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NORMAL SELECTORS FOR THE NORMAL SPACES

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1. Introduction. Terminology and Notations. Definition 1. Let $\mathscr{K}$ be a class of topological spaces and $\mathscr{Z}(X)$ and $\mathscr{M}(X)$ be families of subsets of a given topological space X. The space X is said to be a $\mathscr{K}-\mathscr{Z}(X)-\mathscr{M}(X)$-selector if every l.s.c. set-valued mapping $\Phi: Y\to\mathscr{Z}(X)$ with $Y\in\mathscr{K}$ has a u.s.c. selection $\psi: Y\to\mathscr{M}(X)$. If, in addition, X is a closed convex subset of a locally convex vector space and every l.s.c. mapping $\Phi: Y\to\overline{\mathrm{Conv}}\,\mathscr{Z}(X)$ with $Y\in\mathscr{K}$ has a u.s.c. selection $\psi: Y\to\mathscr{M}(X)$, then X is said to be a convex-$\mathscr{K}-\mathscr{Z}(X)$-$\mathscr{M}(X)$-selector (here $\overline{\mathrm{Conv}}\,\mathscr{Z}(X)$ stands for the family of the closed convex hulls of the members of $\mathscr{Z}(X)$).

In what follows $\mathscr{F}(X)$ stands for the family of all non-void closed subsets of X, $\mathscr{C}(X)=\{F\in\mathscr{F}(X): F$ is compact$\}$ and $\mathscr{C}'(X)=\mathscr{C}(X)\cup\{X\}$; $\mathscr{N}$ stands for the class of all normal (always Hausdorff) spaces: The class of all τ-paracompact members of $\mathscr{N}$ is denoted by $\mathscr{P}_\tau$ and the class of all p-paracompact (in the sense of A. Arkhangel'skii, [1]) members of $\mathscr{N}$, by $p\mathscr{P}$.

The (convex-) $\mathscr{K}-\mathscr{F}(X)-\mathscr{C}(X)$-selectors were called in [7] (convex-) $\mathscr{K}$-selectors. As in [7], several particular cases of the following question are discussed in the present paper:

Let $\mathscr{K}$ be a class of topological spaces with $\mathscr{C}\subset\mathscr{K}\subset\mathscr{N}$, where $\mathscr{C}$ denotes the class of all compact Hausdorff spaces. Is every (convex-) $\mathscr{K}$-selector which belongs to $\mathscr{K}$ a (complete) metric space?

Let us recall that in [7] the positive answers to the above question were obtained in the case of $\mathscr{K}=\mathscr{C}$, as well as of $\mathscr{K}=p\mathscr{P}$.

2. Statements of the Results. Theorem 1. Let X be a normal space which is a (convex-) $\mathscr{N}-\mathscr{Z}(X)-\mathscr{F}(X)-$ selector, where $\mathscr{C}'(X)\subset\mathscr{Z}(X)\subset\mathscr{F}(X)$. Then

(a) X is a complete separable metric space;

(b) every decreasing sequence of members of $\mathscr{Z}(X)$ ($\overline{\mathrm{Conv}}\,\mathscr{Z}(X)$) has no empty intersection.

Corollary 1. The class of all normal $\mathscr{N}-\mathscr{C}'(X)-\mathscr{F}(X)$-selectors coincides with the class of all completely metrizable separable spaces.

From the proof of Theorem 1 (below) we also get:

Corollary 2. The class of all $\aleph_0$-paracompact and normal $\mathscr{P}_{\aleph_0}-\mathscr{F}(X)-\mathscr{F}(X)$-selectors coincides with the class of all completely metrizable separable spaces.

Analogous results follow for convex-selectors of corresponding types too.

Corollary 3. The class of all normal $\mathscr{N}-\mathscr{F}(X)-\mathscr{F}(X)$-selectors coincides with the class of all compact metrizable spaces.

Corollary 4. The normal space X is a convex-$\mathcal{N}-\mathcal{F}(X)-\mathcal{F}(X)$-selector iff X is a closed, convex, separable, weakly compact subspace of a Frechet space.

Corollary 5. If X is a normal convex-$\mathcal{N}-\mathcal{F}(X)-\mathcal{F}(X)$-selector and Y is normal, then every l. s. c. mapping $\Phi: Y \to \overline{\text{Conv}}\,\mathcal{F}(X))$ has a single-valued continuous selection. This corollary is in agreement with the following remark of S. Nedev.

If X is a T_1-space and every l. s. c. $\Phi: X \to \overline{\text{Conv}}\,(\mathcal{F}(Y))$ with Y — a Banach space, has a u. s. c. selection $\psi: X \to \mathcal{F}(Y)$, then X is paracompact and (which is the same) every such Φ has a continuous single-valued selection. (The collectionwise-normal version, as well as different generalizations of the last assertion, hold too).

Open question: Is the converse of Theorem 1 true?

3. Proofs. The Corollaries 1., 2.. 3., 5. and one half of Corollary 4. follow immediately from Theorem 1. (and its proof below) and from some known results on selections of set-valued mappings (see, for instance, [4,6]). All we have to prove, therefore, are Theorem 1 and the selection theorem formulated in Cor. 4.

Proof of Theorem 1. (a) (V. Valov). That X is a separable completely metrizable space. In fact, we shall prove something more. Namely, call the normal space X a multi-valued absolute retract for normal spaces (MAR $(\mathcal{N})$) iff, whenever the normal space Y contains X as a closed subset, there exists a u. s. c. retraction $r: Y \to \mathcal{F}(X)$ (i. e. $r(x)=\{x\}$ for every $x \in X$). We shall show that the class of all normal spaces X in which every member of $\mathcal{C}'(X)$ (every member of $\overline{\text{Conv}}\,(\mathcal{C}'(X))$ in the case when X is a closed convex subspace of a locally convex vector space) is MAR$(\mathcal{N})$, coincides with the class of all separable completely metrizable spaces (fulfilling the corresponding requirements of convexity, respectively). We begin with the following lemma.

Lemma 1. Let A be a set and, for each $\alpha \in A$, $Y_\alpha \subset X_\alpha$ be topological spaces, Y_α being first countable. Suppose also Y is dense in X, where $X=\Pi\{X_\alpha: \alpha \in A\}$ and $Y=\Pi\{Y_\alpha: \alpha \in A\}$. Next, let $x \in Y$ and let H be a closed subset of X containing x. If every closed G_δ-subset of X which contains x meets $X \setminus H$ then there a discrete space $T \subset Y \setminus H$ of cardinality $|T|=\aleph_1$ exists, such that $T \cup \{x\}$ is the one-point compactification of T.

Proof of Lemma 1. We shall exploit Efimov's arguments [3], the proof of Lemma 5. For $\mu<\omega_1$ finite set $B(\mu) \subset A$ and a point $x^\mu \in Y \setminus H$ are defined inductively, under the conditions: (i) $B(\mu)=\{\alpha \in A: x^\mu_\alpha \neq x_\alpha\}$; (ii) $x^\mu_\alpha = x_\alpha$ for every $\alpha \in A(\mu)=\cup\{B(\nu): \nu<\mu\}$ and (iii) $\{B(\mu): \mu<\omega_1\}$ is a disjoint family. Let $\mu<\omega_1$ and suppose $B(\nu)$ and x^ν have already been defined for each $\nu<\mu$. Put $H(\mu)=\pi^{-1}_{A(\mu)}\pi_{A(\mu)}(x)$, where $\pi_{A(\mu)}: X \to \Pi\{X_\alpha: \alpha \in A(\mu)\}$ is the natural projection. As $H(\mu)$ is a closed G_δ-set containing x, we can take $y^\mu \in (H(\mu) \cap Y) \setminus H$ and et $C(\mu)=\{\alpha_1, \ldots \alpha_n\} \subset A$ and the open sets $U_i \subset X_{\alpha_i}$ be such that $y^\mu \in U(y^\mu) = \Pi\{U_i: i=1, 2, \ldots n\} \times \Pi\{X_\alpha: \alpha \in A \setminus C(\mu)\} \subset X \setminus H$. Define $x^\mu \in U(y^\mu) \cap Y$ by setting $x^\mu_\alpha = y^\mu_\alpha$ for $\alpha \in C(\mu)$ and $x^\mu_\alpha = x_\alpha$ otherwise. Put $B(\mu)=\{\alpha \in A: x^\mu_\alpha \neq x_\alpha\}$ and $T=\{x^\mu: \mu<\omega_1\}$. One checks easily that T is discrete (since $B(\mu) \cap A(\mu)=\varnothing$ for every $\mu<\omega_1$) and $T \cup \{x\}$ is compact,

Lemma 2. Suppose X is MAR$(\mathcal{N})$ and X contains the one-point compactification of no uncountable discrete space. Then X is separable and completely metrizable.

Proof of Lemma 2. Let A be the set of all continuous mappings $f: X \to [0,1]=I$. We consider X as imbedded in the cube I^A so that the closure of

X in I^A is the Štone-Čech compactification βX of X. Denote by Y the space obtained from I^A by means of making the points of $I^A \setminus X$ isolated. Y is a normal space containing X as a closed subset, so that there is a u. s. c. retraction $r_1: Y \to \mathcal{F}(X)$ (i. e. $r_1(x)=\{x\}$ for every $x \in X$). For each open $U \subset X$ put $e(U)=\mathrm{Int}_{I^A}(r_1^{\#}(U))=\mathrm{Int}_{I^A}(\{x \in I^A : r_1(x) \subset U\})$. Clearly, (i) $e(U) \cap X = U$ and (ii): $U_1 \cap U_2 = \varnothing$ implies $e(U_1) \cap e(U_2) = \varnothing$. Thus we have

Remark 1. If X is MAR$(\mathcal{N})$, then $c(X)=\sup\{\mathrm{Card}(\gamma):\gamma$ is a disjoint family of open subsets of $X\} \leq \aleph_0$, because $c(I^A) \leq \aleph_0$.

Next, let $r: I^A \to C(\beta X)$ be defined as follows: $r(x) = \bigcap\{\overline{U}^{I^A} : x \in e(U)\}$ for $x \in I^A$. One checks easily that r is u.s.c., that $r_1(x) \subset r(x)$ for each $x \in I^A$ and $r(x)=\{x\}$ for $x \in X$. Suppose now that for some $x \in X$, every closed G_δ-subset of I^A containing x meets $I^A \setminus r^{-1}(x)$. Then, by Lemma 1, there is a discrete space $T \subset I^A \setminus r^{-1}(x)$ with $\mathrm{Card}(T)=\aleph_1$ and such that $P = T \cup \{x\}$ is compact. Note that $x \notin r^{-1}(r(y))$ for every $y \in T$ since $T \subset P \setminus r^{-1}(x)$. Hence $r^{-1}(r(y)) \cap P$ is finite for each $y \in T$, thus there is a set $T' \subset T$ with Card $(T')=\aleph_1$ such that the family $\{r(y): y \in T'\}$ is disjoint. Fix a point $x_y \in r(y) \cap X$ for every $y \in T'$. It follows from the u. s. continuity of r that the set $\{x, x_y : y \in T'\}$ is the one-point compactification of the uncountable discrete space $\{x_y : y \in T'\}$; this contradicts the assumption of Lemma 2. Hence we can fix a closed G_δ-set $H(x)$ with $x \in H(x) \subset r^{-1}(x)$ for each $x \in X$. By results of Pol a. Pol [8], Theorem 1, there is a countable subset $\{x_i : i=1, 2, \ldots\}$ of X such that $\bigcup\{H(x_i): i=1, 2, \ldots\}$ is dense in $Z=\bigcup\{H(x): x \in X\}$. For each n fix a countable subset $B(n)$ of A such that $H(x_n)=\pi_{B(n)}^{-1}(\pi_{B(n)}(H(x_n)))$ where $\pi_{B(n)}: I^A \to I^{B(n)}$ is the natural projection. Put $B=\bigcup\{B(n): n=1, 2, \ldots\}$. One checks easily that if $x \in X$ and $\pi_B(x)=\pi_B(y)$, then $x \in r(y)$, so that the restriction $f=\pi_B/X$ is a continuous bijection from X onto the separable metrizable space $\pi_B(X)$. Actually, however, f is a homeomorphism. Indeed, for every open subset U of X we have $f(U)=\pi_B(r^{\#}(\widetilde{U})) \cap \pi_B(X)$, where $\widetilde{U}$ is open in βX with $\widetilde{U} \cap X = U$. The Čech-completeness of X follows now from [7], Proposition 1. Lemma 2 has thus been proved.

Now, if for a normal space X each member of $\mathcal{C}'(X)$ is MAR$(\mathcal{N})$, then X is separable and completely metrizable by Lemma 2 and Remark 1. In the convex case, one has also to apply Proposition 2 from [7] in a suitable manner (by passing to the completion). Thus, proposition (a) of Theorem 1 has been proved.

(b) The intersection of every decreasing sequence of members of $\mathcal{Z}(X)$ is not empty.

Suppose this is not the case and let $F_1 \underset{\neq}{\supset} F_2 \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} F_n \underset{\neq}{\supset}$ be a strictly (without loss of generality) decreasing sequence of members F_n of $\mathcal{Z}(X)$ with an empty intersection. Yet, let Y be a normal space with an open covering $\omega=\{U_1 \subset U_2 \subset \ldots\}$ the last having no locally finite open (nor closed) refinement (for the existence of such an Y, see [9]). For $y \in Y$ define $\Phi(y)=F_{n_y}$, where $n_y=\min\{n: y \in U_n\}$. Obviously, the mapping $\Phi: Y \to \mathcal{Z}(X)$ is l.s.c., so there exists an u.s.c. selection $\psi: Y \to \mathcal{F}(X)$ for Φ. Next, fix a point $x_n \in F_n \setminus F_{n+1}$ for every $n=1, 2, \ldots$. Since $\bigcap\{F_n : n=1, 2, \ldots\}=\varnothing$ the set $H=\{x_n : n=1, 2, \ldots\}$ is closed and discrete in F_1 (even in X). For $x \in F_1$ define $\varphi(x)=\{x_{n_x}, x_{n_x+1}, \ldots\}$, where $n_x=\max\{n: x \in F_n\}$. The mapping $\varphi: F_1 \to \mathcal{F}(H)$ is l.s.c., so that there exists [5,6] a u.s.c. selection $r: F_1 \to \mathcal{C}(H)$ for φ (remember that F_1 is metrizable and hence a paracompact space). Now put $P_n=(r \circ \psi)^{-1}$ $\times(\{x_1, x_2, \ldots x_n\})$. One checks easily that for each n the set P_n is closed in

$Y, P_n \subset U_n$ and $Y = \cup\{P_n : n = 1, 2, \ldots\}$. But this enables us to construct an open locally finite refinement for ω [2], which is a contradiction.

Proof of the Selection Theorem from Corollary 4. Let $\Phi : Y \to \overline{\text{Conv}}\,(\mathcal{F}(X))$ be l.s.c. where Y is a normal space. In order to construct a single-valued continuous selection for Φ we are going to show that, for each $\varepsilon > 0$, the covering $\{\Phi^{-1}(O_\varepsilon(x)) : x \in X\}$ of Y has an open locally finite refinement and then to apply Michael's arguments from [4]. The proof of Theorem 3.2″, a) $\to$ b)] ($O_\varepsilon(x)$ denotes the open ball of center x and radius ε). Let $\{O_\varepsilon(x_i) : i = 1, 2, \ldots\}$ be countable covering of X and denote $\omega = \{\Phi^{-1}(O_\varepsilon(x_i)) : i = 1, 2, \ldots\}$. As is well known (and can easily be seen), it suffices to show that ω has an index-refinement consisting of F_σ-sets in order to prove that ω has an open locally finite refinement. To this end, denote by X' the set X endowed with the weak topology. Obviously X' may be considered as a compact convex subspace of a certain Frechet space and hence the l.s.c. mapping $\Phi : Y \to \overline{\text{Conv}}\,(\mathcal{F}(X'))$ has a single-valued continuous selection $f : Y \to X'$. Put $H_{i,n} = f^{-1}(B_{(1-n^{-1})\varepsilon}(x_i))$, where $B_\delta(x)$ denotes the closed ball of center x and radius δ, and let $H_i = \cup\{H_{i,n} : n = 1, 2, \ldots\}$. It is clear that H_i is an F_σ-subset of Y (note that $B_\delta(x)$ being closed and convex in X, is a closed subset of X'), $H_i \subset \Phi^{-1}(O_\varepsilon(x_i))$ for every $i = 1, 2, \ldots$ and $\{H_i, i = 1, 2, \ldots\}$ covers Y. The next steps being obvious, the proof is complete.

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