Topology and its Applications 159 (2012) 1670–1677

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Embeddings of finite-dimensional compacta in Euclidean spaces

S. Bogataya [a], S. Bogatyi [b,1], V. Valov [c,*,2]

[a] High School of Economics, Moscow 119992, Russia
[b] Faculty of Mechanics and Mathematics, Moscow State University, Vorob'evy gory, Moscow 119899, Russia
[c] Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada

ARTICLE INFO

MSC:
primary 54C10
secondary 54F45

Keywords:
Compact spaces
Algebraically independent sets
General position
Dimension
Euclidean spaces

ABSTRACT

If g is a map from a space X into $\mathbb{R}^m$ and q is an integer, let $B_{q,d,m}(g)$ be the set of all planes $\Pi^d \subset \mathbb{R}^m$ such that $|g^{-1}(\Pi^d)| \geqslant q$. Let also $\mathcal{H}(q,d,m,k)$ denote the maps $g: X \to \mathbb{R}^m$ such that $\dim B_{q,d,m}(g) \leqslant k$. We prove that for any n-dimensional metric compactum X each of the sets $\mathcal{H}(3,1,m,3n+1-m)$ and $\mathcal{H}(2,1,m,2n)$ is dense and G_δ in the function space $C(X,\mathbb{R}^m)$ provided $m \geqslant 2n+1$ (in this case $\mathcal{H}(3,1,m,3n+1-m)$ and $\mathcal{H}(2,1,m,2n)$ can consist of embeddings). The same is true for the sets $\mathcal{H}(1,d,m,n+d(m-d)) \subset C(X,\mathbb{R}^m)$ if $m \geqslant n+d$, and $\mathcal{H}(4,1,3,0) \subset C(X,\mathbb{R}^3)$ if $\dim X \leqslant 1$. This result complements an authors' result from Bogatyi and Valov (2005) [5]. A parametric version of the above theorem, as well as a partial answer of a question from Bogatyi (2008) [4] and Bogatyi and Valov (2005) [5] are also provided.

1. Introduction

In this paper we assume that all topological spaces are metrizable and all single-valued maps are continuous.

Everywhere below by $M_{m,d}$ we denote the space of all affine d-dimensional subspaces Π^d (briefly, d-planes) of $\mathbb{R}^m$. If g is a map from a space X into $\mathbb{R}^m$ and q is an integer, let $B_{q,d,m}(g) = \{\Pi^d \in M_{m,d}\colon\ |g^{-1}(\Pi^d)| \geqslant q\}$. There is a metric topology on $M_{m,d}$, see [7], and we consider $B_{q,d,m}(g)$ as a subspace of $M_{m,d}$ with this topology. For a given space X we consider the set $\mathcal{H}(q,d,m,k)$ of all maps $g: X \to \mathbb{R}^m$ such that $\dim B_{q,d,m}(g) \leqslant k$.

It follows from authors' result [5, Corollary 1.6 with $T = m = 2n+1$ and $t = 0$] that if X is metrizable compactum with $\dim X \leqslant n$, then all maps $g: X \to \mathbb{R}^{2n+1}$ such that for every $\Pi^1 \in M_{2n+1,1}$ the preimage $g^{-1}(\Pi^1)$ contains at most 4 points form a dense and G_δ-subset of $C(X,\mathbb{R}^{2n+1})$ (here $C(X,\mathbb{R}^m)$ is the space of all maps from X into $\mathbb{R}^m$ with the uniform convergence topology). This result can be complemented as follows:

Theorem 1.1. *Let X be a metrizable compactum of dimension $\leqslant n$. Then:*

(a) *The set $\mathcal{H}(3,1,m,3n+1-m)$ is dense and G_δ in $C(X,\mathbb{R}^m)$ provided $m \geqslant 2n+1$.*
(b) *The set $\mathcal{H}(2,1,m,2n)$ is dense and G_δ in $C(X,\mathbb{R}^m)$ provided $m \geqslant 2n+1$.*
(c) *The set $\mathcal{H}(1,d,m,n+d(m-d))$ is dense and G_δ in $C(X,\mathbb{R}^m)$ provided $m \geqslant n+d$.*
(d) *The set $\mathcal{H}(4,1,3,0)$ is dense and G_δ in $C(X,\mathbb{R}^3)$ if $n = 1$.*

* Corresponding author.
E-mail addresses: svetbog@mail.ru (S. Bogataya), bogatyi@inbox.ru (S. Bogatyi), veskov@nipissingu.ca (V. Valov).
1 The author was supported by Grants NSH 1562.2008.1 and RFFI 09-01-00741-a.
2 The author was supported by NSERC Grant 261914-08.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.07.028

If $f: X \to Y$ is a perfect surjection, we denote by $\mathcal{P}(q,d,m,k)$ the set of all maps $g: X \to \mathbb{R}^m$ such that $\dim B_{q,d,m}(g|f^{-1}(y)) \leqslant k$ for all $y \in Y$, where $g|f^{-1}(y)$ is the restriction of the map g on $f^{-1}(y)$.

We apply Theorem 1.1 to prove the following its parametric version.

Theorem 1.2. *Let $f: X \to Y$ be a perfect n-dimensional map between metrizable spaces with $\dim Y = 0$. Then the following conditions are satisfied, where $C(X, \mathbb{R}^m)$ is equipped with the source limitation topology*:

(a) *The set $\mathcal{P}(3, 1, m, 3n+1-m)$ is dense and G_δ in $C(X, \mathbb{R}^m)$ provided $m \geqslant 2n+1$.*
(b) *The set $\mathcal{P}(2, 1, m, 2n)$ is dense and G_δ in $C(X, \mathbb{R}^m)$ provided $m \geqslant 2n+1$.*
(c) *The set $\mathcal{P}(1, d, m, n+d(m-d))$ is dense and G_δ in $C(X, \mathbb{R}^m)$ provided $m \geqslant n+d$.*
(d) *The set $\mathcal{P}(4, 1, 3, 0)$ is dense and G_δ in $C(X, \mathbb{R}^3)$ if $n = 1$.*

For any map $g \in C(X, \mathbb{R}^m)$ we also consider the set $C_{q,d,m}(g)$ consisting of points $y = (y_1, \ldots, y_q) \in (\mathbb{R}^m)^q$ such that all y_i belongs to a d-plane in $\mathbb{R}^m$ and there exist q different points $x_i \in X$ with $g(x_i) = y_i$, $i = 1, \ldots, q$. The set of all maps $g \in C(X, \mathbb{R}^m)$ with $\dim C_{q,d,m}(g) \leqslant k$ is denoted by $\mathcal{Q}(q,d,m,k)$.

Theorem 1.3 below follows from the proof of Theorem 1.2 by considering the sets $C_{q,d,m}(g)$ instead of $B_{q,d,m}(g)$.

Theorem 1.3. *Let X, Y and f satisfy the hypotheses of Theorem 1.2. Then all items of Theorem 1.2 remain true if the corresponding sets $\mathcal{P}(q,d,m,k)$ are replaced by $\mathcal{Q}(q,d,m,k)$.*

Theorem 1.3 provides a partial answer of [4, Question 5] and [5, Conjecture 3] in the case when Y is a point, $m = T = 3$, $d = 1$ and $t = 0$.

2. A preliminary information

We are going to consider some general statements before proving Theorem 1.1. Suppose $q \geqslant 1$ is an integer, X is a metric compactum. Let $\Gamma = \{B_1, B_2, \ldots, B_q\}$ be a disjoint family consisting of q closed subsets of X and $g \in C(X, \mathbb{R}^m)$. We denote

$$B_\Gamma(g,m,d) = \{\Pi^d \in M_{m,d}\colon\ g^{-1}(\Pi^d) \cap B_i \neq \varnothing \text{ for each } i = 1, \ldots, q\},$$

where $0 \leqslant d \leqslant m$. Now, define the set-valued map

$$\Phi_{\Gamma,m,d}: C(X, \mathbb{R}^m) \to M_{m,d}, \quad \Phi_{\Gamma,m,d}(g) = B_\Gamma(g,m,d).$$

Proposition 2.1. *$\Phi_{\Gamma,m,d}$ is a closed-valued map and $\Phi^\sharp_{\Gamma,m,d}(W) = \{g \in C(X, \mathbb{R}^m)\colon \Phi_{\Gamma,m,d}(g) \subset W\}$ is open in $C(X, \mathbb{R}^m)$ for every open $W \subset M_{m,d}$.*

Proof. Suppose $g_0 \in \Phi^\sharp_{\Gamma,m,d}(W)$ with $W \subset \mathbb{R}^m_d$ being open. It suffices to show there exists $\delta > 0$ such that for any $g \in C(X, \mathbb{R}^m)$ which is δ-close to g_0 we have $B_\Gamma(g,m,d) \subset W$. Assume this is not true. So, for each n there exists $g_n \in C(X, \mathbb{R}^m)$ which is $(1/n)$-close to g_0 and $\Pi^d_n \in B_\Gamma(g_n,m,d)$ with $\Pi^d_n \notin W$. For any $i \leqslant q$ and $n \geqslant 1$ there exists a point $x^i_n \in B_i \cap g_n^{-1}(\Pi^d_n)$. Since $P = \bigcup_{i \leqslant q} g_0(B_i) \subset \mathbb{R}^m$ is compact, we take a closed ball K in $\mathbb{R}^m$ with center the origin containing P in its interior. Because every $\Pi^d \in B_\Gamma(g_0,m,d)$ intersects P, we can identify $B_\Gamma(g_0,m,d)$ with $\{\Pi^d \cap K\colon\ \Pi^d \in B_\Gamma(g_0,m,d)\}$ considered as a subspace of $\exp(K)$ (here $\exp(K)$ is the hyperspace of all compact subset of K equipped with the Vietoris topology).

Having in mind that for any $x \in X$ the distance in $\mathbb{R}^m$ between $g_0(x)$ and each $g_n(x)$ is $\leqslant 1$, we can assume that K contains each set $\bigcup_{i \leqslant q} g_n(B_i)$, $n \geqslant 1$. Hence, $g_n(x^i_n) \in K \cap \Pi^d_n$ for all $i \leqslant q$ and $n \geqslant 1$. Therefore, passing to subsequences, we may suppose that there exist points $x^i_0 \in B_i$, $i \leqslant q$, and a plane $\Pi^d_0 \in M_{m,d}$ such that each sequence $\{x^i_n\}_{n \geqslant 1}$, $i = 1, 2, \ldots, q$, converges to $x^i_0 \in B_i$ and $\{\Pi^d_n \cap K\}_{n \geqslant 1}$ converges to $\Pi^d_0 \cap K$. So, $\lim\{g_0(x^i_n)\}_{n \geqslant 1} = g_0(x^i_0)$, $i = 1, 2, \ldots, q$. Then each $\{g_n(x^i_n)\}_{n \geqslant 1}$ also converges to $g_0(x^i_0)$. Consequently, $g_0(x^i_0) \in \Pi^d_0$ for all i, so $\Pi^d_0 \in B_\Gamma(g_0,m,d)$. Hence, $\Pi^d_0 \in W$. Since W is open in $M_{m,d}$ and $\lim\{\Pi^d_n \cap K\}_{n \geqslant 1} = \Pi^d_0 \cap K$ implies that $\{\Pi^d_n\}_{n \geqslant 1}$ converges to Π^d_0 in $M_{m,d}$, $\Pi^d_n \in W$ for almost all n, a contradiction.

The above arguments also show that each $B_\Gamma(g,m,d)$ is closed in $M_{m,d}$. So, $\Phi_{\Gamma,m,d}$ is a closed-valued map. □

Corollary 2.2. *Let X and the integers $0 \leqslant d \leqslant m$ be as in Proposition 2.1. Then $\mathcal{H}(q,d,m,n)$ is a G_δ-subset of $C(X, \mathbb{R}^m)$ for any $n \geqslant 0$ and $q \geqslant 1$.*

Proof. We choose a countable family $\mathcal{B}$ of closed subsets of X such that the interiors of the elements of $\mathcal{B}$ form a base for the topology of X. Let $\epsilon > 0$ and Γ be a disjoint family of q elements of $\mathcal{B}$. Denote by $\mathcal{H}_\Gamma(q,d,m,n,\epsilon)$ the set of all maps $g: X \to \mathbb{R}^m$ such that $B_\Gamma(g,m,d)$ can be covered by an open in $M_{m,d}$ family ω satisfying the following conditions:

(1) $\mathrm{mesh}(\omega) < \epsilon$;
(2) the order of ω is $\leqslant n$ (i.e., each point from $M_{m,d}$ is contained in at most $n+1$ elements of ω).

Let $g_0 \in \mathcal{H}_\Gamma(q,d,m,n,\epsilon)$ and $W = \bigcup\{U\colon U \in \omega\}$. Then $B_\Gamma(g_0,m,d) \subset W$. According to Proposition 2.1, the set $G = \{g \in C(X,\mathbb{R}^m)\colon B_\Gamma(g,m,d) \subset W\}$ is open in $C(X,\mathbb{R}^m)$, it contains g_0 and G contains $\mathcal{H}_\Gamma(q,d,m,n,\epsilon)$. Hence, each $\mathcal{H}_\Gamma(q,d,m,n,\epsilon)$ is also open in $C(X,\mathbb{R}^m)$.

We claim that

$$\bigcap\bigl\{\mathcal{H}_\Gamma(q,d,m,n,1/k)\colon k \geqslant 1 \text{ and } \Gamma \in \mathcal{B}(q)\bigr\} = \mathcal{H}(q,d,m,n), \tag{3}$$

where $\mathcal{B}(q)$ is the family of all disjoint subsets of $\mathcal{B}$ having q elements. Indeed, according to Proposition 2.1, each $B_\Gamma(g,m,d)$ is a closed subset of $M_{m,d}$. Moreover, $B_{q,d,m}(g) = \bigcup\{B_\Gamma(g,m,d)\colon \Gamma \in \mathcal{B}(q)\}$. So, by the countable sum theorem for dim, we have $\dim B_{q,d,m}(g) \leqslant n$ if and only if $\dim B_\Gamma(g,m,d) \leqslant n$ for every $\Gamma \in \mathcal{B}(q)$. This easily implies equality (3). Therefore, $\mathcal{H}(q,d,m,n)$ is G_δ in $C(X,\mathbb{R}^m)$. □

3. Grassmann manifolds and general position of points and planes

Let V^m be a vector space of dimension m. The Grassmann manifold $G_{V^m,d}$ (briefly, $G_{m,d}$) is the set of all d-dimensional (vector) subspaces V^d of V^m. It is well known that $G_{m,d}$ has the structure of a smooth compact manifold, which can be identified with the quotient space $O(m)/(O(d) \times O(m-d))$ (for example, see [7, Part II, Chapter 1]). Here, $O(m)$ is the orthogonal group of degree m. Since $\dim O(k) = k(k-1)/2$, this implies $\dim G_{m,d} = (m-d)d$. Suppose $V_i^{n_i}$ and $0 \leqslant r_i \leqslant n_i$, $i = 1,2,\dots,k$, are fixed subspaces of V^m and integers, respectively. Then we denote

$$G_{V^m,d;V_1^{n_1},r_1;\dots;V_k^{n_k},r_k} = \bigl\{V^d \in G_{m,d}\colon \dim V^d \cap V_i^{n_i} = r_i,\ i = 1,2,\dots,k\bigr\}.$$

Sometimes, instead of $G_{V^m,d;V_1^{n_1},r_1;\dots;V_k^{n_k},r_k}$ we use the simpler notation $G_{m,d;V_1^{n_1},r_1;\dots;V_k^{n_k},r_k}$. If $\dim(V_1^{n_1} + \dots + V_k^{n_k}) = n_1 + \dots + n_k$, we write $G_{m,d;n_1,r_1;\dots;n_k,r_k}$ instead of $G_{m,d;V_1^{n_1},r_1;\dots;V_k^{n_k},r_k}$. We have $G_{m,d;n,r} \neq \varnothing$ if and only if

$$0 \leqslant r \leqslant d \leqslant n+d-r \leqslant m. \tag{4}$$

We are going to consider also the sets

$$G_{m,d;n_1,\geqslant r_1;\dots;n_k,\geqslant r_k} = \bigcup\bigl\{G_{m,d;n_1,r_1';\dots;n_k,r_k'}\colon r_i' \geqslant r_i,\ i = 1,\dots,k\bigr\},$$

which are closed in $G_{m,d}$. Since $G_{m,d;n,\geqslant r} = G_{m,d;n,\geqslant r+1} \cup G_{m,d;n,r}$, we have

$$\dim G_{m,d;n,\geqslant r} = \max\{\dim G_{m,d;n,\geqslant r+1}, \dim G_{m,d;n,r}\}. \tag{5}$$

Recall that $M_{m,d}$ stands the set of all d-planes $\Pi^d \subset \mathbb{R}^m$. If $\Pi_i^{n_i}$ and $-1 \leqslant r_i \leqslant n_i$, $i = 1,2,\dots,k$, are fixed planes and integers, we denote

$$M_{m,d;\Pi_1^{n_1},r_1;\dots;\Pi_k^{n_k},r_k} = \bigl\{\Pi^d \in M_{m,d}\colon \dim \Pi^d \cap \Pi_i^{n_i} = r_i,\ i = 1,2,\dots,k\bigr\}.$$

Identifying every d-plane in $\mathbb{R}^m$ with a $(d+1)$-dimensional subspace in $\mathbb{R}^{m+1}$, we obtain the inclusion

$$M_{m,d;\Pi_1^{n_1},r_1;\dots;\Pi_k^{n_k},r_k} \subset G_{m+1,d+1;V_1^{n_1+1},r_1+1;\dots;V_k^{n_k+1},r_k+1},$$

where $V_i^{n_i+1}$ is the subspace of $\mathbb{R}^{m+1}$ corresponding to $\Pi_i^{n_i}$. Therefore,

$$\dim M_{m,d;\Pi_1^{n_1},r_1;\dots;\Pi_k^{n_k},r_k} \leqslant \dim G_{m+1,d+1;V_1^{n_1+1},r_1+1;\dots;V_k^{n_k+1},r_k+1}.$$

$\Pi(S)$, where S is a subset of $\mathbb{R}^m$, denotes the affine hull of S, i.e., the smallest affine subspace of $\mathbb{R}^m$ containing S. We say that the planes $\Pi_i^{n_i}$, $i = 1,\dots,k$, are *jointly skew* provided

$$\dim \Pi\bigl(\Pi_1^{n_1} \cup \Pi_2^{n_2} \cup \dots \cup \Pi_k^{n_k}\bigr) = n_1 + \dots + n_k + k - 1.$$

In such a case we use the notation $M_{m,d;n_1,r_1;\dots;n_k,r_k}$ instead of the general one $M_{m,d;\Pi_1^{n_1},r_1;\dots;\Pi_k^{n_k},r_k}$.

Proposition 3.1. *If the integers m, d, n, r satisfy the inequalities* (4), *then* $\dim G_{m,d;n,r} = (n-r)r + (m-d)(d-r)$.

Proof. Let $r = d$. In this case $G_{m,d;n,r}$ consists of all r-dimensional subspaces of V^n, i.e., $G_{m,d;n,r} = G_{n,d}$. Hence, $\dim G_{m,d;n,r} = (n-r)r$.

If $r = 0$, then $n + d \leqslant m$ and $G_{m,d;n,0}$ is a non-empty open subset of $G_{m,d}$. Consequently, $\dim G_{m,d;n,0} = (m-d)d$.

Suppose $0 < r < d$ and consider the map $\varphi : G_{m,d;V^n,r} \to G_{V^n,r}$, $\varphi(V^d) = V^d \cap V^n$. This map is a locally trivial bundle whose fibre is the space $\varphi^{-1}(V^r) = G_{(V^r)^\perp,d-r;V^n\cap(V^r)^\perp,0} = G_{m-r,d-r;n-r,0}$, where $(V^r)^\perp$ is the orthogonal complement of V^r in V^m. Therefore, according to the previous two cases, $\dim G_{m,d;n,r} = \dim G_{n,r} + \dim G_{m-r,d-r;n-r,0} = (n-r)r + (m-d)(d-r)$. $\square$

Proposition 3.2. *If the integers m, d, n, r satisfy the inequalities* (4), *then* $\dim G_{m,d;n,\geqslant r} = (n-r)r + (m-d)(d-r)$.

Proof. Indeed, according to (4) and Proposition 3.1, $\dim G_{m,d;n,r+1} < \dim G_{m,d;n,r}$. Then the required inequality follows from (5). $\square$

Corollary 3.3. *If the integers m, d, n, r satisfy the inequalities $-1 \leqslant r \leqslant d \leqslant n + d - r \leqslant m$, then* $\dim M_{m,d;n,\geqslant r} \leqslant (n-r)(r+1) + (m-d)(d-r)$.

Proof. This follows from Proposition 3.2 and the inclusion $M_{m,d;n,\geqslant r} \subset G_{m+1,d+1;n+1,\geqslant r+1}$. $\square$

Proposition 3.4. *If the integers m, n_1, r_1, n_2, r_2 satisfy the equalities $0 \leqslant r_1 \leqslant n_1$, $0 \leqslant r_2 \leqslant n_2$ and $n_1 + n_2 \leqslant m$, then*

$$\dim G_{m,r_1+r_2;n_1,r_1;n_2,r_2} = (n_1 - r_1)r_1 + (n_2 - r_2)r_2.$$

Proof. Define the map $\varphi : G_{V^m,r_1+r_2;V_1^{n_1},r_1;V_2^{n_2},r_2} \to G_{V_1^{n_1},r_1} \times G_{V_2^{n_2},r_2}$, $\varphi(V^{r_1+r_2}) = (V^{r_1+r_2} \cap V_1^{n_1}, V^{r_1+r_2} \cap V_2^{n_2})$. This map is bijection, its inverse is the map $\psi : G_{V_1^{n_1},r_1} \times G_{V_2^{n_2},r_2} \to G_{V^m,r_1+r_2;V_1^{n_1},r_1;V_2^{n_2},r_2}$ given by $\psi(V^{r_1}, V^{r_2}) = V^{r_1} \oplus V^{r_2}$. So, $G_{m,r_1+r_2;n_1,r_1;n_2,r_2}$ is homeomorphic to $G_{n_1,r_1} \times G_{n_2,r_2}$. Consequently, $\dim G_{m,r_1+r_2;n_1,r_1;n_2,r_2} = \dim G_{n_1,r_1} + \dim G_{n_2,r_2} = (n_1 - r_1)r_1 + (n_2 - r_2)r_2$. $\square$

Remark 3.5. Observe that the following equality was established: $G_{m,r_1+r_2;n_1,\geqslant r_1;n_2,\geqslant r_2} = G_{m,r_1+r_2;n_1,r_1;n_2,r_2}$.

Because $M_{m,r_1+r_2+1;n_1,\geqslant r_1;n_2,\geqslant r_2} \subset G_{m+1,r_1+r_2+2;n_1+1,\geqslant r_1+1;n_2+1,\geqslant r_2+1}$, Proposition 3.4 and Remark 3.5 imply the next corollary.

Corollary 3.6. *Let $\Pi_1^{n_1}$ and $\Pi_2^{n_2}$ be two skew planes in $\mathbb{R}^m$. Then* $\dim M_{m,r_1+r_2+1;n_1,\geqslant r_1;n_2,\geqslant r_2} \leqslant (n_1 - r_1)(r_1 + 1) + (n_2 - r_2)(r_2 + 1)$.

Let us note that because the planes $\Pi_1^{n_1}$ and $\Pi_2^{n_2}$ from Corollary 3.6 are skew, we have $m \geqslant n_1 + n_2 + 1$.

Proposition 3.7. *Suppose the zero is the only common element of any two of the subspaces $V_1^{n_1}, V_2^{n_2}, V^r \subset V^m$. If V^r is contained in a subspace $V^{2r} \subset V^m$ such that $\dim V^{2r} \cap V_1^{n_1} = \dim V^{2r} \cap V_2^{n_2} = r$, then $V^r \subset V_1^{n_1} \oplus V_2^{n_2}$. Moreover, V^{2r} is uniquely determined by the above conditions.*

Proof. For every such a space V^{2r} the following inclusions hold

$$V^r \subset V^{2r} = \left(V^{2r} \cap V_1^{n_1}\right) \oplus \left(V^{2r} \cap V_2^{n_2}\right) \subset V_1^{n_1} \oplus V_2^{n_2}.$$

Thus, $V^r \subset V_1^{n_1} \oplus V_2^{n_2}$.

Next, consider the subspaces $W_1 = V^r \oplus V_1^{n_1}$ and $W_2 = V^r \oplus V_2^{n_2}$. Since V^r is in a general position with respect to each $V_i^{n_i}$, $\dim W_i = r + n_i$, $i = 1, 2$. Then $W = W_1 \cap W_2$ is the required $(2r)$-dimensional subspace. Indeed, $V^r \subset W$ and, because $V^r \subset V_1^{n_1} \oplus V_2^{n_2}$, we have $W_1 + W_2 \subset V_1^{n_1} \oplus V_2^{n_2}$. Hence, $\dim W = \dim W_1 + \dim W_2 - \dim(W_1 + W_2) = (r + n_1) + (r + n_2) - (n_1 + n_2) = 2r$. We also have that $V_1^{n_1} \cap W = V_1^{n_1} \cap W_2$ and $V_1^{n_1} + W_2 = V_1^{n_1} \oplus V_2^{n_2}$. Consequently, $\dim(V_1^{n_1} \cap W) = \dim V_1^{n_1} + \dim W_2 - \dim(V_1^{n_1} \cap W_2) = n_1 + r + n_2 - (n_1 + n_2) = r$. Similarly, we can obtain that $\dim(V_2^{n_2} \cap W) = r$. $\square$

Corollary 3.8. *Let any two of the planes $\Pi_1^{n_1}, \Pi_2^{n_2}, \Pi^r \subset \mathbb{R}^m$ be skew. Then there exists at most one $(2r+1)$-plane $\Pi^{2r+1} \subset \mathbb{R}^m$ containing Π^r such that $\dim(\Pi^{2r+1} \cap \Pi_i^{n_i}) \geqslant r$ for each $i = 1, 2$.*

Proposition 3.9. *Suppose the intersection of any two of the subspaces $V_1^{n_1}, V_2^{n_2}, V_3^{n_3} \subset V^m$ is the zero vector. If $V_1^{n_1} + V_2^{n_2} + V_3^{n_3} = V^m$ and $0 \leqslant r \leqslant n_1 + n_2 + n_3 - m$, then* $\dim G_{m,2r;V_1^{n_1},\geqslant r;V_2^{n_2},\geqslant r;V_3^{n_3},\geqslant r} = (n_1 + n_2 + n_3 - m - r)r$.

Proof. According to Remark 3.5, the set $G_{m,2r;V_1^{n_1},\geqslant r;V_2^{n_2},\geqslant r;V_3^{n_3},\geqslant r}$ coincide with $G_{m,2r;V_1^{n_1},r;V_2^{n_2},r;V_3^{n_3},r}$. So, we need to find the dimension of the last set. Let $W=(V_1^{n_1}\oplus V_2^{n_2})\cap V_3^{n_3}$. Then $\dim W=\dim(V_1^{n_1}\oplus V_2^{n_2})+\dim V_3^{n_3}-\dim(V_1^{n_1}+V_2^{n_2}+V_3^{n_3})=n_1+n_2+n_3-m\geqslant r$. By Proposition 3.7, $G_{m,2r;V_1^{n_1},r;V_2^{n_2},r;V_3^{n_3},r}$ is homeomorphic to $G_{W,r}$. Therefore, $\dim G_{m,2r;V_1^{n_1},r;V_2^{n_2},r;V_3^{n_3},r}=\dim G_{W,r}=(n_1+n_2+n_3-m-r)r$. □

Remark 3.10. We can suppose that Proposition 3.9 is also true provided $r>n_1+n_2+n_3-m$ because in this case $G_{m,2r;V_1^{n_1},\geqslant r;V_2^{n_2},\geqslant r;V_3^{n_3},\geqslant r}=\varnothing$.

Corollary 3.11. *Suppose any two of the planes $\Pi_1^{n_1},\Pi_2^{n_2},\Pi_3^{n_3}\subset\mathbb{R}^m$ are skew. If $\Pi(\Pi_1^{n_1}\cup\Pi_2^{n_2}\cup\Pi_3^{n_3})=\mathbb{R}^m$ and $m\leqslant n_1+n_2+n_3+1-r$, then the dimension of the set $\{\Pi^{2r+1}\subset\mathbb{R}^m\colon \dim(\Pi^{2r+1}\cap\Pi_i^{n_i})\geqslant r,\ i=1,2,3\}$ is $\leqslant(n_1+n_2+n_3+1-m-r)(r+1)$.*

Recall that a real number v is called algebraically dependent on the real numbers $u_1,\dots,u_k$ if v satisfies the equation $p_0(u)+p_1(u)v+\dots+p_n(u)v^n=0$, where $p_0(u),\dots,p_n(u)$ are polynomials in $u_1,\dots,u_k$ with rational coefficients, not all of them 0. A finite set of real numbers is *algebraically independent* if none of them depends algebraically on the others. The idea to use algebraically independent sets for proving general position theorems was originated by Roberts in [11]. This idea was also applied by Berkowitz and Roy in [3]. A proof of the Berkowitz–Roy theorem was provided by Goodsell in [9, Theorem A.1] (see [5, Corollary 1.2] for a generalization of the Berkowitz–Roy theorem and [8] for another application of this theorem). Let us note that any finitely many points in a Euclidean space $\mathbb{R}^n$ whose set of coordinates is algebraically independent are in general position.

It is well known (see for example [6]) that any hyperboloid of one sheet H in $\mathbb{R}^3$ is *doubly ruled*. This means that through every one of its points there are two distinct lines that lie on H. So, there are two families of lines on H (we call them family I and family II) such that any two lines on H are skew iff they belong to the same family.

Proposition 3.12. *For any six points A_i, $i=1,\dots,6$, from $\mathbb{R}^3$ whose set of coordinates is algebraically independent there exists a hyperboloid of one sheet* H *such that*:

(a) *The lines $\Pi_1^1=A_1A_2$, $\Pi_2^1=A_3A_4$ and $\Pi_3^1=A_5A_6$ lie on* H *and belong to one family, say family* I;
(b) *If a line $\Pi^1\subset\mathbb{R}^3$ meets each Π_i^1, $i=1,2,3$, then $\Pi^1\subset$ H and Π^1 belongs to family* II;
(c) H *has an equation whose coefficients are polynomials in the coordinates of A_i, $i=1,\dots,6$, with rational coefficients.*

Proof. Since the set of all coordinates of A_i, $i=1,\dots,6$, is algebraically independent, the following conditions hold:

(6) there is no 2-dimensional plane in $\mathbb{R}^3$ containing four of the points A_i, $i=1,\dots,6$;
(7) there is no 2-dimensional plane in $\mathbb{R}^3$ parallel to each line Π_i^1, $i=1,2,3$.

Then, according to [6], there exists a hyperboloid of one sheet H containing the lines Π_i^1, $i=1,2,3$. Since conditions (6) and (7) imply that any two of the lines Π_i^1, $i=1,2,3$, are skew, all they belong to one family, say family I.

If a line $\Pi^1\subset\mathbb{R}^3$ meets each Π_i^1, $i=1,2,3$, then Π^1 has three common points with H. So, $\Pi^1\subset$ H. Moreover, Π^1 belongs to family II because each Π_i^1 belongs to family I.

To prove item (c), observe that the general equation of the quadratic surface H has 10 coefficients a_j, $j=1,\dots,10$. Since $A_i\in$ H, for each $i=1,2,\dots,6$ we obtain a linear with respect to a_j equation with coefficients c_j^i, $1\leqslant j\leqslant 10$, such that any c_j^i is a polynomial in the coordinates of A_i with coefficients 1 or -1. Moreover, $\Pi_i^1\subset$ H, $i=1,2,3$, yields that each of the vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_3A_4}$ and $\overrightarrow{A_5A_6}$ has an asymptotic direction. In this way, there are another three linear with respect to a_j equations whose coefficients are polynomials in the coordinates of A_i, $i=1,2,\dots,6$ with rational coefficients. So, we have a linear system of nine equations with respect to a_j, $j=1,\dots,10$. The system has a unique (up to proportions) a non-zero solution. According to Cramer's rule, this solution can be expressed by rational functions of the coefficients of the equations. Finally, the proof of (c) follows from the fact that each of the system's coefficients are polynomials in the coordinates of A_i, $i=1,2,\dots,6$, with rational coefficients. □

Corollary 3.13. *Let $\{A_1,\dots,A_8\}$ be eight points in $\mathbb{R}^3$ whose set of coordinates is algebraically independent. Then at most two lines in $\mathbb{R}^3$ meets each of the segments $[A_1,A_2]$, $[A_3,A_4]$, $[A_5,A_6]$ and $[A_7,A_8]$.*

Proof. Consider a hyperboloid of one sheet H satisfying Proposition 3.12. Suppose a line $\Pi^1\subset\mathbb{R}^3$ meets each segment $[A_1A_2]$, $[A_3A_4]$ and $[A_5A_6]$. Then $\Pi^1\subset$ H and Π^1 belongs to family II. Since the equation of H has coefficients which are polynomials with rational coefficients in the coordinates of the points A_i, $i=1,\dots,6$, the coordinates of A_7 and A_8 don't satisfy the equation of H. So, both A_7 and A_8 are outside H. Then the line A_7A_8 has at most two common points with H. Because there exists exactly one line from family II passing through a given point of H, we can have at most two lines from family II meeting the line A_7A_8. This completes the proof of Corollary 3.13. □

4. Proof of Theorem 1.1

We are going first to prove Theorem 1.1(a). In this case we have to show that the set $\mathcal{H}(3,1,m,3n+1-m)$ of all maps $g \in C(X,\mathbb{R}^m)$ such that $\dim B_{3,1,m}(g) \leqslant 3n+1-m$ is dense and G_δ in $C(X,\mathbb{R}^3)$. Fix a countable family $\mathcal{B}$ of closed subsets of X such that the interiors of its elements are a base for X. Since $\mathcal{H}(3,1,m,3n+1-m)$ is the intersection of the open family $\{\mathcal{H}_\Gamma(3,1,m,3n+1-m,1/k)\colon k \geqslant 1\}$ (see the proof of Corollary 2.2), it suffices to show that each $\mathcal{H}_\Gamma(3,1,m,3n+1-m,\epsilon)$ is dense in $C(X,\mathbb{R}^3)$ (recall that if $\Gamma = \{B_1,B_2,B_3\} \subset \mathcal{B}$ is a disjoint family of three elements, then $\mathcal{H}_\Gamma(3,1,m,3n+1-m,\epsilon)$ consists of all maps $g \in C(X,\mathbb{R}^m)$ such that $B_\Gamma(g,m,1)$ can be covered by an open in $M_{m,1}$ family ω with $\mathrm{mesh}(\omega) < \epsilon$) and order $\leqslant 3n+1-m$.

To this end, observe that each map $g \in C(X,\mathbb{R}^m)$ can be approximated by maps $f = h \circ p$ with $p : X \to K$ and $h : K \to \mathbb{R}^m$, where K is a finite polyhedron of dimension $\leqslant n$. Actually, K can be supposed to be a nerve of a finite open cover β of X. Moreover, if we choose β such that any its element meets at most one element of Γ, then we have $p(B_i) \cap p(B_j) = \varnothing$ for $i \neq j$. Further, taking sufficiently small barycentric subdivision of K, we can find disjoint subpolyhedra K_i of K with $p(B_i) \subset K_i$, $i = 1,2,3$. Obviously, $B_\Gamma(h \circ p,m,1)$ is contained in the set $B_\Lambda(h,m,1) = \{\Pi^1 \in M_{m,1}\colon h^{-1}(\Pi^1) \cap K_i \neq \varnothing,\ i = 1,2,3\}$, where Λ is the family $\{K_1,K_2,K_3\}$. Therefore, the density of $\mathcal{H}_\Gamma(3,1,m,3n+1-m,\epsilon)$ in $C(X,\mathbb{R}^m)$ is reduced to show that the maps $h \in C(K,\mathbb{R}^m)$ such that $B_\Lambda(h,m,1)$ is covered by an open family in $M_{m,1}$ of mesh $< \epsilon$ and order $\leqslant 3n+1-m$ form a dense subset of $C(K,\mathbb{R}^m)$. And this follows from the next proposition.

Proposition 4.1. *Let K_i, $i = 1,2,3$, be disjoint at most n-dimensional subpolyhedra of a finite polyhedron K and $m \geqslant 2n+1$. Then the maps $h \in C(K,\mathbb{R}^m)$ such that $\dim B_\Lambda(h,m,1) \leqslant 3n+1-m$ form a dense subset of $C(K,\mathbb{R}^m)$, where $\Lambda = \{K_1,K_2,K_3\}$.*

Proof. Let $h_0 \in C(K,\mathbb{R}^m)$ and $\delta > 0$. We take a subdivision of K such that $\mathrm{diam}\, h_0(\sigma) < \delta/2$ for all simplexes σ. Let $K^{(0)} = \{a_1,a_2,\dots,a_k\}$ be the vertexes of K and $v_j = h_0(a_j)$, $j = 1,\dots,k$. Then, by [3], there are points $b_j \in \mathbb{R}^m$ such that the distance between v_j and b_j is $< \delta/2$ for each j and the coordinates of all b_j, $j = 1,\dots,k$, form an algebraically independent set. Define a map $h : K \to \mathbb{R}^m$ by $h(a_j) = b_j$ and h is linear on every simplex of K. It is easily seen that h is δ-close to h_0. Without loss of generality, we may assume that each K_i, $i = 1,2,3$, is a simplex. Since the coordinates of all vertexes b_j form an algebraically independent set, each $h(K_i)$ generates a plane $\Pi_i^{n_i} \subset \mathbb{R}^n$ such that $n_i = \dim h(K_i) \leqslant n$ and any two of the planes $\{\Pi_1^{n_1},\Pi_2^{n_2},\Pi_3^{n_3}\}$ are skew. Then, by Corollary 3.11, the set

$$A(h) = \{\Pi^1 \in M_{m,1}\colon \Pi^1 \cap \Pi_i^{n_i} \neq \varnothing,\ i = 1,2,3\}$$

is of dimension $n_1+n_2+n_3+1-m \leqslant 3n+1-m$. Because $A(h) = B_\Lambda(h,m,1)$, we have $\dim B_\Lambda(h,m,1) \leqslant 3n+1-m$. This completes the proof. □

As above, the proof of the other three items of Theorem 1.1 is reduced to the proof of the following proposition.

Proposition 4.2. *Let K be a finite polyhedron. Then we have*:

(a) *If $\Lambda = \{K_1,K_2\}$ is a disjoint pair of at most n-dimensional subpolyhedra of K and $m \geqslant 2n+1$, then the maps $h \in C(K,\mathbb{R}^m)$ with $\dim B_\Lambda(h,m,1) \leqslant 2n$ form a dense subset of $C(K,\mathbb{R}^m)$.*
(b) *If $\Lambda = \{K_1\}$ and $m \geqslant n+1$, where $K_1 \subset K$ is a subpolyhedron with $\dim K_1 \leqslant n$, then the maps $h \in C(K,\mathbb{R}^m)$ such that $\dim B_\Lambda(h,m,d) \leqslant n+d(m-d)$ form a dense subset of $C(K,\mathbb{R}^m)$, where $B_\Lambda(h,m,d) = \{\Pi^d \in M_{m,d}\colon h^{-1}(\Pi^d) \cap K_1 \neq \varnothing\}$.*
(c) *If $\Lambda = \{K_1,K_2,K_3,K_4\}$ is a disjoint family of at most 1-dimensional subpolyhedra of K, then the maps $h \in C(K,\mathbb{R}^3)$ with $\dim B_\Lambda(h,3,1) \leqslant 0$ form a dense subset of $C(K,\mathbb{R}^3)$.*

Proof. The same arguments as in the proof of Proposition 4.1 can be used. The only difference is that, instead Corollary 3.11, we apply now Corollary 3.6 (with $r_1 = r_2 = 0$) for item (a), Corollary 3.3 (with $r = 0$) for item (b) and Corollary 3.13 for item (c), respectively. □

5. Proof of Theorem 1.2

We fix a metric ρ generating the topology of X. Let $d \in [1,m]$ and $q \geqslant 1$ be integers, $g \in C(X,\mathbb{R}^m)$, $y \in Y$ and $\eta > 0$. We define $B^\eta_{q,d,m}(g,y)$ to be the set of all $\Pi^d \in M_{m,d}$ such that there exist q points $x^i \in g^{-1}(\Pi^d) \cap f^{-1}(y)$, $i = 1,\dots,q$, with $\rho(x^i,x^j) \geqslant \eta$ for all $i \neq j$. Obviously, $B^\eta_{q,d,m}(g,y) \subset B_{q,d,m}(g|f^{-1}(y))$ and $B_{q,d,m}(g|f^{-1}(y)) = \bigcup_{k=1}^\infty B^{1/k}_{q,d,m}(g,y)$.

Lemma 5.1. *Each $B^\eta_{q,d,m}(g,y)$ is closed in $B_{q,d,m}(g|f^{-1}(y))$.*

Proof. Suppose we have a sequence $\{\Pi^d_k\}_{k\geqslant 1} \subset B^\eta_{q,d,m}(g,y)$ converging in $M_{m,d}$ to a plane Π^d_0. Then for every k we have a q points $x^i_k \in g^{-1}(\Pi^d_k) \cap f^{-1}(y)$, $i = 1,\dots,q$, such that $\rho(x^i_k,x^j_k) \geqslant \eta$ for $i \neq j$. Since $f^{-1}(y)$ is a metric compactum, we can suppose that each sequence $\{x_k^i\}_{k\geqslant 1}$ converges to a point $x_0^i \in f^{-1}(y)$. Then $\lim g(x_k^i) = g(x_0^i) \in \Pi_0^d$, $i = 1, \dots, q$. Moreover, $\rho(x_0^i, x_0^j) \geqslant \eta$ for all $i \neq j$. Hence, $\Pi_0^d \in B_{q,d,m}^{\eta}(g, y)$. $\square$

Next, if $y \in Y$, $\eta, \epsilon > 0$ and $1 \leqslant k$ is an integer, let $\mathcal{P}_y^{\eta}(q, k, d, \epsilon)$ be the set of all maps $g \in C(X, \mathbb{R}^m)$ such that $B_{q,d,m}^{\eta}(g, y)$ can be covered by an open in $M_{m,d}$ family of order $\leqslant k$ and mesh $< \epsilon$. If $F \subset Y$, we consider the set $\mathcal{P}_F^{\eta}(q, k, d, \epsilon) = \bigcap_{y \in F} \mathcal{P}_y^{\eta}(q, k, d, \epsilon)$. Obviously the intersection of all $\mathcal{P}_Y^{\eta}(q, k, d, 1/s)$, $s \geqslant 1$, is the set

$$\mathcal{P}^{\eta}(q, k, d) = \bigl\{g \in C\bigl(X, \mathbb{R}^m\bigr)\colon \dim B_{q,d,m}^{\eta}(g, y) \leqslant k \text{ for all } y \in Y\bigr\}.$$

Moreover, since $B_{q,d,m}(g|f^{-1}(y)) = \bigcup_{s=1}^{\infty} B_{q,d,m}^{1/s}(g, y)$ and each $B_{q,d,m}^{1/s}(g, y)$ is closed in $B_{q,d,m}(g|f^{-1}(y))$ (see Lemma 5.1), the countable sum theorem for the dimension dim yields that $\bigcap_{s=1}^{\infty} \mathcal{P}^{1/s}(q, k, d)$ coincides with the set

$$\mathcal{P}(q, d, m, k) = \bigl\{g \in C\bigl(X, \mathbb{R}^m\bigr)\colon \dim B_{q,d,m}\bigl(g\big|f^{-1}(y)\bigr) \leqslant k,\ y \in Y\bigr\}.$$

Therefore,

$$\mathcal{P}(q, d, m, k) = \bigcap \bigl\{\mathcal{P}_Y^{1/s}(q, k, d, 1/p)\colon\ p \geqslant 1,\ s \geqslant 1\bigr\}.$$

So, in order to show that $\mathcal{P}(q, d, m, k)$ is dense and G_δ in $C(X, \mathbb{R}^m)$, it suffices to show that each $\mathcal{P}_Y^{\eta}(q, k, d, \epsilon)$ is open and dense in $C(X, \mathbb{R}^m)$.

We are going first to show that any $\mathcal{P}_Y^{\eta}(q, k, d, \epsilon)$ is open in $C(X, \mathbb{R}^m)$.

Lemma 5.2. *Let $g_0 \in \mathcal{P}_{y_0}^{\eta}(q, k, d, \epsilon)$ for some $y_0 \in Y$. Then there exists a neighborhood V of y_0 in Y and $\delta > 0$ such that $g \in \mathcal{P}_V^{\eta}(q, k, d, \epsilon)$ for all $g \in C(X, \mathbb{R}^m)$ such that the restrictions $g|f^{-1}(V)$ and $g_0|f^{-1}(V)$ are δ-close.*

Proof. Since $g_0 \in \mathcal{P}_{y_0}^{\eta}(q, k, d, \epsilon)$, there exists an open in $M_{m,d}$ family ω of order $\leqslant k$ and $\mathrm{mesh}(\omega) < \epsilon$ which covers $B_{q,d,m}^{\eta}(g_0, y_0)$. Let $W = \bigcup\{U\colon\ U \in \omega\}$. It suffices to show that we can find a neighborhood V of y_0 in Y and $\delta > 0$ such that $B_{q,d,m}^{\eta}(g, y) \subset W$ for all $y \in V$ and all maps $g \in C(X, \mathbb{R}^m)$ such that the restrictions $g|f^{-1}(V)$ and $g_0|f^{-1}(V)$ are δ-close. Suppose this is not true. Then, for each $i = 1, \dots, q$ there exist sequences $\{V_s\}_{s\geqslant 1}$, $\{y_s\}_{s\geqslant 1} \subset Y$, $\{g_s\}_{s\geqslant 1} \subset C(X, \mathbb{R}^m)$, $\{\Pi_s^d\}_{s\geqslant 1} \subset M_{m,d}$ and $\{x_s^i\}_{s\geqslant 1} \subset X$ satisfying the following conditions for every $s \geqslant 1$:

- $\{V_s\}_{s\geqslant 1}$ is a local base of neighborhoods at y_0;
- $y_s \in V_s$;
- $g_s|f^{-1}(V_s)$ and $g_0|f^{-1}(V_s)$ are $(1/s)$-close;
- $\Pi_s^d \in B_{q,d,m}^{\eta}(g_s, y_s)\backslash W$;
- $x_s^i \in g_s^{-1}(\Pi_s^d) \cap f^{-1}(y_s)$ for all i;
- $\rho(x_s^i, x_s^j) \geqslant \eta$ for all $i \neq j$.

As in Proposition 2.1, we can suppose that there exist points $x_0^i \in f^{-1}(y_0)$, $i = 1, \dots, q$, and a plane $\Pi_0^d \in M_{m,d}$ such that $\lim x_s^i = x_0^i$ and $\lim \Pi_s^d = \Pi_0^d$. It is easily seen that $\Pi_0^d \in B_{q,d,m}^{\eta}(g_0, y_0)$ which implies $\Pi_0^d \in W$. This is a contradiction because $\lim \Pi_s^d = \Pi_0^d$ and $\Pi_s^d \notin W$. $\square$

Proposition 5.3. *Any $\mathcal{P}_Y^{\eta}(q, k, d, \epsilon)$ is open in $C(X, \mathbb{R}^m)$ with respect to the source limitation topology.*

Proof. We follow the arguments from the proof of [5, Proposition 3.3] (see also the proof of [12, Proposition 2.3]). For every $y \in Y$ there exists a neighborhood V_y and a positive $\delta_y \leqslant 1$ satisfying Lemma 5.2. We suppose that the family $\{V_y\colon\ y \in Y\}$ is locally finite. Define a lower semi-continuous convex-valued map $\varphi : Y \to (0, 1]$ by $\varphi(y) = \bigcup\{(0, \delta_z]\colon\ y \in V_z\}$. According to [10, Theorem 6.2, p. 116], φ admits a continuous selection $\beta : Y \to (0, 1]$. Let $\alpha = \beta \circ f$. Using the choice of the neighborhoods V_y it is easily seen that if $\rho_m(g(x), g_0(x)) < \alpha(x)$ for all $x \in X$, where $g \in C(X, \mathbb{R}^m)$ and ρ_m is the Euclidean metric on $\mathbb{R}^m$, then $g \in \mathcal{P}_Y^{\eta}(q, k, d, \epsilon)$. Therefore, $\mathcal{P}_Y^{\eta}(q, k, d, \epsilon)$ is open in $C(X, \mathbb{R}^m)$. $\square$

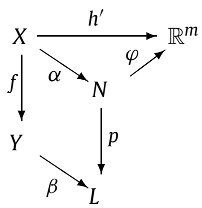
Proposition 5.4. *Suppose X, Y and f satisfy the hypotheses of Theorem* 1.2 *and $\eta, \epsilon > 0$. Then the following holds*:

(a) *Each of the sets $\mathcal{P}_Y^{\eta}(3, 3n + 1 - m, 1, \epsilon)$ and $\mathcal{P}_Y^{\eta}(2, 2n, 1, \epsilon)$ is dense in $C(X, \mathbb{R}^m)$ provided $m \geqslant 2n + 1$.*
(b) *The set $\mathcal{P}_Y^{\eta}(1, n + d(m - d), d, \epsilon)$ is dense in $C(X, \mathbb{R}^m)$ provided $m \geqslant n + d$.*
(c) *The set $\mathcal{P}_Y^{\eta}(4, 0, 1, \epsilon)$ is dense in $C(X, \mathbb{R}^3)$ if $n = 1$.*

Proof. We are going to show only that $\mathcal{P}_Y^\eta(3, 3n+1-m, 1, \epsilon)$ is dense in $C(X, \mathbb{R}^m)$ if $m \geqslant 2n+1$, the other proofs are similar. Let $g \in C(X, \mathbb{R}^m)$ and $\delta \in C(X, (0, 1])$. We are going to find $h \in \mathcal{P}_Y^\eta(q, 3n+1-m, 1, \epsilon)$ such that $\rho_m(g(x), h(x)) < \delta(x)$ for all $x \in X$. By [1, Proposition 4], g can be supposed to be simplicially factorizable. This means that there exists a simplicial complex D and maps $g_D : X \to D$, $g^D : D \to M$ with $g = g^D \circ g_D$. Following the proof of [2, Proposition 3.4], we can find an open cover $\mathcal{U}$ of X, simplicial complexes N, L and maps $\alpha : X \to N$, $\beta : Y \to L$, $p : N \to L$, $\varphi : N \to \mathbb{R}^m$ and $\delta_1 : N \to (0, 1]$ satisfying the following conditions, where $h' = \varphi \circ \alpha$:

- α is an $\mathcal{U}$-map and for any $x_1, x_2 \in X$ with $\rho(x_1, x_2) \geqslant \eta$ we have $\alpha(x_1) \neq \alpha(x_2)$;
- $\beta \circ f = p \circ \alpha$;
- p is a perfect *PL*-map with $\dim p \leqslant n$ and $\dim L = 0$;
- h' is $(\delta/2)$-close to g;
- $\delta_1 \circ \alpha \leqslant \delta$.

So, we have the following commutative diagram:



Since L is a 0-dimensional simplicial complex and p is a perfect *PL*-map, N is a discrete union of the finite complexes $K_z = p^{-1}(z)$, $z \in L$. Because $\dim p \leqslant n$, $\dim K_z \leqslant n$, $z \in L$. Applying Theorem 1.1(a) to each complex K_z, we can find a map $\varphi_1 : N \to \mathbb{R}^m$ such that $\dim B_{q,1,m}(\varphi_1|p^{-1}(z)) \leqslant 3n+1-m$ and $\varphi_1|p^{-1}(z)$ is θ_z-close to $\varphi|p^{-1}(z)$, where $\theta_z = \min\{\delta_1(u)\colon u \in p^{-1}(z)\}$. Moreover, the map $h = \varphi_1 \circ \alpha$ is δ-close to g. We claim that $h \in \mathcal{P}_Y^\eta(q, 3n+1-m, 1, \epsilon)$. Indeed, let $y \in Y$ and $\Pi^1 \in B_{q,1,m}^\eta(h, y)$. Then there exist q points $x^i \in h^{-1}(\Pi^1) \cap f^{-1}(y)$, $i = 1, \ldots, q$, with $\rho(x^i, x^j) \geqslant \eta$ for all $i \neq j$. According to the choice of the cover $\mathcal{U}$, we have $\alpha(x^i) \neq \alpha(x^j)$ for $i \neq j$. Since $\varphi_1^{-1}(\Pi^1) \cap p^{-1}(\beta(y))$ contains the points $\alpha(x^i)$, $i \leqslant q$, we obtain that $\Pi^1 \in B_{q,1,m}(\varphi_1|p^{-1}(\beta(y)))$. Thus, we established the inclusion $B_{q,1,m}^\eta(h, y) \subset B_{q,1,m}(\varphi_1|p^{-1}(\beta(y)))$ which implies $\dim B_{q,1,m}^\eta(h, y) \leqslant 3n+1-m$ for every $y \in Y$. Consequently, $h \in \mathcal{P}_Y^\eta(q, 3n+1-m, 1, \epsilon)$. □

Acknowledgements

The results from this paper were obtained during the second author's visit of Department of Computer Science and Mathematics (COMA), Nipissing University in August 2010. He acknowledges COMA for the support and hospitality.

References

[1] T. Banakh, V. Valov, General position properties in fiberwise geometric topology, arXiv:1001.2494v1 [math.GT].
[2] T. Banakh, V. Valov, Approximation by light maps and parametric Lelek maps, Topology Appl. 157 (2010) 2325–2341.
[3] H. Berkowitz, P. Roy, General position and algebraic independence, in: L.C. Glaser, T.B. Rushing (Eds.), Geometric Topology, Proceedings of the Geometry Topology Conference Held at Park City, UT, Springer, New York, 1975, pp. 9–15.
[4] S.A. Bogatyi, Finite-to-one maps, Topology Appl. 155 (2008) 1876–1887.
[5] S. Bogatyi, V. Valov, Roberts' type embeddings and conversion of the transversal Tverberg's theorem, Mat. Sb. 196 (11) (2005) 33–52 (in Russian); transl. in: Sb. Math. 196 (11–12) (2005) 1585–1603.
[6] R. Courant, H. Robbins, What is Mathematics? An Elementary Approach to Ideas and Methods, Oxford University Press, New York, 1979.
[7] B. Dubrovin, S. Novikov, A. Fomenko, Modern Geometry, Nauka, Moscow, 1979 (in Russian).
[8] T. Goodsell, Strong general position and Menger curves, Topology Appl. 120 (2002) 47–55.
[9] T. Goodsell, Projections of compacta in $\mathbb{R}^n$, PhD thesis, Brigham Young University, Provo, UT, 1997.
[10] D. Repovš, P. Semenov, Continuous Selections of Multivalued Mappings, Math. Appl., vol. 455, Kluwer, Dordrecht, 1998.
[11] J. Roberts, A theorem on dimension, Duke Math. J. 8 (1941) 565–574.
[12] V. Valov, Parametric bing and Krasinkiewicz maps, Topology Appl. 155 (8) (2008) 916–922.