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GENERALIZED CANTOR MANIFOLDS AND HOMOGENEITY

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Abstract. A classical theorem of Alexandroff states that every n-dimensional compactum X contains an n-dimensional Cantor manifold. This theorem has a number of generalizations obtained by various authors. We consider extension-dimensional and infinite dimensional analogs of strong Cantor manifolds, Mazurkiewicz manifolds, and V^n-continua, and prove corresponding versions of the above theorem. We apply our results to show that each homogeneous metrizable continuum which is not in a given class $\mathcal{C}$ is a strong Cantor manifold (or at least a Cantor manifold) with respect to $\mathcal{C}$. Here, the class $\mathcal{C}$ is one of four classes that are defined in terms of dimension-like invariants. A class of spaces having bases of neighborhoods satisfying certain special conditions is also considered.

Contents

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1. Introduction

All spaces in this paper are assumed to be at least normal.

Cantor manifolds were introduced by Urysohn [39] as a generalization of Euclidean manifolds. Recall that a space X is a *Cantor n-manifold* if X cannot be separated by a closed $(n-2)$-dimensional subset. In other words, X cannot be the union of two proper closed sets whose intersection is of covering dimension $\leq n-2$. Alexandroff [2] introduced the stronger notion of V^n-continua: a compactum X is a *V^n-continuum* if for every two closed disjoint subsets X_0, X_1 of X, both having non-empty interior in X, there exists an open cover ω of X such that there is no partition P in X between X_0 and X_1 admitting an ω-map into a space Y with $\dim Y \leq n-2$. Another specification of Cantor manifolds was considered by Hadžiivanov [15]: X is a *strong Cantor n-manifold* if for arbitrary representation $X = \bigcup_{i=1}^{\infty} F_i$, where all F_i are proper closed subsets of X, we have $\dim(F_i \cap F_j) \geq n-1$ for some $i \neq j$.

Obviously, strong Cantor n-manifolds are Cantor n-manifolds. Moreover, every V^n-continuum is a strong Cantor n-manifold [17] and none of the above inclusions is reversible (see [16], [29] and the Appendix).

In the present paper we generalize these notions by considering a general dimension function $D_{\mathcal{K}}$ which captures the covering dimension, cohomological dimension $\dim_G$ with respect to any Abelian group G, as well as the extraordinary dimension $\dim_L$ with respect to a given CW-complex L.

More precisely, a sequence $\mathcal{K} = \{K_0, K_1, ..\}$ of CW-complexes is called a *stratum* for a dimension theory [8] if

- for each space X admitting a perfect map onto a metrizable space, $K_n \in AE(X)$ implies both $K_{n+1} \in AE(X \times \mathbb{I})$ and $K_{n+j} \in AE(X)$ for all $j \geq 0$.

Here, $K_n \in AE(X)$ means that K_n is an absolute extensor for X. Given a stratum $\mathcal{K}$, we can define a dimension function $D_{\mathcal{K}}$ in a standard way:

(1) $D_{\mathcal{K}}(X) = -1$ iff $X = \emptyset$;
(2) $D_{\mathcal{K}}(X) \leq n$ if $K_n \in AE(X)$ for $n \geq 0$; if $D_{\mathcal{K}}(X) \leq n$ and $K_m \notin AE(X)$ for all $m < n$, then $D_{\mathcal{K}}(X) = n$;
(3) $D_{\mathcal{K}}(X) = \infty$ if $D_{\mathcal{K}}(X) \leq n$ is not satisfied for any n.

Since every CW-complex K with the weak topology is homotopically equivalent to K equipped with the metric topology, we can assume that all $K_i \in \mathcal{K}$ are considered with the metric topology.

If $\mathcal{K} = \{\mathbb{S}^0, \mathbb{S}^1, ..\}$, we obtain the covering dimension dim. The stratum $\mathcal{K} = \{\mathbb{S}^0, K(G,1), .., K(G,n), ..\}$, $K(G,n)$, $n \geq 1$, being the Eilenberg-MacLane complexes for a given group G, determines the cohomological dimension $\dim_G$. Moreover, if L is a fixed CW-complex and $\mathcal{K} = \{L, \Sigma(L), .., \Sigma^n(L), ..\}$, where $\Sigma^n(L)$ denotes the n-th iterated suspension of L, we obtain the extraordinary dimension $\dim_L$ introduced recently by Shchepin [36] and considered in details by Chigogidze [6].

According to the countable sum theorem in extension theory, it follows directly from the above definition that $D_{\mathcal{K}}(X) \leq n$ implies $D_{\mathcal{K}}(A) \leq n$ for any F_σ-subset $A \subset X$.

Now, it is clear how to define Cantor n-manifolds, strong Cantor n-manifolds and V^n-continua with respect to $D_{\mathcal{K}}$, where $\mathcal{K}$ is a fixed stratum. Furthermore, we consider quite general concepts of Mazurkiewicz manifolds, strong Cantor manifolds and Cantor manifolds with respect to some classes of finite or infinite-dimensional spaces. We define them following the idea and some terminology from [16, 17].

A non-empty class of spaces $\mathcal{C}$ is said to be *admissible* if it satisfies the following conditions:

(i) $\mathcal{C}$ contains all topological copies of any element $X \in \mathcal{C}$;
(ii) if $X \in \mathcal{C}$, then each F_σ-subset of X belongs to $\mathcal{C}$.

Definition 1. A space X is a *Mazurkiewicz manifold with respect to an admissible class* $\mathcal{C}$ if for every two closed, disjoint subsets $X_0, X_1 \subset X$, both having non-empty interiors in X, and every F_σ-subset $F \subset X$ with $F \in \mathcal{C}$, there exists a continuum in $X \setminus F$ joining X_0 and X_1.

The notion of a Mazurkiewicz manifold has its roots in the classical Mazurkiewicz theorem saying that no region in the Euclidean n-space can be cut by a subset of dimension $\leq n-2$ [11]. Recall that a set P (not necessarily closed) *cuts* a space X between two subsets X_0 and X_1 of X if X_0, X_1, and P are disjoint, and for any continuum C such that $C \cap X_i \neq \emptyset$, $i = 0, 1$, we have $C \cap P \neq \emptyset$; P *cuts* X if it cuts X between a pair of distinct points.

One can easily prove, using Lemma 2.5, that if no F_σ-subset from an admissible class $\mathcal{C}$ cuts a compact space X, then X is a Mazurkiewicz manifold with respect to $\mathcal{C}$; the converse implication holds for locally connected compact spaces X.

Definition 2. A space X is a *strong Cantor manifold with respect to an admissible class* $\mathcal{C}$ if X can not be represented as the union

$$X = \bigcup_{i=0}^{\infty} F_i \quad \text{with} \quad \bigcup_{i \neq j} (F_i \cap F_j) \in \mathcal{C} \tag{1.1}$$

where all F_i are proper closed subsets of X.

Definition 3. A space X is a *Cantor manifold with respect to an admissible class* $\mathcal{C}$ if X cannot be separated by a closed subset which belongs to $\mathcal{C}$.

Four specifications of $\mathcal{C}$ will be considered:

(1) the class $\mathcal{D}^k_{\mathcal{K}}$ of at most k-dimensional spaces with respect to dimension $D_{\mathcal{K}}$,
(2) the class $\mathcal{D}^{<\infty}_{\mathcal{K}}$ of strongly countable $D_{\mathcal{K}}$-dimensional spaces, i.e. all spaces represented as a countable union of closed finite-dimensional subsets with respect to $D_{\mathcal{K}}$,
(3) the class **C** of paracompact C-spaces,
and
(4) the class $\mathcal{WID}$ of weakly infinite-dimensional spaces.

Recall that X is said to be *strongly infinite-dimensional* if there exists a sequence $\{(A_n, B_n)\}_{n\geq 1}$ of pairs of disjoint closed sets in X such that for every sequence of closed partitions $C_n \subset X$ separating A_n and B_n the intersection $\bigcap_{n\geq 1} C_n$ is non-empty. Spaces which are not strongly infinite-dimensional are called *weakly infinite-dimensional*.

A space X is said to be a *C-space* (or has *property C*) [11] if for every sequence $\{\omega_n\}_{n\geq 1}$ of open covers of X there exists a sequence $\{\gamma_n\}_{n\geq 1}$ of open disjoint families in X such that each γ_n refines ω_n and $\bigcup_{n\geq 1} \gamma_n$ is a cover of X.

Every finite-dimensional paracompact space as well as every countable-dimensional metrizable space is a C-space, but there exist metrizable C-spaces which are not countable-dimensional [32]. Moreover, compact C-spaces form a proper subclass of weakly infinite-dimensional compact spaces [5].

Every compact Mazurkiewicz manifold with respect to any admissible class $\mathcal{C}$ is a strong Cantor manifold with respect to $\mathcal{C}$ (see Proposition 2.1) and strong Cantor manifolds with respect to $\mathcal{C}$ are Cantor manifolds with respect to $\mathcal{C}$.

The following theorems are amongst the main results of the paper.

Theorem 2.6. *Any compact space X with $D_{\mathcal{K}}(X) = n$ contains a closed subset M such that $D_{\mathcal{K}}(M) = n$ and M is both a V^n-continuum and a Mazurkiewicz manifold with respect to the class $\mathcal{D}^{n-2}_{\mathcal{K}}$.*

Theorem 3.1. *If a compact space X has dimension $D_{\mathcal{K}}(X) = \infty$, then either X contains closed subsets of arbitrary large finite dimension $D_{\mathcal{K}}$ or X contains a compact Mazurkiewicz manifold with respect to the class $\mathcal{D}_{\mathcal{K}}^{<\infty}$.*

Theorem 3.4. *Any compact space without property C contains a closed set which is a Mazurkiewicz manifold with respect to the class* $\mathbf{C}$.

Theorem 3.6. *Any metrizable strongly infinite-dimensional compact space contains a closed set which is a Mazurkiewicz manifold with respect to the class $\mathcal{WID}$.*

Based on these theorems, we prove the following result.

Theorem 4.7. *Each metrizable homogeneous continuum $X \notin \mathcal{C}$ is a strong Cantor manifold with respect to class $\mathcal{C}$ provided that:*

(1) *$\mathcal{C}$ is any of the following three classes: $\mathcal{WID}$, $\mathbf{C}$, $\mathcal{D}_{\mathcal{K}}^{n-2}$ (in the latter case we additionally assume $D_{\mathcal{K}}(X) = n$);*
or

(2) *$\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{<\infty}$ and X does not contain closed subsets of arbitrary large finite dimension $D_{\mathcal{K}}$.*

Theorem 3.4 is totally new, while some particular weaker cases of Theorems 2.6, 3.1 and 3.6 were proved by different authors. Let us mention the classical result that every compact space X with the covering dimension $\dim X = n$ contains an n-dimensional Cantor n-manifold (with respect to dim) established independently by Hurewicz-Menger [22] and Tumarkin [37] for metrizable spaces, and by Alexandroff [1] for any compact spaces. For V^n-continua with respect to dim, this theorem was obtained by Alexandroff [2] (metrizable compact spaces) and Kuz'minov [28] (arbitrary compact spaces). Both Alexandroff's and Kuz'minov's proofs are based on cohomological methods. An elementary proof was given by Hamamdžiev [18]. For strong Cantor n-manifolds with respect to $\dim_G$, Theorem 2.6 appeared in [19].

A classical counterpart of Theorem 3.1 saying that each infinite-dimensional compact space X contains either closed subsets of arbitrary large finite dimension or a Cantor ∞-manifold M (i.e., no finite-dimensional subset separates M) was proved by Tumarkin [38].

The fact that each strongly infinite-dimensional compact metric space contains a compact strongly infinite-dimensional Cantor manifold M (i.e., no weakly infinite-dimensional closed subset of M separates M) is due to Skljarenko (see [3, p. 550]).

One of the main technical tools in proving Theorem 2.6 is an extension theorem, see Proposition 2.3. In its turn, Proposition 2.3 implies another general extension theorem (Proposition 2.4) whose analogues were established by Holsztyński [21], Hadžiivanov [14] and Dijkstra [7] for covering dimension. Hadžiivanov-Shchepin [19, Theorem 1] also formulated similar to Proposition 2.4 statement for cohomological dimension. However, we were not able to verify some details in their proof. Instead of following the arguments of the above authors, we base our proofs of Proposition 2.3 and Proposition 2.4 on a completely different idea.

It was proved in [25] that every homogeneous metrizable, locally compact, connected space X with the covering dimension $\dim X = n \leq \infty$ is a Cantor n-manifold; in case where X is strongly infinite-dimensional, it is a strongly infinite-dimensional Cantor manifold. Theorem 4.7 significantly generalizes those results.

The final section contains examples distinguishing the following four classes (with respect to dim): Cantor n-manifolds, strong Cantor n-manifolds, Mazurkiewicz n-manifolds and V^n-continua.

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2. Mazurkiewicz n-manifolds and V^n-continua with respect to dimension $D_{\mathcal{K}}$

Proposition 2.1. *Let $\mathcal{C}$ be an admissible class of spaces. Then every compact Mazurkiewicz manifold with respect to $\mathcal{C}$ is a strong Cantor manifold with respect to $\mathcal{C}$.*

Proof. Suppose X is a compact Mazurkiewicz manifold with respect to $\mathcal{C}$ but not a strong Cantor manifold with respect to $\mathcal{C}$. Then $X = \bigcup_{i\geq 0} F_i$ with F_i being proper closed subsets of X such that $F_i \cap F_j \in \mathcal{C}$ for all $i \neq j$. Let $F = \bigcup_{i\neq j} F_i \cap F_j$. Shrinking F_i, $i \geq 0$, to smaller closed subsets and re-indexing these sets, if necessary, we can assume that there exist $n \neq m$ and two closed disjoint subsets X_0 and X_1 of X both having non-empty interiors in X with $X_0 \subset F_n \setminus \bigcup_{i\neq n} F_i$ and $X_1 \subset F_m \setminus \bigcup_{i\neq m} F_i$. This can be done using arguments similar to the Baire theorem. Then $X_0 \cup X_1$ is disjoint from the set F. Since X is a Mazurkiewicz manifold with respect to $\mathcal{C}$, there exists a continuum $C \subset X \setminus F$ joining X_0 and X_1. This implies that C is covered by the family $F_i \cap C$ of its disjoint closed subsets. Hence, according to the Sierpiński theorem [10, p. 440], $C = F_i \cap C$ for some i, which contradicts the conditions on X_0 and X_1. □

The following lemma is a variation on the countable sum theorem and is required in the proof of Proposition 2.3.

Lemma 2.2. *Let X be a compact space, A be a closed subspace of X, and $Y = \bigcup_{n\geq 1} Y_n$ with all Y_n being closed in X. Let L be a CW-complex such that $L \in AE(Y_n)$, $n \geq 1$. Then any map $f\colon A \to L$ is extendable over some open neighborhood of $A \cup Y$ in X.*

Proof. We may suppose that $Y_n \subset Y_{n+1}$, $n \geq 1$. The required map can be obtained by gradually extending f to maps $f_n\colon \mathrm{cl}(U_n) \to L$, where U_n is an open neighborhood of $A \cup Y_n$ in X such that $\mathrm{cl}(U_{n-1}) \subset U_n$ and $f_n|\mathrm{cl}(U_{n-1}) = f_{n-1}$ for every $n \geq 2$. Suppose f_n has been already constructed. Since $L \in AE(Y_{n+1})$, we can extend f_n to a map $\overline{f}_n\colon \mathrm{cl}(U_n) \cup Y_{n+1} \to L$. Using that L is an absolute neighborhood extensor for compact spaces, we extend $\overline{f}_n$ to a map $f_{n+1}\colon \mathrm{cl}(U_{n+1}) \to L$, where U_{n+1} is an open neighborhood of $\mathrm{cl}(U_n) \cup Y_{n+1}$. The sequence $\{f_n\}_{n\geq 0}$ gives rise to a map

$$\overline{f}\colon U = \bigcup_{n\geq 0} U_n \to L$$

extending f. □

Propositions 2.3 and 2.4 below are among the main technical tools. As we noted in the Introduction, particular cases of Proposition 2.4 were established by various authors.

Proposition 2.3. *Let L be a CW-complex and X be a compact space. Let $\{F_i\}_{i\geq 0}$ be a family of closed subsets of X such that $L \in AE\big(F_i \times \mathbb{I}\big)$ for all i and $F = \bigcup_{i\geq 0} F_i$ cuts X between two closed subsets X_0 and X_1 both having non-empty interiors. Let $A \subset X$ be a closed set and $f\colon A \to L$ be a map extendable over $A \cup Y$ for every proper closed subset Y of X. Then f is extendable over X.*

Proof. We may assume that $X_0 = \mathrm{cl}(U_0)$ and $X_1 = \mathrm{cl}(U_1)$, where U_0 and U_1 are non-empty open subsets of X. Let $Y_0 = X \setminus U_1$ and $Y_1 = X \setminus U_0$. Then both Y_0 and Y_1 are proper closed subsets of X. Therefore there exist two maps $f_0\colon Y_0 \cup A \to L$ and $f_1\colon Y_1 \cup A \to L$ both extending f.

Consider the map $G\colon (Y_0 \times \{0\}) \cup (Y_1 \times \{1\}) \cup A \times \mathbb{I} \to L$ defined as follows:

$$G(x,t) = \begin{cases} f_0(x), & \text{if } x \in Y_0 \text{ and } t = 0; \\ f_1(x), & \text{if } x \in Y_1 \text{ and } t = 1; \\ f(x), & \text{if } x \in A. \end{cases}$$

According to Lemma 2.2, the map G can be extended to a map $H\colon W \to L$, where W is an open neighborhood of $(Y_0 \times \{0\}) \cup (Y_1 \times \{1\}) \cup (A \times \mathbb{I}) \cup (F \times \mathbb{I})$ in $X \times \mathbb{I}$. Since $\mathbb{I}$ is compact, there is an open set $V \subset X$ containing F such that $V \times \mathbb{I} \subset W$ and $V \cap (X_0 \cup X_1) = \varnothing$.

If $X \setminus V$ were connected between X_0 and X_1, then there would be a continuum $C \subset X \setminus V$ such that $C \cap X_k \neq \varnothing$, $k = 0, 1$ (see, e.g., [27, §47.II, Theorem 3]), contradicting the fact that F cuts X between X_0 and X_1. Therefore, the set V contains a closed partition P between X_0 and X_1 in X. Thus $X = X_0' \cup X_1'$, where X_0' and X_1' are closed subsets of X such that $X_0' \cap X_1' = P$ and $X_k \subset X_k'$, $k = 0, 1$. According to the definition of Y_0 and Y_1, we have $X_k' \subset Y_k$, $k = 0, 1$.

Let $f_k' = f_k|_{A \cup P}$, $k = 0, 1$. Note that the map $H|_{(A \cup P) \times \mathbb{I}}$ is a homotopy between f_0' and f_1'. Then, by the Homotopy Extension Theorem, there exists a map from X into L extending f. □

Proposition 2.4. *Let L be a CW complex and X be a compact space admitting a cover $\{F_i\}_{i \geq 0}$ by closed subsets $F_i \subset X$ such that $L \in AE\big((F_i \cap F_j) \times \mathbb{I}\big)$ for all $i \neq j$. Let $A \subset X$ be a closed set and $f \colon A \to L$ a map extendable over $A \cup F_i$ for every i. Then f is extendable over X.*

Proof. Suppose the opposite. Let $\mathcal{A}$ be the family of all closed subsets Y of X containing A such that f is not extendable over Y. Note that $\mathcal{A}$ is partially ordered by inclusion and $X \in \mathcal{A}$. We show that $\mathcal{A}$ satisfies the Zorn's lemma. Indeed, suppose $\{Y_\alpha : \alpha \in \Lambda\}$ is a decreasing net of sets from $\mathcal{A}$ and $Y = \bigcap\{Y_\alpha : \alpha \in \Lambda\}$ is not in $\mathcal{A}$. If there exists a map $\overline{f} \colon Y \to L$ extending f, then $\overline{f}$ can be extended to a map $g \colon U \to L$ with U being an open neighborhood of Y in X. Due to the compactness, U contains Y_α for some α, which is a contradiction.

Let M be a minimal element of $\mathcal{A}$. Let $C_i = M \cap F_i$. Since f is extendable over each $A \cup F_i$ but not extendable over M, all $A \cup C_i$, $i \geq 0$, are proper subsets of M. Using this fact and the Baire theorem, we can assume that there exist open sets U_0 and U_1 in M such that

$$\mathrm{cl}(U_0) \subset C_0 \setminus A, \quad \mathrm{cl}(U_1) \subset C_1 \setminus A, \quad \mathrm{cl}(U_0) \cap \mathrm{cl}(U_1) = \varnothing$$

and

$$U_0 \cap C_0 \cap C_1 = \varnothing, \quad U_1 \cap C_0 \cap C_1 = \varnothing.$$

Denote

$$B_0 = C_0, \quad B_1 = C_1, \quad B_i = C_i \setminus (U_0 \cup U_1) \quad \text{for } i \geq 2$$

and let $B = \cup_{i \neq j}(B_i \cap B_j)$. Shrinking U_0 and U_1, if necessary, we may also assume that

$$\mathrm{cl}(U_0) \cap \mathrm{cl}(B) = \varnothing \quad \text{and} \quad \mathrm{cl}(U_1) \cap \mathrm{cl}(B) = \varnothing.$$

We claim that B cuts M between $\mathrm{cl}(U_0)$ and $\mathrm{cl}(U_1)$. Indeed, suppose not. Then there exists a continuum $C \subset M \setminus B$ such that $C \cap \mathrm{cl}(U_k) \neq \varnothing$, $k = 0, 1$.

Note that $\{B_i \cap C\}_{i \geq 0}$ is a cover of C by closed disjoint proper sets. Hence, by the Sierpiński theorem [10, p. 440], $C \subset B_i$ for some i, which contradicts the choice of $\mathrm{cl}(U_0)$ and $\mathrm{cl}(U_1)$.

Therefore, due to the minimality of M, we can apply Proposition 2.3 to M, the collection $B_i \cap B_j$, $i, j \geq 0$, $i \neq j$, and the sets $\mathrm{cl}(U_0)$ and $\mathrm{cl}(U_1)$ to obtain a map $\overline{f}\colon M \to L$ extending f. This contradicts $M \in \mathcal{A}$. □

The following technical lemma will help us to work with Mazurkiewicz manifolds.

Lemma 2.5. *Let X be a compact space, X_0 and X_1 be two closed disjoint subsets of X with non-empty interiors, and S be a subset of X. Suppose that for any continuum C with $C \cap X_0 \neq \varnothing \neq C \cap X_1$ we have $C \cap S \neq \varnothing$. Then there exist open non-empty sets U_k and V_k with $V_k \subset \mathrm{cl}(V_k) \subset U_k \subset X_k$, $k = 0, 1$, such that for any continuum C with $C \cap \mathrm{cl}(V_0) \neq \varnothing \neq C \cap \mathrm{cl}(V_1)$ we have $C \cap (S \setminus (U_0 \cup U_1)) \neq \varnothing$.*

Proof. Since X_0 and X_1 have non-empty interiors, we can find open non-empty sets U_k and V_k such that $\mathrm{cl}(V_k) \subset U_k \subset X_k$, $k = 0, 1$. Consider a continuum C such that $C \cap \mathrm{cl}(V_0) \neq \varnothing \neq C \cap \mathrm{cl}(V_1)$. Note that $C \cap \mathrm{bd}(U_k) \neq \varnothing$, $k = 0, 1$. Since C is a continuum, there exists a component C' of the compact space $C \setminus (U_0 \cup U_1)$ such that $C' \cap \mathrm{bd}(U_k) \neq \varnothing$, $k = 0, 1$. Then C' is a continuum joining X_0 and X_1 and therefore $C' \cap S \neq \varnothing$. Since $C' \subset C \setminus (U_0 \cup U_1)$, we have $C \cap (S \setminus (U_0 \cup U_1)) \neq \varnothing$, as required. □

Now we are ready to prove our first main result.

Theorem 2.6. *Every compact space X with $D_{\mathcal{K}}(X) = n \geq 1$ contains a closed subset M such that $D_{\mathcal{K}}(M) = n$ and M is both a V^n-continuum and a Mazurkiewicz manifold (and hence a strong Cantor n-manifold) with respect to $\mathcal{D}_{\mathcal{K}}^{n-2}$.*

Proof. Since $D_{\mathcal{K}}(X) = n$, we have $K_n \in AE(X)$ but $K_{n-1} \notin AE(X)$. Therefore there exists a closed subset $A \subset X$ and a map $f\colon A \to K_{n-1}$ which cannot be extended to a map from X into K_{n-1}. Consider the family $\mathcal{B}$ of all closed sets $B \subset X$ such that there is no map from $A \cup B$ to K_{n-1} extending f. Obviously, $X \in \mathcal{B}$. As in the proof of Proposition 2.4, one verifies that $\mathcal{B}$ is partially ordered by inclusion and satisfies the condition of the Zorn's lemma. Let M be a minimal element of $\mathcal{B}$. Then, $D_{\mathcal{K}}(M) \leq D_{\mathcal{K}}(X) = n$. Since the map $f|_{A \cap M}$ cannot be extended to a map from M into K_{n-1}, $D_{\mathcal{K}}(M) > n - 1$. Thus, $D_{\mathcal{K}}(M) = n$.

Suppose M is not a V^n-continuum with respect to $D_{\mathcal{K}}^{n-2}$. Then, without loss of generality, we can assume that there exist two disjoint sets $X_0 = \mathrm{cl}(U_0) \subset M$

and $X_1 = \mathrm{cl}(U_1) \subset M$, where U_0 and U_1 are open non-empty subsets of M, with the following property:

- for any open cover ω of M there exists a partition P_ω in M between X_0 and X_1 such that P_ω admits an ω-map into a compact space Y_ω of dimension $D_{\mathcal{K}}(Y_\omega) \leq n-2$.

Since both $M_0 = M \setminus U_1$ and $M_1 = M \setminus U_0$ are proper closed subsets of M and M is a minimal element of $\mathcal{B}$, there exist maps $f_i\colon M_i \to K_{n-1}$ extending $f|_{A\cap M_i}$, $i = 0, 1$. Let

$$Z = (M_0 \times \{0\}) \cup (M_1 \times \{1\}) \cup \big((M \cap A) \times \mathbb{I}\big).$$

Define a map $F\colon Z \to K_{n-1}$ by

$$F(x,t) = \begin{cases} f_0(x), & \text{if } x \in M_0 \text{ and } t = 0; \\ f_1(x), & \text{if } x \in M_1 \text{ and } t = 1; \\ f(x), & \text{if } x \in A \cap M. \end{cases}$$

Let γ be an open cover of K_{n-1} such that any two γ-close maps to K_{n-1} are homotopic.

Next claim follows easily from the fact that K_{n-1}, as a metrizable ANR, is a neighborhood retract of a locally convex space (see [4, Lemma 8.1] for a similar proof).

Claim 2.7. There exists an open cover ν of Z satisfying the following condition: for any closed $B \subset Z$ and any ν-map $\varphi\colon B \to Y$ into a paracompact space Y, there exists a map $g\colon \varphi(B) \to K_{n-1}$ such that $F|_B$ and $g \circ \varphi$ are γ-close in K_{n-1}.

Let ω be an open cover of M such that each set $(W \times \{t\}) \cap Z$, $W \in \omega$ and $t \in \mathbb{I}$, is contained in some element of ν. There exists a partition P_ω in M between X_0 and X_1 admitting an ω-map $\varphi_\omega\colon P_\omega \to Y_\omega$ into a compact space Y_ω of dimension $D_{\mathcal{K}}(Y_\omega) \leq n-2$.

Let

$$B = P_\omega \times \{0,1\} \cup \big((P_\omega \cap A) \times \mathbb{I}\big)$$

and $\varphi\colon B \to Y = Y_\omega \times \mathbb{I}$ be defined as

$$\varphi(x,t) = (\varphi_\omega(x), t) \quad \text{for all } (x,t) \in B.$$

Note that φ is a ν-map. Applying the above claim we obtain a map $g\colon \varphi(B) \to K_{n-1}$ such that $F|_B$ and $g \circ \varphi$ are γ-close in K_{n-1}. The map

$$\Phi\colon P_\omega \times \mathbb{I} \to Y_\omega \times \mathbb{I}, \quad \Phi(x,t) = (\varphi_\omega(x), t),$$

is an extension of φ. Since $D_{\mathcal{K}}(Y_\omega \times \mathbb{I}) \le n-1$, the map g can be extended to a map $G\colon Y_\omega \times \mathbb{I} \to K_{n-1}$. Note that $F|_B$ and $(G \circ \Phi)|_B = g \circ \varphi$ are γ-close, and therefore homotopic by the choice of γ. The Homotopy Extension Theorem implies the existence of a map $H\colon P_\omega \times \mathbb{I} \to K_{n-1}$ extending $F|_B$. Note that H is a homotopy between $f_0|_{P_\omega}$ and $f_1|_{P_\omega}$ such that $H(x,t) = f(x)$ for all $x \in P_\omega \cap A$. Since P_ω is a partition between X_0 and X_1, there exist two closed subsets M_0' and M_1' of M such that $X_i \subset M_i' \subset M_i$, $i = 0, 1$, $M_0' \cup M_1' = M$ and $M_0' \cap M_1' = P_\omega$. Applying the Homotopy Extension Theorem to the space M_1', its closed subset $P = P_\omega \cup (A \cap M_1')$, and the maps f_0 and f_1, we get a map $f_0'\colon M_1' \to K_{n-1}$ extending $f_1|_P$ over M_1'. By pasting f_0 and f_0' we finally obtain an extension of $f|_{M\cap A}$ over M. This yields a contradiction with $M \in \mathcal{B}$. Thus, M is a V^n-continuum with respect to $D_{\mathcal{K}}^{n-2}$.

Now we show that M is a Mazurkiewicz manifold with respect to $\mathcal{D}_{\mathcal{K}}^{n-2}$. Assuming the opposite and applying Lemma 2.5, we find closed subsets F_i of M, $i \ge 0$, such that $F = \bigcup_{i\ge 0} F_i$ cuts M between two closed disjoint subsets of M with non-empty interiors and $D_{\mathcal{K}}(F) \le n-2$.

Note that $K_{n-2} \in AE(F_i)$ for each i. So, according to the definition of a stratum, $K_{n-1} \in AE(F_i \times \mathbb{I})$. Moreover, since M is a minimal element of $\mathcal{B}$, the map $f|(A \cap M)$ can be extended to a map from $(A \cap M) \cup Y$ into K_{n-1} for any proper closed subset Y of M. Then, by Proposition 2.3, there exists a map $g\colon M \to K_{n-1}$ extending $f|(A\cap M)$, which contradicts $M \in \mathcal{B}$. $\square$

3. Infinite-dimensional Mazurkiewicz manifolds

In this section we consider Mazurkiewicz manifolds with respect to classes $\mathcal{D}_{\mathcal{K}}^{<\infty}$, $\mathcal{WID}$ and $\mathbf{C}$ (of strongly countable $\mathcal{D}_{\mathcal{K}}$-dimensional spaces, weakly infinite-dimensional spaces and C-spaces, respectively).

Theorem 3.1. *If a compact space X has dimension $D_{\mathcal{K}}(X) = \infty$, then either X contains closed subsets of arbitrary large finite dimensions $D_{\mathcal{K}}$ or X contains a compact Mazurkiewicz manifold with respect to the class $\mathcal{D}_{\mathcal{K}}^{<\infty}$.*

Proof. We have $K_n \notin AE(X)$ for all $n \ge 0$. Suppose there exists $n_0 \in \mathbb{N}$ such that X contains no closed subset of finite dimension $D_{\mathcal{K}} \ge n_0$. We follow the idea from the proof of Theorem 2.6. First, choose a closed subset $A \subset X$ and a map $f\colon A \to K_{n_0}$ which cannot be extended over X. Then, there exists M minimal in the family $\mathcal{B}$ of all closed subsets $B \subset X$ for which there is no extension of f over $A \cup B$. It follows that $D_{\mathcal{K}}(M) \ge n_0 + 1$, hence $D_{\mathcal{K}}(M) = \infty$.

Suppose M is not a Mazurkiewicz manifold with respect to the class $\mathcal{D}_{\mathcal{K}}^{<\infty}$. Then, by Lemma 2.5, there exist closed subsets $F_i \subset M$ such that $F = \bigcup_{i\ge 1} F_i$

cuts M between two closed, disjoint subsets of M with non-empty interiors and $D_{\mathcal{K}}(F) = n < \infty$ for some $n < n_0$. It follows that $K_n \in AE(F_i)$, so $K_{n+1} \in AE(F_i \times \mathbb{I})$ for each i. Since $n+1 \leq n_0$, $K_{n_0} \in AE(F_i \times \mathbb{I})$, $i \geq 1$. The minimality of M implies that the map $f|(A \cap M) : A \cap M \to K_{n_0}$ extends over $(A \cap M) \cup Y$ for any proper closed subset $Y \subset M$. Now, by Proposition 2.3, there exists an extension of $f|(A \cap M)$ over M, a contradiction with $M \in \mathcal{B}$. $\square$

Recall that a set-valued map $\Phi\colon X \to Y$ is lower semi-continuous (resp., upper semi-continuous) if the set $\{x \in X : \Phi(x) \cap U \neq \varnothing\}$ (resp., $\{x \in X : \Phi(x) \subset U\}$) is open in X for every open $U \subset Y$. We say that Φ is continuous provided it is both lower semi-continuous and upper semi-continuous. Recall also that a closed subset $F \subset \mathbb{I}^\infty$ is said to be a Z-set in $\mathbb{I}^\infty$ if for every compact space X the set $\{g \in C(X, \mathbb{I}^\infty) : g(X) \cap F = \varnothing\}$ is dense in $C(X, \mathbb{I}^\infty)$ in the compact-open topology.

Proposition 3.2. *A compact space X does not have property C if and only if there exists a continuous set-valued map $\Phi\colon X \to \mathbb{I}^\infty$ satisfying the following conditions: each $\Phi(x)$ is a Z-set in $\mathbb{I}^\infty$ and for any single-valued map $g\colon X \to \mathbb{I}^\infty$ we have $g(x) \in \Phi(x)$ for some $x \in X$.*

Proof. This proposition is a direct consequence of the following result of Uspenskij [40, Theorem 1.4] that characterizes compact C-spaces: a compact space has the property C if and only if for every continuous $\Phi\colon X \to \mathbb{I}^\infty$ with each $\Phi(x)$ being a Z-set in $\mathbb{I}^\infty$ there exists a single-valued map $g\colon X \to \mathbb{I}^\infty$ such that $g(x) \notin \Phi(x)$ for all $x \in X$. $\square$

Lemma 3.3. *Let X be a compact space and $\Phi\colon X \to \mathbb{I}^\infty$ a continuous set-valued map with each $\Phi(x)$ being a Z-set in $\mathbb{I}^\infty$. Suppose $A \subset X$ is closed and $F = \bigcup_{i \geq 1} F_i$ such that all F_i are closed C-subspaces of X. Then any map $f\colon A \to \mathbb{I}^\infty$ with $f(x) \notin \Phi(x)$, $x \in A$, can be extended to a map $g\colon W \to \mathbb{I}^\infty$, where W is a neighborhood of $A \cup F$, such that $g(x) \notin \Phi(x)$ for any $x \in W$.*

Proof. Consider the sets

$$C(f) = \{h \in C(X, \mathbb{I}^\infty) : h|A = f\}$$

and

$$C_i(f) = \{h \in C(f) : h(x) \notin \Phi(x) \text{ for all } x \in F_i\}, \quad i \geq 1.$$

Here, $C(X, \mathbb{I}^\infty)$ is the space of all continuous maps from X into $\mathbb{I}^\infty$ equipped with the metric $d(g_1, g_2) = \max\{\rho\big(g_1(x), g_2(x)\big) : x \in X\}$, where ρ is the standard convex metric on $\mathbb{I}^\infty$.

We claim that each $C_i(f)$ is open and dense in $C(f)$. Indeed, let $h \in C_i(f)$ and observe that $\epsilon = \min\{\rho\big(h(x), \Phi(x)\big) : x \in F_i\}$ is positive because Φ is continuous. Then, any map in $C(f)$ which is ϵ-close to h is contained in $C_i(f)$. Thus $C_i(f)$ is open in $C(f)$.

To prove $C_i(f)$ is dense in $C(f)$, fix $h \in C(f)$ and $\epsilon = 2\eta > 0$, and consider the set-valued map

$$\phi\colon F_i \to \mathbb{I}^\infty, \quad \phi(x) = \begin{cases} f(x) & \text{for } x \in A \cap F_i, \\ \overline{B}\big(h(x), \eta\big) & \text{for } x \in F_i \setminus A, \end{cases}$$

where $\overline{B}\big(h(x), \eta\big)$ is the closed ball in $(\mathbb{I}^\infty, \rho)$ with radius η and center $h(x)$. This is a lower semi-continuous convex-valued map. Since all $\overline{B}\big(h(x), \eta\big)$ are convex and $\Phi(x)$ are Z-sets in $\mathbb{I}^\infty$, it is easily seen that $\overline{B}\big(h(x), \eta\big) \cap \Phi(x)$ is a Z-set in $\overline{B}\big(h(x), \eta\big)$, $x \in F_i \setminus A$. Since F_i is a C-space, by [13, Theorem 1.1], ϕ admits a continuous selection

$$h_1\colon F_i \to \mathbb{I}^\infty \quad \text{with} \quad h_1(x) \notin \Phi(x) \quad \text{for all} \quad x \in F_i.$$

Now, define

$$h_2\colon A \cup F_i \to \mathbb{I}^\infty \quad \text{by} \quad h_2|A = f \quad \text{and} \quad h_2|F_i = h_1.$$

Finally, extend h_2 to a map $h_3 \in C(X, \mathbb{I}^\infty)$ in such a way that h_3 is η-close to h. According to the convex-valued selection theorem of Michael [31], the map h_3 can be obtained as a selection of the convex-valued lower semi-continuous map

$$\varphi\colon X \to \mathbb{I}^\infty, \quad \varphi(x) = \begin{cases} h_2(x) & \text{if } x \in A \cup F_i, \\ \overline{B}\big(h(x), \eta\big) & \text{otherwise.} \end{cases}$$

Obviously, $h_3 \in C_i(f)$ and it is ϵ-close to h.

Since $C(f)$ is complete (as a closed subset of $C(X, \mathbb{I}^\infty)$), by the Baire theorem, there exists a map $g \in \bigcap_{i \geq 1} C_i(f)$. Then $g(x) \notin \Phi(x)$ for all $x \in F \cup A$ and $g|A = f$. Moreover, by the continuity of Φ, one can show that every point $x \in F \cup A$ has a neighborhood $O(x)$ in X with $g(y) \notin \Phi(y)$ for all $y \in O(x)$. Then $W = \bigcup_{x \in F \cup A} O(x)$ is a neighborhood of $A \cup F$ such that $g(x) \notin \Phi(x)$ for $x \in W$. □

Theorem 3.4. *Every compact space X which is not a C-space contains a compact Mazurkiewicz manifold with respect to the class* **C**.

Proof. Let $\Phi\colon X \to \mathbb{I}^\infty$ be a continuous set-valued map satisfying Proposition 3.2. Consider the family $\mathcal{B}_\Phi$ of all closed subsets $B \subset X$ such that for every map $g\colon B \to \mathbb{I}^\infty$ there exists a point $x \in B$ with $g(x) \in \Phi(x)$. Let us show that

$\mathcal{B}_\Phi$ has a minimal element. Indeed, if $\{B_\alpha : \alpha \in \Lambda\}$ is a decreasing net of sets from $\mathcal{B}_\Phi$ and $B_0 = \bigcap\{B_\alpha : \alpha \in \Lambda\}$, then every $g\colon B_0 \to \mathbb{I}^\infty$ can be extended to a map $\overline{g}\colon X \to \mathbb{I}^\infty$. For every $\alpha \in \Lambda$ choose $x_\alpha \in B_\alpha$ such that $\overline{g}(x_\alpha) \in \Phi(x_\alpha)$ and let x_0 be a limit point of a subnet of $\{x_\alpha\}$. Obviously, $x_0 \in B_0$ and since both Φ and $\overline{g}$ are continuous, $g(x_0) \in \Phi(x_0)$. Thus, by the Zorn lemma, $\mathcal{B}_\Phi$ has a minimal element M. Since $M \in \mathcal{B}_\Phi$, Proposition 3.2 yields that M is not a C-space.

We will show that M is a Mazurkiewicz manifold with respect to the class **C**. Suppose not. Then, by Lemma 2.5, there exist closed subsets F_i, $i = 0, 1, 2, \dots$, of M such that $F = \bigcup_{i\geq 0} F_i$ cuts M between two closed disjoint subsets $X_0 = \mathrm{cl}(U_0)$ and $X_1 = \mathrm{cl}(U_1)$, where U_0 and U_1 are non-empty open subsets of M, and F is a C-space.

Let $Y_0 = M \setminus U_1$ and $Y_1 = M \setminus U_0$. Then both Y_0 and Y_1 are proper closed subsets of M. Therefore there exist two maps $g_i\colon Y_i \to \mathbb{I}^\infty$ such that $g_i(x) \notin \Phi(x)$ for all $x \in Y_i$, $i = 0, 1$. Consider the map $g\colon (Y_0 \times \{0\}) \cup (Y_1 \times \{1\}) \to \mathbb{I}^\infty$ defined as follows:

$$g(x,t) = \begin{cases} g_0(x), & \text{if } x \in Y_0 \text{ and } t = 0; \\ g_1(x), & \text{if } x \in Y_1 \text{ and } t = 1. \end{cases}$$

Applying Lemma 3.3 to the closed subset $A = (Y_0 \times \{0\}) \cup (Y_1 \times \{1\})$ of $M \times \mathbb{I}$ and to $F \times \mathbb{I}$ (which is a C-space), we obtain an extension $G\colon W \to \mathbb{I}^\infty$ of g over some open neighborhood W of $A \cup (F \times \mathbb{I})$ in $M \times \mathbb{I}$, such that $G(x,t) \notin \Phi(x)$ for all $(x,t) \in W$. Due to the compactness of $\mathbb{I}$, we can find an open subset V of M containing F such that $V \times \mathbb{I} \subset W$ and $V \cap (X_0 \cup X_1) = \varnothing$. As in the proof of Proposition 2.3 we conclude that V is an open partition between X_0 and X_1 in M. Then $M \setminus V = M_0 \cup M_1$, where M_i are disjoint closed subsets of M and $X_i \subset M_i \subset Y_i$, $i = 0, 1$. Let $\theta\colon M \to \mathbb{I}$ be a function such that $\theta(M_i) = i$, $i = 0, 1$. Then the map $f(x) = G(x, \theta(x))$ is well-defined for all $x \in M$ and $f(x) \notin \Phi(x)$ for any $x \in M$. The last condition contradicts $M \in \mathcal{B}_\Phi$.

Thus, M is a Mazurkiewicz manifold with respect to the spaces having property C. □

The next theorem is an analogue of Theorem 3.4 for strongly infinite-dimensional spaces. We say that a (single-valued) map $f\colon X \to \mathbb{I}^\infty$ is *universal* [20] if for any map $g\colon X \to \mathbb{I}^\infty$ there exists a point $x \in X$ with $g(x) = f(x)$.

Proposition 3.5. *A compact space X is strongly infinite-dimensional if and only if there exists a universal map $f\colon X \to \mathbb{I}^\infty$.*

Proof. By [3], a compact space X is strongly infinite-dimensional if and only if there exists an essential map $f\colon X \to \mathbb{I}^\infty$. Recall that a map $f\colon X \to \mathbb{I}^\infty$ is essential if for every n the composition $\pi_n \circ f$ is essential, i.e. there is no map $g\colon X \to \mathbb{S}^{n-1}$ with $g|(\pi_n \circ f)^{-1}(\mathbb{S}^{n-1}) = (\pi_n \circ f)|(\pi_n \circ f)^{-1}(\mathbb{S}^{n-1})$. Here, $\pi_n\colon \mathbb{I}^\infty \to \mathbb{I}^n$ is the projection onto $\mathbb{I}^n$ and $\mathbb{S}^{n-1}$ is the boundary of $\mathbb{I}^n$. On the other hand, a map $f\colon X \to \mathbb{I}^\infty$ is essential if and only if f is universal (this fact was established in [12] for metrizable compact spaces, but the proof works for arbitrary compact spaces). □

Theorem 3.6. *Every strongly infinite-dimensional metrizable compact space X contains a Mazurkiewicz manifold with respect to the class $\mathcal{WID}$.*

Proof. We fix a (single-valued) universal map $\Phi\colon X \to \mathbb{I}^\infty$. Observe that the values of Φ, being points, are Z-sets in $\mathbb{I}^\infty$. Since X contains a strongly infinite-dimensional closed set Y such that every subset of Y is either 0-dimensional or strongly infinite-dimensional (see [34] or [30]), we can assume that every subset of X is 0-dimensional provided it is not strongly infinite-dimensional. Then, as in the proof of Theorem 3.4, we can obtain a closed strongly infinite-dimensional set $M \subset X$ such that the map $\Phi|M\colon M \to \mathbb{I}^\infty$ is universal, but $\Phi|H$ is not universal for any closed proper subset H of M. Following the ideas from the proof of Theorem 3.4, we can show that M is a Mazurkiewicz manifold with respect to the class $\mathcal{WID}$. Indeed, if $\{F_i\}_{i\geq 1}$ is a sequence of closed subsets of M with F_i being weakly infinite-dimensional, then F_i should be 0-dimensional. Note that every 0-dimensional compact space is a C-space, so we can apply the arguments from the proof of Theorem 3.4. □

4. Applications to homogeneous continua

All spaces in this section are metrizable and the dimension of a space X means any dimension $D_{\mathcal{K}}(X)$ if not stated otherwise.

Remark 4.1. Recall that a connected, locally compact metrizable space X is second-countable. Thus, by the Countable Sum Theorem, if X contains a closed n-dimensional subset, then X contains compact n-dimensional subsets of arbitrary small diameters.

A topological group H acts transitively on a space X if the action $H \times X \to X$ is continuous and for each two points $x, y \in X$ there is $h \in H$ such that $h(x) = y$. We denote by $H(X)$ the group of homeomorphisms of a space X onto itself with a compact-open topology. A space X is homogeneous if $H(X)$ acts transitively on X, i.e. for each two points $x, y \in X$ there exists $h \in H(X)$ such that $h(x) = y$;

X is called locally homogeneous if for each $x, y \in X$ there exist neighborhoods U and V of x and y, respectively, and a homeomorphism $h : U \to V$ such that $h(x) = y$.

Theorem 4.2 (Effros' Theorem [9]). *If $H(X)$ acts transitively on a closed subset Y of a compact space (X, d), then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in Y$ and $d(x, y) < \delta$, then there exists $h \in H(X)$ such that $h(x) = y$ and $d(h(z), z) < \epsilon$ for every $z \in X$.*

A homeomorphism h in the above theorem will be called an ϵ-homeomorphism.

The following simple observation explains a role of the class $\mathcal{D}_{\mathcal{K}}^{<\infty}$ for infinite-dimensional homogeneous continua.

Proposition 4.3. *A homogeneous continuum is infinite-dimensional if and only if it is not strongly countable dimensional.*

Proof. Suppose X is a homogeneous continuum and $X = \bigcup_{i=1}^{\infty} F_i$, where F_i is a closed finite-dimensional closed subset of X for each i. There exists k such that $\text{int}F_k \neq \emptyset$, by the Baire theorem. By the homogeneity, finitely many homeomorphic copies of F_k covers continuum X, so it is finite-dimensional. The converse implication is obvious. □

Theorem 4.4. *Each homogeneous continuum $X \notin \mathcal{C}$ is a Cantor manifold with respect to class $\mathcal{C}$ where $\mathcal{C}$ is any of the following four classes: $\mathcal{D}_{\mathcal{K}}^{<\infty}$, $\mathcal{WID}$, $\mathbf{C}$, $\mathcal{D}_{\mathcal{K}}^{n-2}$ (in the latter case we additionally assume $D_{\mathcal{K}}(X) = n$).*

Proof. Theorem 4.4 was proved in [25] for the covering dimension and weak infinite dimension. The proof was based on the classical Cantor Manifold Theorem that any compact n-dimensional space contains a Cantor n-manifold (see [11]), the corresponding Tumarkin's result for infinite-dimensional compacta and Skljarenko's theorem for the case of strongly infinite-dimensional compacta (both mentioned in the Introduction). Due to Theorems 2.6, 3.1, 3.4, the same idea applies. In the case of class $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{<\infty}$, however, we have to consider an extra situation when there is no Cantor manifold with respect to $\mathcal{D}_{\mathcal{K}}^{<\infty}$ in X but X contains closed subsets of arbitrary large finite dimension (see Theorem 3.1). In particular, X is not a Cantor manifold with respect to $\mathcal{D}_{\mathcal{K}}^{<\infty}$ which means that there is a closed set $F = \bigcup_{n=1}^{\infty} F_n$ which separates X, where F_n is a finite-dimensional closed set. We can assume that $X \setminus F = U \cup V$, where U, V are non-empty disjoint open subsets of X and $F = \text{bd}U = \text{bd}V$. By the Baire theorem, one can find n_0 such that $\text{int}_F(F_{n_0}) \neq \emptyset$. Since each finite-dimensional nondegenerate compactum contains arbitrary small compacta of the same finite dimension (Remark 4.1), there are arbitrary small Cantor manifolds in X with respect to this finite dimension (Theorem 2.6). Let $D_{\mathcal{K}}(F_{n_0}) = n$ and pick up a point $x \in \mathrm{int}_F(F_{n_0})$. By the homogeneity, there is a compact Cantor manifold K with respect to $D^k_{\mathcal{K}}$ for some $k > n$ satisfying

(1) $D_{\mathcal{K}}(K) = k + 2$,
(2) $x \in K$,
(3) $\mathrm{diam} K < \eta = d(x, F \setminus \mathrm{int}_F(F_{n_0}))$.

Since the set K is not contained in F, we can assume that there is a point $a \in K \cap U$. Then, for $0 < \epsilon < \min\{\eta, d(a, X \setminus U)\}$ there is $\delta > 0$ as in Theorem 4.2. Choosing a point $b \in V$, $d(x, b) < \delta$, we obtain an ϵ-homeomorphism $h : X \to X$ such that $h(x) = b$. Then $h(K)$ is a Cantor manifold with respect to $D^k_{\mathcal{K}}$ which is separated by a subset of F_{n_0}, a contradiction.

□

Next two propositions are easy consequences of the definition of a strong Cantor manifold.

Proposition 4.5. *Let X be a space satisfying condition* (1.1) *and let $K \subset X$ be a strong Cantor manifold with respect to an admisible class $\mathcal{C}$. Then there exists exactly one i such that $K \subset F_i$.*

Proposition 4.6. *Let X be a locally compact Cantor manifold with respect to an admissible class $\mathcal{C}$. Assume X is not a strong Cantor manifold with respect to $\mathcal{C}$ and no open non-empty subspace of X belongs to $\mathcal{C}$. Then X satisfies condition* (1.1), *i.e.,*

$$X = \bigcup_{i=0}^{\infty} F_i \quad \text{with} \quad \bigcup_{i \neq j} (F_i \cap F_j) \in \mathcal{C},$$

where the sets F_i are proper, closed subsets of X which additionally satisfy

(i) *no finite sum of F_i's covers X,*
(ii) $\mathrm{int} F_i \neq \varnothing$ *for each i,*
(iii) $F_i \cap \mathrm{int} F_j = \varnothing$ *for each $i \neq j$.*

Proof. Part (i) is a direct consequence of X being a Cantor manifold with respect to $\mathcal{C}$. To prove (ii) and (iii) we can assume that $\mathrm{int} F_0 \neq \varnothing$ by the Baire Category Theorem. Then, since $F_0 \neq X$, the open set $U_0 = X \setminus \mathrm{cl}(\mathrm{int} F_0)$ is non-empty and is contained in the union $(F_0 \setminus \mathrm{cl}(\mathrm{int} F_0)) \cup F_1 \cup F_2 \cup \dots$, so there exits $n_1 > n_0 = 0$ such that $\mathrm{int} F_{n_1} \neq \varnothing$. The open set $U_1 = X \setminus (\mathrm{cl}(\mathrm{int} F_0) \cup \mathrm{cl}(\mathrm{int} F_{n_1}))$ is non-empty by (i) and it is contained in $(F_0 \setminus \mathrm{cl}(\mathrm{int} F_0)) \cup (F_{n_1} \setminus \mathrm{cl}(\mathrm{int} F_{n_1})) \cup F_2 \cup \dots$, etc. We obtain a subsequence $n_0 < n_1 < n_2 < \dots$ such that the sets F_{n_i} have non-empty interiors. Redefining $F'_0 = F_0$, $F'_i = F_{n_{i-1}+1} \cup \dots \cup F_{n_i}$, we get the representation $X = \bigcup_{i=0}^{\infty} F'_i$ satisfying (ii). Notice that $\mathrm{int}F'_i \cap \mathrm{int}F'_j = \varnothing$ if $i \neq j$. Indeed, otherwise this intersection would be a non-empty open subset of X, so it does not belong to $\mathcal{C}$. Since this open set is an F_σ-subset of $F'_i \cap F'_j$, we get $F'_i \cap F'_j \notin \mathcal{C}$, a contradiction with $\bigcup_{i \neq j}(F'_i \cap F'_j) \in \mathcal{C}$. Therefore, putting $F''_i = F'_i \setminus \bigcup_{j \neq i}(\mathrm{int}F'_j)$ we obtain the representation $X = \bigcup_{i=1}^{\infty} F''_i$ with (i–iii) satisfied. □

Theorem 4.7. *Each homogeneous continuum $X \notin \mathcal{C}$ is a strong Cantor manifold with respect to class $\mathcal{C}$ provided that:*

(1) *$\mathcal{C}$ is any of the following three classes: $\mathcal{WID}$, $\mathbf{C}$, $\mathcal{D}^{n-2}_{\mathcal{K}}$ (in the latter case we additionally assume $D_{\mathcal{K}}(X) = n$);*
or
(2) *$\mathcal{C} = \mathcal{D}^{<\infty}_{\mathcal{K}}$ and X does not contain closed subsets of arbitrary large finite dimension.*

Proof. Suppose X is not a strong Cantor manifold with respect to $\mathcal{C}$. Then, by Theorem 4.4, X has a representation $X = \bigcup_{i=0}^{\infty} F_i$ as in Proposition 4.6. By Theorems 2.6, 3.1, 3.4, 3.6 and Proposition 2.1 X contains a strong Cantor manifold with respect to $\mathcal{C}$. By homogeneity, we can assume that any point of X belongs to such a strong Cantor manifold in X.

Claim 4.8. If a strong Cantor manifold $K \subset X$ with respect to $\mathcal{C}$ intersects $Y_i = \mathrm{bd}(\mathrm{int}F_i)$, then $K \subset Y_i$.

Indeed, let $x \in K \cap \mathrm{bd}(\mathrm{int}F_i)$ and suppose K is not a subset of $\mathrm{bd}(\mathrm{int}F_i)$. In the case where there exists $a \in K \cap \mathrm{int}F_i$, we can apply the Effros Theorem for $0 < \epsilon < d(a, (X \setminus \mathrm{int}F_i)$ to find a $\delta > 0$ such that if a point $y \in X \setminus F_i$ is chosen with $d(x, y) < \delta$, then there exists an ϵ-homeomorphism $h\colon X \to X$ that maps x onto y. Then $h(K)$ is a strong Cantor manifold with respect to $\mathcal{C}$ which intersects $\mathrm{int}F_i$ and another set F_j. This is however impossible by Propositions 4.5 and 4.6 (iii). In the case where there is a point $a \in K \cap (X \setminus \mathrm{cl}(\mathrm{int}F_i))$, we use an ϵ-homeomorphism h which maps x to a point in $\mathrm{int}F_i$ for $\epsilon < d(a, \mathrm{cl}(\mathrm{int}F_i))$. The continuum $h(K)$, containing points in $\mathrm{int}F_i$ and in $X \setminus \mathrm{cl}(\mathrm{int}F_i))$, must intersect $\mathrm{bd}(\mathrm{int}F_i)$. Since $h(K)$ is a strong Cantor manifold, we come to the former case above.

Let $K \subset Y_0 = Y_X$ be a strong Cantor manifold with respect to $\mathcal{C}$. Define by transfinite induction:

$$K_0 = K, \quad K_{\alpha+1} = \operatorname{cl}\left(\bigcup\{h(K_\alpha) : h(K_\alpha) \cap K_\alpha \neq \varnothing, h \in H(X)\}\right) \tag{4.1}$$

$$\text{and} \quad K_\alpha = \operatorname{cl}\Big(\bigcup_{\beta<\alpha} K_\beta\Big) \quad \text{for limit ordinals } \alpha.$$

There exists a countable ordinal γ such that $K_\gamma = K_{\gamma+1} = \dots$ [26, Theorem 3, p. 258]. Denote $G_0 = K_\gamma$.

Claim 4.9. G_0 is a continuum contained in Y_X and the group $H(X)$ acts transitively on G_0.

This follows from (4.1), by the homogeneity of X and by Claim 4.8.

Claim 4.10. G_0 is a strong Cantor manifold with respect to $\mathcal{C}$.

Suppose not. Then Claim 4.9 allows us to repeat all the above considerations substituting G_0 for X as the underlying space but keeping the whole group $H(X)$ to act transitively on G_0. In particular, since $K \subset G_0$, we get $K \subset Y_{G_0} \subsetneq G_0$ and definition (4.1) gives $G_0 \subset Y_{G_0} \subsetneq G_0$, a contradiction.

Claim 4.11. The collection $\mathcal{G} = \{h(G_0) : h \in H(X)\}$ is a continuous decomposition of X.

Observe that each two distinct $G, G' \in \mathcal{G}$ are disjoint (see (4.1)) and if $h(G) \cap G' \neq \varnothing$, then $h(G) = G'$ for any $h \in H(X)$. The continuity of the decomposition easily follows from the Effros Theorem (cf. [33]).

To get a final contradiction, consider a correspondence $s : \mathcal{G} \to \{F_1, F_2, \dots\}$ such that $G \subset s(G)$. By Proposition 4.5, s is a well defined function. Notice that

$$s^{-1}(F_i) \subset F_i, \quad \text{for each } i, \quad \text{and} \quad X = s^{-1}(F_1) \cup s^{-1}(F_2) \cup \dots$$

Since the decomposition $\mathcal{G}$ is continuous, the sets $s^{-1}(F_i)$ are closed in X. It follows from the Sierpiński Theorem [10, p. 440] that the intersection $s^{-1}(F_i) \cap s^{-1}(F_j)$ is nonempty for some $i \neq j$. Thus, the intersection contains an element of $\mathcal{G}$ which is a strong Cantor manifold with respect to $\mathcal{C}$, a contradiction with Proposition 4.5. □

We do not know if one can omit, in general, the extra hypothesis in Theorem 4.7(2) that X does not contain closed subsets of arbitrary large finite dimension for $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{<\infty}$.

Definition 4. The property (α) of an n-dimensional space X (originally considered by Hurewicz in [23] for the covering dimension of separable spaces) means that any n-dimensional closed subset of X has the non-empty interior.

It is known that all topological n-manifolds have property (α) and it was observed in [35] that n-dimensional locally compact, locally homogeneous ANR's also have this property for the covering dimension.

It is proved in [35, Theorem C] that, for the covering dimension, an n-dimensional, locally compact, connected, locally homogeneous ANR-space X is a Cantor n-manifold. Actually, the assumption in that proof that X is an ANR reduces just to property (α) and the reasoning is applicable to dimension $D_{\mathcal{K}}$, so we have the following proposition.

Proposition 4.12 ([35])**.** *If X is a locally compact, connected, locally homogeneous space with property (α) and $D_{\mathcal{K}}(X) = n$, then X is an n-dimensional Cantor manifold with respect to $\mathcal{D}_{\mathcal{K}}^{n-2}$.*

The following theorem generalizes this result.

Theorem 4.13. *Under the hypotheses of Proposition* 4.12*, the space X is a locally connected strong Cantor manifold with respect to $\mathcal{D}_{\mathcal{K}}^{n-2}$.*

Proof. By Remark 4.1, there exist arbitrary small n-dimensional compact subsets of X. Therefore X contains arbitrary small compact, n-dimensional, strong Cantor manifolds with respect to $\mathcal{D}_{\mathcal{K}}^{n-2}$ (Theorem 2.6). The local homogeneity and property (α) guarantee that X has a basis consisting of such strong Cantor manifolds. In particular, X is locally connected. Moreover, by Proposition 4.12, X is an n-dimensional Cantor n-manifold with respect to $\mathcal{D}_{\mathcal{K}}^{n-2}$. Suppose that X is not a strong Cantor manifold with respect to $\mathcal{D}_{\mathcal{K}}^{n-2}$. Then we can apply Proposition 4.6. Since each point of X is contained in the interior of an n-dimensional strong Cantor manifold K and K is contained in only one F_i (see Proposition 4.5), it follows that $F_i = \mathrm{int}F_i$ for each i. This contradicts the connectedness of X. □

5. Remarks on property (α) for dimension $D_{\mathcal{K}}$

We are going to propose an extension property (H) which implies property (α) (see Definition 4.12) for dimension $D_{\mathcal{K}}$ in the class of compact spaces. It is extracted from a proof in [24] and seems to be a natural and convenient criterion for deriving property (α) in many cases.

Definition 5. Let $\mathcal{K}$ be a given stratum. A space X with an open basis $\mathcal{U}$ has property (H) if

(*H*) $D_{\mathcal{K}}(\mathrm{bd}U) \leq n-1$ and any mapping $f : \mathrm{bd}U \to K_{n-1}$ extends over $(\mathrm{cl}U) \setminus V$ for each $U \in \mathcal{U}$ and any open, nonempty subset V of U.

As in the case of property (α), natural examples of spaces with property (H) are manifolds (with or without boundaries). Other examples include n-manifolds from which a sequence (or finite number of sequences) of mutually disjoint open n-cells converging to a point (or to different points, resp.) is removed. A simple triod T and the product $T \times \mathbb{I}$ have property (α) but they do not have property (H).

The proof of Theorem 5.2 follows the idea of [24, Theorem VI 12]. A key ingredient of the proof is the following lemma (cf. [24, A), p.96]).

Lemma 5.1. *Suppose a space X has an open basis $\mathcal{U}$ satisfying (H). Let Y be a closed subset of X. If $a \in Y$ is a boundary point of Y and $a \in U \in \mathcal{U}$, then any map of $Y \setminus U$ into K_{n-1} can be extended over Y.*

Proof. Let

$$U' = U \cap Y \quad \text{and} \quad B = \mathrm{bd}U.$$

If $f\colon Y \setminus U' \to K_{n-1}$, then the restricted map $f|_{(Y\setminus U')\cap B}$ extends to a map f' over B, since $(Y \setminus U') \cap B$ is a closed subset of B and $D_{\mathcal{K}}(B) \leq n-1$. Next, the map f' can be extended to a map

$$g : (\mathrm{cl}U) \setminus (U \setminus Y) \to K_{n-1}.$$

Finally, the map $\overline{f} : Y \to K_{n-1}$ given by

$$\overline{f}(x) = \begin{cases} g(x) & \text{for } x \in U', \\ f(x) & \text{for } x \in Y \setminus U' \end{cases}$$

extends f. □

Theorem 5.2. *If an n-dimensional compact space X satisfies (H), then X has property (α).*

Proof. Let Y be a closed n-dimensional subset of X. There exist a closed subset C of Y and a map $f\colon C \to K_{n-1}$ which cannot be extended over Y. By the compactness of Y and Zorn Lemma, there exists a minimal (with respect to the inclusion) closed subset K of Y such that f is not extendable over $C \cup K$. Then the set $K \setminus C$ is non-empty and we will show that it is open in X. Let $a \in K \setminus C$. Take $U \in \mathcal{U}$ such that $a \in U \subset \mathrm{cl}U \subset X \setminus C$. Since $K \setminus U$ is a closed proper subset of K, there exists an extension $F\colon C \cup (K \setminus U) \to K_{n-1}$ of f. The map $F|_{\mathrm{cl}(K\setminus C)\setminus U}$ cannot be extended over $\mathrm{cl}(K \setminus C)$ because if F' were such an extension, then the map $G\colon C \cup K \to K_{n-1}$ given by

$$G(x) = \begin{cases} F'(x) & \text{if } x \in K \setminus C, \\ F(x) & \text{if } x \in C \end{cases}$$

would extend f. By Lemma 5.1, $a \in \mathrm{int}(\mathrm{cl}(K \setminus C))$, hence $a \in \mathrm{int}(K \setminus C) \subset \mathrm{int}Y$. □

Question 5.3. *Are properties* (α) *and* (H) *equivalent for finite-dimensional (locally) homogeneous compact spaces?*

6. Appendix

In this section we provide examples which distinguish different subclasses of Cantor manifolds we already considered.

Example 6.1. We construct below an example of a Cantor manifold which is not a strong Cantor manifold with respect to the covering dimension dim.

Let A_1 be the set

$$A_1 = \left[\frac{1}{2}, 1\right] \times [0,1] \cup [-1,1] \times \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \times \{0\}$$

and let $A_k = A_k^1 \cup A_k^2 \cup A_k^3$ for any $k \geq 2$, where

$$A_k^1 = \left[\frac{1}{3k-1}, \frac{1}{3k-2}\right] \times \left[0, \frac{1}{3k-2}\right],$$

$$A_k^2 = \left[-1, \frac{1}{3k-2}\right] \times \left[\frac{1}{3k-1}, \frac{1}{3k-2}\right],$$

and

$$A_k^3 = \left[\frac{1}{3k}, \frac{1}{3k-3}\right] \times \{0\}.$$

Furthermore, we put $A = \bigcup_{k=1}^{\infty} A_k$ and $X = \{0,0\} \cup A \cup (-A)$, where as usual, $-A = \{x \in \mathbb{R}^2 : -x \in A\}$.

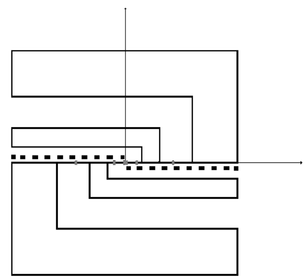

Fig 1.

Thus, one obtains a 2-dimensional compact space X and a representation $X = \bigcup_{m=0}^{\infty} F_m$, where $F_0 = (0,0)$ and for any integer m, $F_m = A_k$ if $m = 2k-1$ and $F_m = -A_k$ if $m = 2k$. Therefore, for $i \neq j$ the intersection $F_i \cap F_j$ is the empty set or a point.

It remains to show that X is a Cantor 2-manifold. Suppose the contrary: $X = U \cup V$ where U and V are non-empty open sets with a zero-dimensional intersection $C = \mathrm{cl}(U) \cap \mathrm{cl}(V)$. Let

$$\bar{A}_1 = [1/2, 1] \times [0,1] \cup [-1,1] \times [1/2, 1] \quad \text{and} \quad \bar{A}_k = A_k^1 \cup A_k^2 \quad \text{for} \quad k \geq 2.$$

Then $\bar{A}_1 \subset U$ or $\bar{A}_1 \subset V$, because $\bar{A}_1$ is a topological square; suppose $\bar{A}_1 \subset U$. Thus, $[1/2, 1] \times [0,1] \subset U$ which implies that $(-\bar{A}_m) \cap U \neq \varnothing$ for every large enough m. Hence, $-\bar{A}_m \subset U$ for almost every m. So, the segment $[0,1] \times \{0\}$ is a subset of U and $\bar{A}_m \cap U \neq \varnothing$ for almost all m. Therefore, $[-1,0] \times \{0\} \subset U$ and U meets all $\bar{A}_m$ and $-\bar{A}_m$. Consequently, $X \subset U$ which contradicts $V \neq \varnothing$.

Remark 6.2. If, in the above example, $\pm\bar{A}_m$ is replaced with $\pm\bar{A}_m \times \left[\frac{1}{3k-1}, \frac{1}{3k-2}\right]^{n-2}$, we obtain an n-dimensional Cantor n-manifold which is not a strong Cantor n-manifold.

Example 6.3. An example of a strong Cantor 2-manifold which is not a Mazurkiewicz manifold was described by Alexandroff [2]. Originally, he showed that it is a Cantor 2-manifold which is not a V^2-continuum.

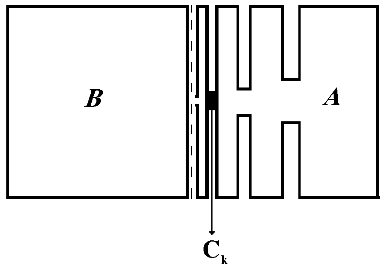

Fig 2.

Recall that this space is a union of the square $B = [-1, 0] \times [0, 1]$ and the set

$$A = \bigcup_{k=1}^{\infty} \left[\frac{1}{2k}, \frac{1}{2k-1}\right] \times [0,1] \ \cup \ \bigcup_{k=1}^{\infty} \left[\frac{1}{2k+1}, \frac{1}{2k}\right] \times \left[\frac{2k-1}{4k}, \frac{2k+1}{4k}\right].$$

It is easy to verify that the point $(0, \frac{1}{2})$ cuts X between every point of A and every point of B. To make sure that X is a strong Cantor 2-manifold, it suffices to notice that X can be represented as the union of two strong Cantor 2-manifolds, the sets B and $\widetilde{A} = \{0\} \times [0, 1] \cup A$, whose intersection is of dimension 1.

Example 6.4. The examples of Cantor n-manifolds, constructed by Lelek ([29, Figures 1 and 3]), which are not V^n-continua are also Mazurkiewicz n-manifolds for $n = 2, 3$. The first one is a modification of Example 6.3 (Fig. 3). Since every point of the segment $\{0\} \times [0, 1]$ is accessible by continua from both regions A and B, X is a Mazurkiewicz 2-dimensional manifold. A similar argument applies to the second Lelek's 3-dimensional example.

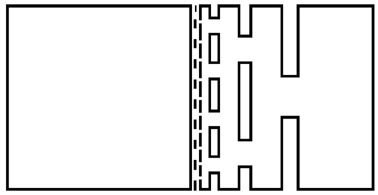

Fig. 3

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