\{U_\alpha : \alpha \in \Lambda(V)\} \times \prod\{X_\alpha : \alpha \notin \Lambda(V)\},$$

where $\Lambda(V)$ is finite. To each $V \in \mathscr{B}_1$ we assign the open set

$$\theta_1(V) \subset \prod\{\mathbb{I}^{A(\alpha)} : \alpha < \omega(\tau)\}$$

defined by

$$\theta_1(V) = \prod\{e_\alpha(U_\alpha) : \alpha \in \Lambda(V)\} \times \prod\{\mathbb{I}^{A(\alpha)} : \alpha \notin \Lambda(V)\}.$$

Consider the regular operator θ from the topology of $\prod\{X_\alpha : \alpha < \omega(\tau)\}$ to the topology of $\prod\{\mathbb{I}^{A(\alpha)} : \alpha < \omega(\tau)\}$ defined by

$$\theta(G) = \bigcup\{\theta_1(V) : V \in \mathscr{B}_1 \text{ and } V \subset G\}.$$

By Statement 2, X is regularly embedded in $\prod\{X_\alpha : \alpha < \omega(\tau)\}$, and γ is the corresponding regular operator. Therefore, the relation $e_1(W) = \theta(\gamma(W))$, where $W \in \mathscr{T}_X$, defines a regular operator from the topology of X to that of $\prod\{\mathbb{I}^{A(\alpha)} : \alpha < \omega(\tau)\}$. Thus, X is regularly embedded in $\mathbb{I}^A$, where $A = \bigcup\{A(\alpha) : \alpha < \omega(\tau)\}$. □

Proposition 3.2. *Let X be an openly generated compact space with a binary subbase $\mathscr{S}$ for closed subsets. Then X is a Dugundji space. Moreover, if, in addition, X is connected and $\mathscr{S}$ is normal, then X is an absolute retract.*

Proof. Let X be an openly generated compact set with a binary subbase $\mathscr{S}$ for closed subsets. The space X is embedded in $\mathbb{I}^\tau$ for some τ. By Lemma 3.1, X is regularly embedded in $\mathbb{I}^\tau$; hence there exists a regular operator $e\colon \mathscr{T}_X \to \mathscr{T}_{\mathbb{I}^\tau}$. Consider the (set-valued) map $r\colon \mathbb{I}^\tau \to X$ defined by

$$r(y) = \begin{cases} \bigcap\{I_{\mathscr{S}}(\overline{U}) : y \in e(U),\ U \in \mathscr{T}_X\} & \text{for } y \in \bigcup\{e(U) : U \in \mathscr{T}_X\}, \\ X & \text{otherwise}, \end{cases} \tag{2}$$

where $\overline{U}$ is the closure of U in X and

$$I_{\mathscr{S}}(\overline{U}) = \bigcap\{S \in \mathscr{S} : \overline{U} \subset S\}.$$

Since e is regular, it follows that the system $\gamma_y = \{U \in \mathscr{T}_X : y \in e(U)\}$ is linked for any $y \in \mathbb{I}^\tau$. Therefore, the system

$$\omega_y = \{S \in \mathscr{S} : \overline{U} \subset S \text{ for some } U \in \gamma_y\}$$

is linked as well, and $r(y) = \bigcap\{S : S \in \omega_y\} \neq \varnothing$.

It is easy to see that $r(x) = x$ for all $x \in X$. Moreover, r is upper semicontinuous. Indeed, take $y \in \mathbb{I}^\tau$ and suppose that $r(y) \subset W$, where $W \in \mathscr{T}_X$. Then there exist finitely many sets $U_i \in \mathscr{T}_X$, where $i = 1, 2, \dots, k$, for which

$$y \in \bigcap_{i=1}^{i=k} e(U_i) \qquad \text{and} \qquad \bigcap_{i=1}^{i=k} I_{\mathscr{S}}(\overline{U}_i) \subset W.$$

Obviously, $r(y') \subset W$ for all $y' \in \bigcap_{i=1}^{i=k} e(U_i)$. Thus, r is an upper semicontinuous compact-valued retraction of $\mathbb{I}^\tau$ to X. According to [16], X is a Dugundji space.

Now, suppose that X is connected and $\mathscr{S}$ is a binary normal subbase for X. According to [7], any set of the form $I_{\mathscr{S}}(F)$ is $\mathscr{S}$-convex (a set $A \subset X$ is said to be *$\mathscr{S}$-convex* if $I_{\mathscr{S}}(x,y) \subset A$ for any $x, y \in A$). Thus, each $r(y)$ is $\mathscr{S}$-convex. By Corollary 1.5.8 from [7], all closed $\mathscr{S}$-convex subsets in X are connected. Hence the map r defined by (2) is connected-valued. Therefore, according to [16], X is an absolute extensor in dimension 1, and there exists a map $r_1 \colon \mathbb{I}^\tau \to \exp X$ such that $r_1(x) = \{x\}$ for all $x \in X$ (see [17, Theorem 3.2]). Here $\exp X$ is the space of all closed subsets of X with the Vietoris topology. On the other hand, since X is a normal supercompact space, there exists a retraction r_2 of $\exp X$ to X (see [7, Corollary 1.5.20]). The composition $r_2 \circ r_1 \colon \mathbb{I}^\tau \to X$ is a (single-valued) retraction. It follows that $X \in \mathrm{AR}$. □

Corollary 3.3 ([9], [10]). *If X is an openly generated compact space, then λX is a Dugundji space. If, in addition, X is connected, then λX is an absolute retract.*

Proof. It is easy to see that λ is a continuous functor preserving open maps (see [8]). Therefore, λX is openly generated. Moreover, λX has a binary normal subbase, and the application of Proposition 3.2 completes the proof. □

A closed subset $\mathscr{A}$ of $\exp X$ is called an *inclusion hyperspace* if, for any $A \in \mathscr{A}$ and $B \in \exp X$, the inclusion $A \subset B$ implies $B \in \mathscr{A}$. The space of all inclusion hyperspaces with the topology induced by $\exp^2 X$ is a closed subspace of $\exp^2 X$; it is denoted by GX [11].

Corollary 3.4 ([11]). *If X is an openly generated compact space, then GX is a Dugundji space. If, in addition, X is connected, then GX is an absolute retract.*

Proof. Since G is a continuous functor preserving open maps [18], it follows that GX is openly generated, provided that so is X. On the other hand, GX admits a binary subbase [19]. Hence, by Proposition 3.2, GX is a Dugundji space. Suppose that X is connected and openly generated. Then GX is a connected Dugundji space. Therefore, $\lambda(GX)$ is an absolute retract. Moreover, there are natural inclusions

$$GX \subset \lambda(GX) \subset G^2X = G(GX)$$

and a retraction of G^2X to GX (see [19, Lemma 2]). Therefore, $GX \in \mathrm{AR}$ as a retract of $\lambda(GX)$. □

4. PROOF OF THEOREM 1.1

We begin this section with the following lemma.

Lemma 4.1. *Let X be a compact regularly embedded subset of a space Y. Then there exists an upper semicontinuous compact-valued map $r \colon Y \to \lambda X$ such that $r(x) = \eta_x$ for all $x \in X$.*

Proof. Let $e \colon \mathscr{T}_X \to \mathscr{T}_Y$ be a regular operator from the topology of X to the topology of Y. Consider the map $r \colon Y \to \lambda X$ defined by

$$r(y) = \begin{cases} \bigcap\{(\overline{U})^+ : y \in e(U),\ U \in \mathscr{T}_X\} & \text{if } y \in \bigcup\{e(U) : U \in \mathscr{T}_X\}, \\ \lambda X & \text{otherwise,} \end{cases}$$

where $\overline{U}$ denotes the closure of U in X and $(\overline{U})^+ \subset \lambda X$ consists of all $\eta \in \lambda X$ for which $\overline{U} \in \eta$. Since e is regular, it follows that the system $\{U \in \mathscr{T}_X : y \in e(U)\}$ is linked for any $y \in Y$. Therefore, the system $\{\overline{U} : y \in e(U)\}$ is linked as well, and it consists of sets closed in X. This implies $\bigcap\{(\overline{U})^+ : y \in e(U),\ U \in \mathscr{T}_X\} \neq \varnothing$. Thus, the values of r are nonempty compact sets. It is easy to see that $r(x) = \eta_x$ for all $x \in X$. It can also be shown (in the same way as in the proof of Proposition 3.2) that r is upper semicontinuous. □

Proof of Theorem 1.1. (i) ⇒ (ii). We follow the proof of Theorem 3 in [3]. Relation (1) defines a homogeneous weakly additive extender $u_1\colon C(X) \to C(\lambda X)$. Consider the space Z obtained by attaching λX to Y at the points of X. The space λX is a closed subset of Z, and since λX is a Dugundji space (see Corollary 3.3), there exists a linear monotone extension operator $u_2\colon C(\lambda X) \to C(Z)$ for which $u_2(1) = 1$. The map $u\colon C(X) \to C(Y)$ defined by $u(f) = u_2(u_1(f))|Y$ for $f \in C(X)$ is a monotone homogeneous weakly additive extender.

(ii) ⇒ (iii). Let X be a subspace of Y, and let $u\colon C(X) \to C(Y)$ be a monotone homogeneous weakly additive extender. Consider the map $r\colon Y \to S(X)$ defined by $r(y)(f) = u(f)(y)$ for $y \in Y$ and $f \in C(X)$. It is easy to see that r is continuous and $r(x) = \delta_x$ for all $x \in X$.

(iii) ⇒ (iv). Let $r_2\colon Y \to S(X)$ be a continuous map such that $r_2(x) = \delta_x$ for all $x \in X$. The formula $u(f)(\varphi) = \varphi(f)$ defines a monotone extender $u\colon C(X) \to C(S(X))$. By virtue of Theorem 4.1 from [20], X is regularly embedded in $S(X)$. Thus, by Lemma 4.1, there exists an upper semicontinuous compact-valued map $r_1\colon S(X) \to \lambda X$ such that $r_1(\delta_x) = \eta_x$ for $x \in X$. The map $r = r_1 \circ r_2\colon Y \to \lambda X$ has the required property.

(iv) ⇒ (i). The space X is embedded in $\mathbb{I}^A$ for some A. Let $r\colon \mathbb{I}^A \to \lambda X$ be an upper semicontinuous compact-valued map such that $r(x) = \eta_x$ for $x \in X$. For any open set $U \subset X$, we put

$$e(U) = \{y \in \mathbb{I}^A : r(y) \subset U^+\}.$$

Since U^+ is open in λX and r is upper semicontinuous, it follows that $e(U)$ is open in $\mathbb{I}^A$. Moreover, $e(U) \cap X = U$ and $e(U) \cap e(V) = \varnothing$, provided that U and V are disjoint. Thus, X is regularly embedded in $\mathbb{I}^A$. Finally, according to [13], X is $\varkappa$-metrizable. □

Now, consider the functor O of order-preserving functionals introduced by Radul in [5]. Recall that a map $\varphi\colon C(X) \to \mathbb{R}$ is said to be *order-preserving* if this map is monotone and weakly additive and $\varphi(1_X) = 1$. Theorem 4.2 stated below follows from Theorem 1.1. Indeed, since S is a subfunctor of the functor O, we have an embedding $X \subset S(X) \subset O(X)$ for any compact set X. This proves the implications (i) ⇒ (ii) and (ii) ⇒ (iii) in Theorem 4.2. The proofs of (iii) ⇒ (vi) and (vi) ⇒ (i) are similar.

Theorem 4.2. *For any compact space X, the following conditions are equivalent*:

- i) *X is $\varkappa$-metrizable*;
- ii) *for any embedding of X in a compact space Y, there exists a monotone weakly additive extender $u\colon C(X) \to C(Y)$ such that $u(1_X) = 1_Y$*;
- iii) *for any embedding of X in a compact space Y, there exists a continuous map $r\colon Y \to O(X)$ such that $r(x) = \delta_x$ for all $x \in X$*;
- iv) *for any embedding of X in a compact space Y, there exists an upper semicontinuous compact-valued map $r\colon Y \to \lambda X$ such that $r(x) = \eta_x$ for all $x \in X$.*

Let F be a covariant functor in the category of compacta and continuous maps. A compact space X is said to be *F-injective [4]* if, for any map $f\colon Y \to X$ and any embedding $i\colon Y \to Z$, there exists a map $g\colon F(Z) \to F(X)$ for which $F(f) = g \circ F(i)$. It is easy to see that if F is a functor with the property that, for any X, there exists an embedding $j_X\colon X \to F(X)$, then X is F-injective if and only if, for any embedding of X in a compact space Y, there exists a map $h\colon Y \to F(X)$ such that $h(x) = j_X(x)$ for all $x \in X$. It follows from Theorem 1.1 (iii) that $\varkappa$-metrizable compacta are precisely S-injective compact sets.

ACKNOWLEDGMENTS

The author expresses gratitude to the referee for valuable comments and suggestions, which completely changed the first version of the paper. The author is also indebted to Taras Radul, who pointed out that S is a subfunctor of O and kindly provided his papers on properties of the functor O.

The first version of the manuscript was prepared during the author's visit to the Faculty of Mathematics and Computer Science, Sofia University, in June and July of 2009. The author thanks the faculty for hospitality.

This work was supported by NSERC (grant no. 261914-08.2010).

REFERENCES

1. A. Pełczyński, "Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions," Diss. Math. **58** (1968).
2. E. V. Shchepin, "Topology of limit spaces of uncountable inverse spectra," Uspekhi Mat. Nauk **31** (5), 191–226 (1976) [Russian Math. Surveys **31** (5), 155–191 (1976)].
3. L. B. Shapiro, "On function extension operators and normal functors," Vestnik Moskov. Univ. Ser. I Mat., Mekh., No. 1, 35–42 (1992) [Moscow Univ. Math. Bull. **47** (1), 34–38 (1992)].
4. E. V. Shchepin, "Functors and uncountable powers of compacta," Uspekhi Mat. Nauk **36** (3), 3–62 (1981) [Russian Math. Surv. **36** (3), 1–71 (1981)].
5. T. Radul, "On the functor of order-preserving functionals," Comment. Math. Univ. Carolin. **39** (3), 609–615 (1998).
6. J. de Groot, "Supercompactness and superextensions," in *Contributions to Extension Theory of Topological Structures. Proceedings of the Symposium Held in Berlin, August 14–19*, Ed. by J. Flachsmeyer, H. Poppe, and F. Terpe (VEB Deutscher Verlag Wiss., Berlin, 1969), pp. 89–90.
7. J. van Mill, *Supercompactness and Wallman Spaces*, in *Math. Centre Tracts* (Mathematisch Centrum, Amsterdam, 1977), Vol. 85.
8. V. V. Fedorchuk and V. V. Filippov, *General Topology. Basic Constructions* (Mosk. Gos. Univ., Moscow, 1988) [in Russian].
9. A. V. Ivanov, "Superextensions of openly generated compact Hausdorff spaces," Dokl. Akad. Nauk SSSR **259** (2), 275–278 (1981) [Soviet Math. Dokl. **24**, 60–64 (1981)].
10. A. V. Ivanov, "Solution of van Mill's problem on the characterization of compacta, the superextensions of which are absolute retracts," Dokl. Akad. Nauk SSSR **262** (3), 526–528 (1982) [Soviet Math. Dokl. **25**, 68–70 (1982)].
11. E. V. Moiseev, "On spaces of closed growth and inclusion hyperspaces," Vestnik Moskov. Univ. Ser. I Mat., Mekh., No. 3, 54–57 (1988) [Moscow Univ. Math. Bull. **43** (3), 48–51 (1988)].
12. T. Radul, "Topology of the space of order-preserving functionals," Bull. Pol. Acad. Sci. Math. **47** (1), 53–60 (1999).
13. L. V. Shirokov, "An extrinsic characterization of Dugundji spaces and $\varkappa$-metrizable compact Hausdorff spaces," Dokl. Akad. Nauk SSSR **263** (5), 1073–1077 (1982) [Soviet Math. Dokl. **25**, 507–510 (1982)].
14. E. V. Shchepin, "On $\varkappa$-metrizable spaces," Izv. Akad. Nauk SSSR Ser. Mat. **43** (2), 442–478 (1979) [Math. USSR-Izv. **14**, 407–440 (1980)].
15. K. Kuratowski, *Topology* (PWN, Warsaw, 1966), Vol. 1.
16. A. N. Dranishnikov, "Many-valued absolute retracts and absolute extensors in dimensions 0 and 1," Uspekhi Mat. Nauk **39** (5), 241–242 (1984) [Russian Math. Surv. **39** (5), 273–274 (1984)].
17. V. V. Fedorchuk, "Certain geometric properties of covariant functors," Uspekhi Mat. Nauk **39** (5), 169–208 (1984) [Russian Math. Surv. **39** (5), 199–249 (1984)].
18. A. Teleiko and M. Zarichnyi, *Categorical Topology of Compact Hausdorff Spaces*, in *Math. Stud. Monogr. Ser.* (VNTL Publ., Lviv, 1999), Vol. 5.
19. T. N. Radul, "The monad of inclusion hyperspaces and its algebras," Ukr. Mat. Zh. **42** (6), 806–811 (1990) [Ukr. Math. J. **42** (6), 712–716 (1990)].
20. E. K. van Douwen, *Simultaneous Extension of Continuous Functions*, Ph. D. Thesis (Vrije Univ., Amsterdam, 1975).