

Special embeddings of finite-dimensional compacta in Euclidean spaces

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ABSTRACT

If g is a map from a space X into \mathbb{R}^m and $z \notin g(X)$, let $P_{2,1,m}(g, z)$ be the set of all lines $\Pi^1 \subset \mathbb{R}^m$ containing z such that $|g^{-1}(\Pi^1)| \geq 2$. We prove that for any n -dimensional metric compactum X the functions $g : X \rightarrow \mathbb{R}^m$, where $m \geq 2n + 1$, with $\dim P_{2,1,m}(g, z) \leq 0$ for all $z \notin g(X)$ form a dense G_δ -subset of the function space $C(X, \mathbb{R}^m)$. A parametric version of the above theorem is also provided.

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1. Introduction

In this paper we assume that all topological spaces are metrizable and all single-valued maps are continuous.

Everywhere below by $M_{m,d}$ we denote the space of all d -dimensional planes Π^d (br., d -planes) in \mathbb{R}^m . If g is a map from a space X into \mathbb{R}^m , q is an integer and $z \notin g(X)$, let $P_{q,d,m}(g, z) = \{\Pi^d \in M_{m,d} : |g^{-1}(\Pi^d)| \geq q \text{ and } z \in \Pi^d\}$. There is a metric topology on $M_{m,d}$, see [6], and we consider $P_{q,d,m}(g, z)$ as a subspace of $M_{m,d}$ with this topology.

One of the results from authors' paper [4] states that if X is a metric compactum of dimension n and $m \geq 2n + 1$, then the function space $C(X, \mathbb{R}^m)$ contains a dense G_δ -subset of maps g such that the set $\{\Pi^1 \in M_{m,1} : |g^{-1}(\Pi^1)| \geq 2\}$ is at most $2n$ -dimensional. The next theorem provides more information concerning the above result:

Theorem 1.1. *Let X be a metric compactum of dimension $\leq n$ and $m \geq 2n + 1$. Then the maps $g : X \rightarrow \mathbb{R}^m$ such that $\dim P_{2,1,m}(g, z) \leq 0$ for all $z \notin g(X)$ form a dense G_δ -subset of $C(X, \mathbb{R}^m)$.*

Theorem 1.1 admits a parametric version.

Theorem 1.2. *Let $f : X \rightarrow Y$ be a perfect n -dimensional map between metrizable spaces with $\dim Y = 0$, and $m \geq 2n + 1$. Then the maps $g : X \rightarrow \mathbb{R}^m$ such that $\dim P_{2,1,m}(g|f^{-1}(y), z) \leq 0$ for all restrictions $g|f^{-1}(y)$, $y \in Y$, and all $z \notin g(f^{-1}(y))$ form a dense G_δ -subset of $C(X, \mathbb{R}^m)$ equipped with the source limitation topology.*

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For any map $g \in C(X, \mathbb{R}^m)$ and $z \notin g(X)$ we also consider the set $D_{2,1,m}(g, z)$ consisting of points $y = (y_1, y_2) \in (\mathbb{R}^m)^2$ such that y_1 and y_2 belong to a line $\Pi^1 \subset \mathbb{R}^m$ with $z \in \Pi^1$, and there exist two different points $x_1, x_2 \in X$ with $g(x_i) = y_i$, $i = 1, 2$.

Theorem 1.3 below follows from the proof of Theorem 1.2 by considering the sets $D_{2,1,m}(g, z)$ instead of $P_{2,1,m}(g, z)$.

Theorem 1.3. *Let X, Y, f and m satisfy the hypotheses of Theorem 1.2. Then the maps $g : X \rightarrow \mathbb{R}^m$ such that $\dim D_{2,1,m}(g|f^{-1}(y), z) \leq 0$ for all restrictions $g|f^{-1}(y)$, $y \in Y$, and all $z \notin g(f^{-1}(y))$ form a dense G_δ -subset of $C(X, \mathbb{R}^m)$.*

Recall that for any metric space (M, ρ) the source limitation topology on $C(X, M)$ can be described as follows: the neighborhood base at a given function $f \in C(X, M)$ consists of the sets $B_\rho(f, \epsilon) = \{g \in C(X, M) : \rho(g, f) < \epsilon\}$, where $\epsilon : X \rightarrow (0, 1]$ is any continuous positive functions on X . The symbol $\rho(f, g) < \epsilon$ means that $\rho(f(x), g(x)) < \epsilon(x)$ for all $x \in X$. It is well known that for metrizable spaces X this topology doesn't depend on the metric ρ and it has the Baire property provided M is completely metrizable.

2. Preliminaries

We need some preliminary information before proving Theorem 1.1. Everywhere in this section we suppose that q, m, d are integers with $0 \leq d \leq m$ and $q \geq 1$. Moreover, the Euclidean space \mathbb{R}^m is equipped with the standard norm $\|\cdot\|_m$. We also suppose that X is a metric compactum and $\Gamma = \{B_1, B_2, \dots, B_q\}$ is a disjoint family consisting of q closed subsets of X . For any $g \in C(X, \mathbb{R}^m)$ and $z \notin g(X)$ we denote by $P_\Gamma(g, z)$ the set

$$\{\Pi^d \in M_{m,d} : g^{-1}(\Pi^d) \cap B_i \neq \emptyset \text{ for each } i = 1, \dots, q \text{ and } z \in \Pi^d\}.$$

Now, consider the open subset \mathcal{R}_X^m of $C(X, \mathbb{R}^m) \times \mathbb{R}^m$ consisting of all pairs (g, z) with $z \notin g(X)$. Define the set-valued map

$$\Phi_\Gamma : \mathcal{R}_X^m \rightarrow M_{m,d}, \quad \Phi_\Gamma(g, z) = P_\Gamma(g, z).$$

Proposition 2.1. Φ_Γ is an upper semi-continuous closed-valued map.

Proof. Suppose $(g_0, z_0) \in \mathcal{R}_X^m$. We need to show that for any open $W \subset M_{m,d}$ containing $\Phi_\Gamma(g_0, z_0)$ there are neighborhoods $O(g_0) \subset C(X, \mathbb{R}^m)$ and $O(z_0) \subset \mathbb{R}^m$ with $O(g_0) \times O(z_0) \subset \mathcal{R}_X^m$ and $\Phi_\Gamma(g, z) \subset W$ for all $(g, z) \in O(g_0) \times O(z_0)$. Assume this is not true. Then there exists a sequence $\{(g_k, z_k)\}_{k \geq 1} \in \mathcal{R}_X^m$ converging to (g_0, z_0) and $\Pi_k^d \in P_\Gamma(g_k, z_k)$ with $\Pi_k^d \notin W$, $k \geq 1$. For any $i \leq q$ and $k \geq 1$ there exists a point $x_k^i \in B_i \cap g_k^{-1}(\Pi_k^d)$. Since $A = \bigcup_{i \leq q} g_0(B_i) \subset \mathbb{R}^m$ is compact, we take a closed ball K in \mathbb{R}^m with center the origin containing A in its interior. Because every $\Pi^d \in P_\Gamma(g_0, z_0)$ intersects A , we can identify $P_\Gamma(g_0, z_0)$ with $\{\Pi^d \cap K : \Pi^d \in P_\Gamma(g_0, z_0)\}$ considered as a subspace of $\exp(K)$ (here $\exp(K)$ is the hyperspace of all compact subset of K equipped with the Vietoris topology).

Since $\{g_k\}_{k \geq 1}$ converges in $C(X, \mathbb{R}^m)$ to g_0 , we can assume that K contains each set $\bigcup_{i \leq q} g_k(B_i)$, $k \geq 1$. Hence, $g_k(x_k^i) \in K \cap \Pi_k^d$ for all $i \leq q$ and $k \geq 1$. Therefore, passing to subsequences, we may suppose that there exist points $x_0^i \in B_i$, $i \leq q$, and a plane $\Pi_0^d \in M_{m,d}$ such that each sequence $\{x_k^i\}_{k \geq 1}$, $i = 1, 2, \dots, q$, converges to x_0^i and $\{\Pi_k^d \cap K\}_{k \geq 1}$ converges to $\Pi_0^d \cap K$. So, $\lim\{g_0(x_k^i)\}_{k \geq 1} = g_0(x_0^i)$, $i = 1, 2, \dots, q$. Then each $\{g_k(x_k^i)\}_{k \geq 1}$ also converges to $g_0(x_0^i)$. Consequently, $g_0(x_0^i) \in \Pi_0^d$ for all i . Moreover, since $z_k \in \Pi_k^d$ for all k , we also have $z_0 \in \Pi_0^d$. Hence, $\Pi_0^d \in P_\Gamma(g_0, z_0)$, i.e., $\Pi_0^d \in W$. On the other hand, W is open in $M_{m,d}$ and $\lim\{\Pi_k^d \cap K\}_{k \geq 1} = \Pi_0^d \cap K$ implies that $\{\Pi_k^d\}_{k \geq 1}$ converges to Π_0^d in $M_{m,d}$. This yields $\Pi_k^d \in W$ for almost all k , a contradiction.

The above arguments also show that $P_\Gamma(g, z)$ is closed in $M_{m,d}$ for all $(g, z) \in \mathcal{R}_X^m$. So, $\Phi_{\Gamma,m,d}$ is a closed-valued map. \square

Let X and the integers q, d, m be as above. We choose a countable family \mathcal{B} of closed subsets of X such that the interiors of the elements of \mathcal{B} form a base for the topology of X . Let also

$$\mathcal{R}_X^m(k) = \{(g, z) \in C(X, \mathbb{R}^m) \times \mathbb{R}^m : \|z\|_m \leq k \text{ and } \rho_m(z, g(X)) \geq 1/k\},$$

where ρ_m is the standard Euclidean metric on \mathbb{R}^m and k an integer. If $\Gamma \subset \mathcal{B}$ is a disjoint family of q elements, for any integers k, s and $\epsilon > 0$ we consider the set $\mathcal{H}_\Gamma(k, s, \epsilon)$ of all maps $g \in C(X, \mathbb{R}^m)$ such that each $P_\Gamma(g, z)$, where $(g, z) \in \mathcal{R}_X^m(k)$, can be covered by an open in $M_{m,d}$ family $\omega(g, z)$ satisfying the following conditions:

- (1) $\text{mesh}(\omega(g, z)) < \epsilon$;
- (2) the order of $\omega(g, z)$ is $\leq s$ (i.e., each point from $M_{m,d}$ is contained in at most $s + 1$ elements of $\omega(g, z)$).

Proposition 2.2. *The set $\mathcal{H}_\Gamma(k, s, \epsilon)$ is open in $C(X, \mathbb{R}^m)$.*

Proof. Assume $g_0 \in \mathcal{H}_\Gamma(k, s, \epsilon)$. For any $(g_0, z) \in \mathcal{R}_X^m(k)$ let $W(g_0, z) = \bigcup\{U : U \in \omega(g_0, z)\}$. Obviously, we have $(g_0, z) \in \mathcal{R}_X^m(k)$ if and only if z belongs to the compact set $B(g_0) = \{z \in \mathbb{R}^m : \|z\|_m \leq k \text{ and } \rho_m(z, g_0(X)) \geq 1/k\}$. Hence, $P_\Gamma(g_0, z) \subset W(g_0, z)$ for every $z \in B(g_0)$. According to Proposition 2.1, for any such z there exists an open neighborhood $O(z) \subset \mathbb{R}^m \setminus g_0(X)$ such that $P_\Gamma(g_0, u) \subset W(g_0, z)$ for all $u \in O(z)$. Next, shrink each $O(z)$ to an open set $V(z)$ such that $z \in V(z) \subset \overline{V(z)} \subset O(z)$. Then $\{V(z) : z \in B(g_0)\}$ is an open cover of $B(g_0)$ and we choose a finite subcover $\{V(z_j) : j = 1, 2, \dots, p\}$. Let η be the distance between $B(g_0)$ and $\mathbb{R}^m \setminus V$, where $V = \bigcup_{j=1}^{j=p} V(z_j)$, and $A(z) = \{j : z \in O(z_j)\}$, $z \in O = \bigcup_{j=1}^{j=p} O(z_j)$. Choosing smaller neighborhoods $V(z_j)$, if necessarily, we may assume that $\eta < 1/k$. According to the choice of $O(z_j)$, we have

$$P_\Gamma(g_0, z) \subset W(g_0, z_j) \quad \text{for any } z \in O \text{ and } j \in A(z). \tag{3}$$

Claim 1. Let $g \in O(g_0, \eta)$ and $\rho_m(z, g(X)) \geq 1/k$, where $O(g_0, \eta)$ consists of all $g \in C(X, \mathbb{R}^m)$ such that $\rho_m(g_0(x), g(x)) < \eta$ for all $x \in X$. Then $\rho_m(z, g_0(X)) \geq (1/k) - \eta$ and $z \in \overline{V} \subset O$.

Indeed, both $\rho_m(z, g_0(X)) < (1/k) - \eta$ and $g \in O(g_0, \eta)$ imply the existence of $x \in X$ with $\rho_m(g_0(x), g(x)) < 1/k$ which contradicts the inequality $\rho_m(z, g(X)) \geq 1/k$. So, for every z satisfying the hypotheses of Claim 1, we have $\rho_m(z, g_0(X)) \geq (1/k) - \eta$. This yields $z \in \overline{V} \subset O$.

Each $W(g_0, z_j)$ is the union of an open family in $M_{m,d}$ of order $\leq s$ and mesh $< \epsilon$. Thus, according to Claim 1, it suffices to show the next claim.

Claim 2. There exists a neighborhood $O(g_0) \subset O(g_0, \eta)$ of g_0 satisfying the following condition: for any $z \in \overline{V}$ with $\rho_m(z, g_0(X)) \geq (1/k) - \eta$ there exists $j \in A(z)$ such that $P_\Gamma(g, z) \subset W(g_0, z_j)$ whenever $g \in O(g_0)$.

Suppose the conclusion of Claim 2 is not true. Then for every $p \geq 1$ there exists a map $g_p \in O(g_0, \eta)$ with $\rho_m(g_0(x), g_p(x)) < 1/p$ for all $x \in X$, a point $z_p \in \overline{V}$ with $\rho_m(z_p, g_0(X)) \geq (1/k) - \eta$, and planes

$$\Pi_p^d \in P_\Gamma(g_p, z_p) \setminus \bigcup\{W(g_0, z_j) : j \in A(z_p)\}. \tag{4}$$

Passing to subsequences, we may assume that the sequence $\{z_p\}_{p \geq 1}$ converges to a point $z_0 \in \overline{V}$ and $\{\Pi_p^d\}_{p \geq 1}$ converges in $M_{m,d}$ to a d -plane Π_0^d . Obviously, $\rho_m(z_0, g_0(X)) \geq (1/k) - \eta$. Since $z_p \in \Pi_p^d$, we also have $z_0 \in \Pi_0^d$. As in the proof of Proposition 2.1, we can see that $g_0^{-1}(\Pi_0^d)$ meets each element of Γ . Consequently, $\Pi_0^d \in P_\Gamma(g_0, z_0)$. So, by (3), $\Pi_0^d \in \bigcap\{W(g_0, z_j) : j \in A(z_0)\}$. This implies that $\Pi_p^d \in \bigcap\{W(g_0, z_j) : j \in A(z_0)\}$ for almost all p . On the other hand, since $\lim z_p = z_0$, there exists p_0 such that $A(z_0) \subset A(z_p)$ for all $p \geq p_0$. So, by (4), $\Pi_p^d \notin \bigcup\{W(g_0, z_j) : j \in A(z_0)\}$ when $p \geq p_0$, a contradiction. \square

Corollary 2.3. All maps $g \in C(X, \mathbb{R}^m)$ such that $\dim P_{q,d,m}(g, z) \leq s$ for all $z \notin g(X)$ form a G_δ -subset $\mathcal{H}_X(q, d, m, s)$ of $C(X, \mathbb{R}^m)$.

Proof. It is easily seen that for each $g \in C(X, \mathbb{R}^m)$ and $z \notin g(X)$ we have $P_{q,d,m}(g, z) = \bigcup\{P_\Gamma(g, z) : \Gamma \subset \mathcal{B} \text{ is disjoint and } |\Gamma| = q\}$. Since the sets $P_\Gamma(g, z)$ are closed in $M_{m,d}$ (by Proposition 2.1), we have $\dim P_{q,d,m}(g, z) \leq s$ if and only if $\dim P_\Gamma(g, z) \leq s$ for all Γ . This implies that $\mathcal{H}_X(q, d, m, s)$ is the intersection of the sets $\mathcal{H}_\Gamma(k, s, 1/p)$, where $k, p \geq 1$ are integers and $\Gamma \subset \mathcal{B}$ is a disjoint family of q elements. \square

3. Proof of Theorems 1.1 and 1.2

Recall that a real number v is called algebraically dependent on the real numbers u_1, \dots, u_k if v satisfies the equation $p_0(u) + p_1(u)v + \dots + p_n(u)v^n = 0$, where $p_0(u), \dots, p_n(u)$ are polynomials in u_1, \dots, u_k with rational coefficients, not all of them 0. A finite set of real numbers is algebraically independent if none of them depends algebraically on the others. The idea to use algebraically independent sets for proving general position theorems was originated by Roberts in [9]. This idea was also applied by Berkowitz and Roy in [3]. A proof of the Berkowitz–Roy main theorem from [3] was provided by Goodsell in [8, Theorem A.1] (see [5, Corollary 1.2] for a generalization of the Berkowitz–Roy theorem and [7] for another application of this theorem). Let us note that any finitely many points in a Euclidean space \mathbb{R}^n whose set of coordinates is algebraically independent are in general position (recall that a finite set $A \subset \mathbb{R}^n$ is in general position if any $k + 1$ points from A , $k \leq n$, span a k -dimensional simplex).

Proof of Theorem 1.1. We have to show that the set $\mathcal{H}_X(2, 1, m, 0)$ of all maps $g \in C(X, \mathbb{R}^m)$ such that $\dim P_{2,1,m}(g, z) \leq 0$ for all $z \notin g(X)$ is dense and G_δ in $C(X, \mathbb{R}^m)$. According to Corollary 2.3, this set is G_δ . So, it remains to show it is also dense in $C(X, \mathbb{R}^m)$. Fix a countable family \mathcal{B} of closed subsets of X such that the interiors of its elements is a base for X . Since $\mathcal{H}_X(2, 1, m, 0)$ is the intersection of the open family

$$\{\mathcal{H}_\Gamma(k, 0, 1/p): \Gamma \subset \mathcal{B} \text{ is disjoint with } |\Gamma| = 2 \text{ and } k, p \geq 1\}$$

(see the proof of Corollary 2.3), it suffices to show that each $\mathcal{H}_\Gamma(k, 0, \epsilon)$ is dense in $C(X, \mathbb{R}^m)$. Recall that $\mathcal{H}_\Gamma(k, 0, \epsilon)$ consists of all maps $g \in C(X, \mathbb{R}^m)$ such that $P_\Gamma(g, z)$ can be covered by a disjoint open in $M_{m,1}$ family ω with $\text{mesh}(\omega) < \epsilon$ for every map g and every point $z \in \mathbb{R}^m$ satisfying the following conditions: $\|z\|_m \leq k$ and $\rho_m(z, g(X)) \geq 1/k$.

To prove that each $\mathcal{H}_\Gamma(k, 0, \epsilon)$ is dense in $C(X, \mathbb{R}^m)$, observe that any map $g \in C(X, \mathbb{R}^m)$ can be approximated by maps $f = h \circ l$ with $l: X \rightarrow K$ and $h: K \rightarrow \mathbb{R}^m$, where K is a finite polyhedron of dimension $\leq n$. Actually, K can be assumed to be the nerve of a finite open cover β of X . Moreover, if we choose β such that any its element meets at most one element of $\Gamma = \{B_1, B_2\}$, then we have $l(B_1) \cap l(B_2) = \emptyset$. Further, taking a sufficiently small barycentric subdivision of K , we can find disjoint subpolyhedra K_i of K with $l(B_i) \subset K_i, i = 1, 2$. Obviously, for any $z \notin h(l(X))$ the set $P_\Gamma(h \circ l, z)$ is contained in $P_\Lambda(h, z) = \{\Pi^1 \in M_{m,1}: h^{-1}(\Pi^1) \cap K_i \neq \emptyset, i = 1, 2 \text{ and } z \in \Pi^1\}$, where $\Lambda = \{K_1, K_2\}$. Therefore, the problem of density of $\mathcal{H}_\Gamma(k, 0, \epsilon)$ in $C(X, \mathbb{R}^m)$ is reduced whether the maps $h \in C(K, \mathbb{R}^m)$ such that every set $P_\Lambda(h, z), z \notin h(K)$, admits a disjoint open cover in $M_{m,1}$ of mesh $< \epsilon$ are dense in $C(K, \mathbb{R}^m)$. This is established in the following proposition.

Proposition 3.1. *Let $K_i, i = 1, 2$, be disjoint n -dimensional subpolyhedra of a finite polyhedron K . Then the maps $h \in C(K, \mathbb{R}^m)$ such that for any $z \notin h(K)$ the set $\{\Pi^1 \in M_{m,1}: h^{-1}(\Pi^1) \cap K_i \neq \emptyset, i = 1, 2 \text{ and } z \in \Pi^1\}$ is of dimension ≤ 0 form a dense subset of $C(K, \mathbb{R}^m)$.*

Proof. Let $h_0 \in C(K, \mathbb{R}^m)$ and $\delta > 0$. We take a subdivision of K such that $\text{diam} h_0(\sigma) < \delta/2$ for all simplexes σ . Let $K^{(0)} = \{a_1, a_2, \dots, a_t\}$ be the vertexes of K and $v_j = h_0(a_j), j = 1, \dots, t$. Then, by [3], there are points $b_j \in \mathbb{R}^m$ such that the distance between v_j and b_j is $< \delta/2$ for each j and the coordinates of all $b_j, j = 1, \dots, t$, form an algebraically independent set. Define a map $h: K \rightarrow \mathbb{R}^m$ by $h(a_j) = b_j$ and h is linear on every simplex of K . It is easily seen that h is δ -close to h_0 . Without loss of generality, we may assume that K_1 and K_2 are two n -dimensional simplexes. Then each $h(K_i)$ is also an n -dimensional simplex in \mathbb{R}^m generating a plane $\Pi_i^n \in M_{m,n}$. Since the coordinates of the points $\{b_j: j = 1, \dots, t\}$ form an algebraically independent set, the planes Π_1^n and Π_2^n are skew. Suppose $z \notin h(K)$. If $z \in \Pi_1^n$ or $z \in \Pi_2^n$, then there is no line $\Pi^1 \subset \mathbb{R}^m$ which contains z and meets both $h(K_1)$ and $h(K_2)$. Suppose $z \notin \Pi_1^n \cup \Pi_2^n$. According to [4, Corollary 3.8], there exists at most one line $\Pi^1 \subset \mathbb{R}^m$ containing z such that $\Pi^1 \cap h(K_i) \neq \emptyset, i = 1, 2$. Hence, for any $z \notin h(K)$ the set $\{\Pi^1 \in M_{m,1}: h^{-1}(\Pi^1) \cap K_i \neq \emptyset, i = 1, 2 \text{ and } z \in \Pi^1\}$ is finite. \square

Proof of Theorem 1.2. We fix a metric d generating the topology of X and for any $g \in C(X, \mathbb{R}^m), y \in Y, \eta > 0$ and $z \notin g(f^{-1}(y))$ let $P^\eta(g, y, z)$ be the set of all $\Pi^1 \in M_{m,1}$ such that $z \in \Pi^1$ and there exist two points $x^1, x^2 \in g^{-1}(\Pi^1) \cap f^{-1}(y)$ with $d(x^1, x^2) \geq \eta$. Obviously,

$$P_{2,1,m}(g|f^{-1}(y), z) = \bigcup_{k=1}^\infty \{P^{1/k}(g, y, z) \text{ for any } z \notin g(f^{-1}(y))\}. \tag{5}$$

Claim 3. *Each $P^\eta(g, y, z)$ is closed in $P_{2,1,m}(g|f^{-1}(y), z)$.*

The proof of Claim 3 follows the arguments from the proof of Proposition 2.1.

Now, for $k \geq 1$ and $y \in Y$ consider the set

$$B_g(y, k) = \{z \in \mathbb{R}^m: \|z\|_m \leq k \text{ and } \rho_m(z, g(f^{-1}(y))) \geq 1/k\}.$$

Next, let $\mathcal{P}_\epsilon^\eta(y, k)$ be the set of all maps $g \in C(X, \mathbb{R}^m)$ such that for each $z \in B_g(y, k)$ the set $P^\eta(g, y, z)$ can be covered by a disjoint open in $M_{m,1}$ family of mesh $< \epsilon$. If $F \subset Y$, we consider the set $\mathcal{P}_\epsilon^\eta(F, k) = \bigcap_{y \in F} \mathcal{P}_\epsilon^\eta(y, k)$. Obviously the intersection of all $\mathcal{P}_{1/s}^\eta(Y, k), s \geq 1$, is the set

$$\mathcal{P}^\eta(Y, k) = \{g \in C(X, \mathbb{R}^m): \dim P^\eta(g, y, z) \leq 0, y \in Y, z \in B_g(y, k)\}.$$

It follows from (5) that the set $\bigcap_{k,s=1}^\infty \mathcal{P}^{1/s}(Y, k)$ coincides with the set

$$\mathcal{P} = \{g \in C(X, \mathbb{R}^m): \dim P_{2,1,m}(g|f^{-1}(y), z) \leq 0, y \in Y, z \notin g(f^{-1}(y))\}.$$

So, in order to show that \mathcal{P} is dense and G_δ in $C(X, \mathbb{R}^m)$, it suffices to show that each $\mathcal{P}_\epsilon^\eta(Y, k)$ is open and dense in $C(X, \mathbb{R}^m)$.

First we are going to show that any $\mathcal{P}_\epsilon^\eta(Y, k)$ is open in $C(X, \mathbb{R}^m)$. This can be done following the arguments from [4, Proposition 5.3] using the next lemma instead of [4, Lemma 5.2].

Lemma 3.2. *Let $g_0 \in \mathcal{P}_\epsilon^\eta(y_0, k)$ for some $y_0 \in Y$. Then there exists a neighborhood V of y_0 in Y and $\delta > 0$ such that $g \in \mathcal{P}_\epsilon^\eta(V, k)$ for all $g \in C(X, \mathbb{R}^m)$ such that the restriction $g|f^{-1}(V)$ is δ -close to $g_0|f^{-1}(V)$.*

Proof. We assume that the conclusion of Lemma 3.2 does not hold true and use the arguments from the proof of Propositions 2.1 and 2.2 to obtain a contradiction. \square

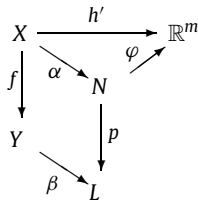
The following proposition completes the proof of Theorem 1.2.

Proposition 3.3. Any set $\mathcal{P}_\epsilon^\eta(Y, k)$ is dense in $C(X, \mathbb{R}^m)$ with respect to the source limitation topology.

Proof. We modify the arguments from the proof of [4, Proposition 5.4]. Let $g \in C(X, \mathbb{R}^m)$ and $\delta \in C(X, (0, 1])$. We are going to find $h \in \mathcal{P}_\epsilon^\eta(Y, k)$ such that $\rho(g(x), h(x)) < \delta(x)$ for all $x \in X$. By [1, Proposition 4], g can be assumed to be simplicially factorizable. This means that there exists a simplicial complex D and maps $g_D : X \rightarrow D$, $g^D : D \rightarrow \mathbb{R}^m$ with $g = g^D \circ g_D$. Following the proof of [2, Proposition 3.4], we can find an open cover \mathcal{U} of X , simplicial complexes N, L , maps $\alpha : X \rightarrow N$, $\beta : Y \rightarrow L$, $p : N \rightarrow L$, $\varphi : N \rightarrow \mathbb{R}^m$, and $\delta_1 : N \rightarrow (0, 1]$ satisfying the following conditions, where $h' = \varphi \circ \alpha$:

- α is a \mathcal{U} -map and for any $x_1, x_2 \in X$ with $d(x_1, x_2) \geq \eta$ we have $\alpha(x_1) \neq \alpha(x_2)$;
- $\beta \circ f = p \circ \alpha$;
- p is a perfect PL -map with $\dim p \leq n$ and $\dim L = 0$;
- h' is $(\delta/2)$ -close to g ;
- $\delta_1 \circ \alpha \leq \delta$.

So, we have the following commutative diagram:



Since L is a 0-dimensional simplicial complex and p is a perfect PL -map, N is a discrete union of the finite complexes $K_l = p^{-1}(l)$, $l \in L$. Since $\dim p \leq n$, it follows that $\dim K_l \leq n$, $l \in L$. Applying Theorem 1.1 to each complex K_l , we can find a map $\varphi_1 : N \rightarrow \mathbb{R}^m$ such that for any $l \in L$ and $z \notin \varphi_1(p^{-1}(l))$ we have $\dim P_{2,1,m}(\varphi_1|_{p^{-1}(l)}, z) \leq 0$ and $\varphi_1|_{p^{-1}(l)}$ is θ_l -close to $\varphi|_{p^{-1}(l)}$, where $\theta_l = \min\{\delta_1(u) : u \in p^{-1}(l)\}$. Moreover, the map $h = \varphi_1 \circ \alpha$ is δ -close to g . We claim that $h \in \mathcal{P}_\epsilon^\eta(Y, k)$. Indeed, let $y \in Y$ and $z \in B_h(y, k)$. If $\Pi^1 \in wP^\eta(h, y, z)$, then there exist two points $x^i \in h^{-1}(\Pi^1) \cap f^{-1}(y)$, $i = 1, 2$, with $d(x^1, x^2) \geq \eta$. According to the choice of the cover \mathcal{U} , we have $\alpha(x^1) \neq \alpha(x^2)$. Since $\varphi_1^{-1}(\Pi^1) \cap p^{-1}(\beta(y))$ contains the points $\alpha(x^i)$, $i = 1, 2$, we obtain that $\Pi^1 \in P_{2,1,m}(\varphi_1|_{p^{-1}(\beta(y))}, z)$. Thus, we established the inclusion $P^\eta(h, y, z) \subset P_{2,1,m}(\varphi_1|_{p^{-1}(\beta(y))}, z)$ which implies the inequality $\dim P^\eta(h, y, z) \leq 0$ for every $y \in Y$ and $z \in B_h(y, k)$. Consequently, $h \in \mathcal{P}_\epsilon^\eta(Y, k)$. \square

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