

GENERALIZED CANTOR MANIFOLDS AND INDECOMPOSABLE CONTINUA

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ABSTRACT. We review results concerning homogeneous compacta and discuss some open questions. It is established that indecomposable continua are Alexandroff (resp., Mazurkiewicz, or strong Cantor) manifolds with respect to the class of all continua. We also provide some new proofs of Bing's theorems about separating metric compacta by hereditarily indecomposable compacta.

1. INTRODUCTION

Cantor manifolds were introduced by Urysohn [31] as a generalization of Euclidean manifolds. Recall that a space X is a *Cantor n -manifold* if X cannot be separated by a closed $(n - 2)$ -dimensional subset. In other words, X cannot be the union of two proper closed sets whose intersection is of covering dimension $\leq n - 2$. Another specification of Cantor manifolds was considered by Hadžiivanov [10]: X is a *strong Cantor n -manifold* if for arbitrary representation $X = \bigcup_{i=1}^{\infty} F_i$, where all F_i are proper closed subsets of X , we have $\dim(F_i \cap F_j) \geq n - 1$ for some $i \neq j$. Hadžiivanov and Todorov [11] introduced the class of Mazurkiewicz n -manifolds, which is a proper sub-class of the strong Cantor n -manifolds: X is a *Mazurkiewicz n -manifold* if for any disjoint closed massive subsets A and B of X (a massive subset of X is a set with non-empty interior in X), and any normally placed set $M \subset X$ with $\dim M \leq n - 2$, there exists a continuum in $X \setminus M$ intersecting A and B (equivalently, no such M is cutting X between A and B). Here, M is normally placed in X provided every two disjoint closed in M sets have disjoint open in X neighborhoods. For example, every F_σ -subset of a normal space is normally placed in that space. The notion of a Mazurkiewicz n -manifold has its roots in the classical Mazurkiewicz theorem saying that no region in the Euclidean n -space can be

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cut by a subset of dimension $\leq n - 2$ [25] (recall that a set $M \subset X$ does not cut X if for every two points $x, y \in X \setminus M$ there is a continuum $K \subset X \setminus M$ joining x and y).

Alexandroff [2] introduced the stronger notion of V^n -continua: a compactum X is a V^n -continuum if for every two closed disjoint subsets X_0, X_1 of X , both having non-empty interior in X , there exists an open cover ω of X such that there is no partition P in X between X_0 and X_1 admitting an ω -map into a space Y with $\dim Y \leq n - 2$ ($f: P \rightarrow Y$ is said to be an ω -map if there exists an open cover γ of Y such that $f^{-1}(\gamma)$ refines ω).

Obviously, strong Cantor n -manifolds are Cantor n -manifolds. Moreover, every V^n -continuum is a Mazurkiewicz n -manifold and Mazurkiewicz n -manifolds are strong Cantor n -manifolds, see [11]. None of the above inclusions is reversible, see [16].

In the present paper we consider quite general concepts of the above notions with respect to some classes of finite or infinite-dimensional spaces.

Let \mathcal{C} be a class of topological spaces.

Definition 1.1. A space X is an *Alexandroff manifold with respect to \mathcal{C}* (br., *Alexandroff \mathcal{C} -manifold*) if for every two closed, disjoint, massive subsets X_0, X_1 of X there exists an open cover ω of X such that there is no partition P in X between X_0 and X_1 admitting an ω -map onto a space $Y \in \mathcal{C}$.

Definition 1.2. A space X is said to be a *Mazurkiewicz manifold with respect to \mathcal{C}* (br., *Mazurkiewicz \mathcal{C} -manifold*) provided for every two closed, disjoint, massive subsets $X_0, X_1 \subset X$, and every set $F = \bigcup_{i=0}^{\infty} F_i \subset X$ with each $F_i \in \mathcal{C}$ being proper closed subset of X , there exists a continuum K in $X \setminus F$ joining X_0 and X_1 .

If, under the above conditions, for every open cover ω of X there exists a set $K_\omega \subset X \setminus F$ joining X_0 and X_1 such that K_ω admits an ω -map onto a continuum, X is called a *weak Mazurkiewicz \mathcal{C} -manifold*.

Definition 1.3. A space X is a *strong Cantor manifold with respect to \mathcal{C}* (br., *strong Cantor \mathcal{C} -manifold*) if X can not be represented as the union

$$(1.1) \quad X = \bigcup_{i=0}^{\infty} F_i \quad \text{with} \quad \bigcup_{i \neq j} (F_i \cap F_j) \in \mathcal{C},$$

where all F_i are proper closed subsets of X .

Definition 1.4. A space X is a *Cantor manifold with respect to \mathcal{C}* (br., *Cantor \mathcal{C} -manifold*) if X cannot be separated by a closed subset which belongs to \mathcal{C} .

Obviously, any strong Cantor \mathcal{C} -manifold is a Cantor \mathcal{C} -manifold. Moreover, if the class \mathcal{C} is hereditary with respect to F_σ -subsets, then compact Mazurkiewicz \mathcal{C} -manifolds are strong Cantor \mathcal{C} -manifolds, see [16].

2. GENERALIZED CANTOR MANIFOLDS AND HOMOGENEOUS CONTINUA

In this section we discuss some properties of homogeneous continua and ask some questions.

We first introduce the general dimension function $D_{\mathcal{K}}$ considered in [16], which captures the covering dimension, cohomological dimension \dim_G with respect to any Abelian group G , as well as the extraordinary dimension \dim_L with respect to a given CW -complex L .

A sequence $\mathcal{K} = \{K_0, K_1, \dots\}$ of CW -complexes is called a *stratum* for a dimension theory [7] if

- for each space X admitting a perfect map onto a metrizable space, $K_n \in AE(X)$ implies both $K_{n+1} \in AE(X \times \mathbb{I})$ and $K_{n+j} \in AE(X)$ for all $j \geq 0$.

Here, $K_n \in AE(X)$ means that K_n is an absolute extensor for X . Given a stratum \mathcal{K} , we can define a dimension function $D_{\mathcal{K}}$ in a standard way:

- (1) $D_{\mathcal{K}}(X) = -1$ iff $X = \emptyset$;
- (2) $D_{\mathcal{K}}(X) \leq n$ if $K_n \in AE(X)$ for $n \geq 0$; if $D_{\mathcal{K}}(X) \leq n$ and $K_m \notin AE(X)$ for all $m < n$, then $D_{\mathcal{K}}(X) = n$;
- (3) $D_{\mathcal{K}}(X) = \infty$ if $D_{\mathcal{K}}(X) \leq n$ is not satisfied for any n .

If $\mathcal{K} = \{\mathbb{S}^0, \mathbb{S}^1, \dots\}$, we obtain the covering dimension \dim . The stratum $\mathcal{K} = \{\mathbb{S}^0, K(G, 1), \dots, K(G, n), \dots\}$, $K(G, n)$, $n \geq 1$, being the Eilenberg-MacLane complexes for a given group G , determines the cohomological dimension \dim_G . Moreover, if L is a fixed CW -complex and $\mathcal{K} = \{L, \Sigma(L), \dots, \Sigma^n(L), \dots\}$, where $\Sigma^n(L)$ denotes the n -th iterated suspension of L , we obtain the extraordinary dimension \dim_L introduced by Shchepin [27] and considered in details by Chigogidze [4].

According to the countable sum theorem in extension theory, it follows directly from the above definition that $D_{\mathcal{K}}(X) \leq n$ implies $D_{\mathcal{K}}(A) \leq n$ for any F_{σ} -subset $A \subset X$.

Henceforth, \mathcal{C} will denote one of the four classes:

- the class $\mathcal{D}_{\mathcal{K}}^k$ of at most k -dimensional spaces with respect to dimension $D_{\mathcal{K}}$,
 - the class $\mathcal{D}_{\mathcal{K}}^{\leq \infty}$ of strongly countable $D_{\mathcal{K}}$ -dimensional spaces, i.e. all spaces represented as a countable union of closed finite-dimensional subsets with respect to $D_{\mathcal{K}}$,
 - the class \mathbf{C} of paracompact C -spaces,
- and
- the class \mathcal{WID} of weakly infinite-dimensional spaces.

For definitions of a weakly (strongly) infinite-dimensional or a C -space, see [9].

It was proved in [19] that every homogeneous metrizable, locally compact, connected space X with the covering dimension $\dim X = n \leq \infty$ is a Cantor n -manifold; in case where X is strongly infinite-dimensional, it is a Cantor \mathcal{WID} -manifold. Next theorem, established in [16], significantly generalizes those results.

Theorem 2.1. *Every metrizable homogeneous continuum $X \notin \mathcal{C}$ is a strong Cantor \mathcal{C} -manifold provided that:*

- (1) \mathcal{C} is any of the following three classes: \mathcal{WID} , \mathbf{C} , $\mathcal{D}_{\mathcal{K}}^{n-2}$ (in the latter case we additionally assume $D_{\mathcal{K}}(X) = n$);
or
- (2) $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{<\infty}$ and X does not contain closed subsets of arbitrary large finite dimension $D_{\mathcal{K}}$.

In case X is locally connected, Theorem 2.1 was generalized in [17].

Theorem 2.2. *Let X be a homogeneous locally compact, locally connected metric space. Suppose U is a region in X with $U \notin \mathcal{C}$, where $\mathcal{C} \in \{\mathcal{WID}, \mathbf{C}, \mathcal{D}_{\mathcal{K}}^{n-2}, \mathcal{D}_{\mathcal{K}}^{<\infty}\}$, $n \geq 1$, and $D_{\mathcal{K}}(U) = n$ in the case $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{n-2}$. Then U can not be cut by any set $\bigcup_{i=0}^{\infty} F_i$ with each $F_i \in \mathcal{C}$ being closed in U .*

Theorems 2.1 and 2.2 are based on the following results from [16]:

Theorem 2.3. *Let X be a compact space.*

- (1) *If $D_{\mathcal{K}}(X) = n$, then X contains a closed subset M such that $D_{\mathcal{K}}(M) = n$ and M is both Alexandroff and a Mazurkiewicz manifold with respect to the class $\mathcal{D}_{\mathcal{K}}^{n-2}$;*
- (2) *If $D_{\mathcal{K}}(X) = \infty$, then either X contains closed subsets of arbitrary large finite dimension $D_{\mathcal{K}}$ or X contains a compact Mazurkiewicz $\mathcal{D}_{\mathcal{K}}^{<\infty}$ -manifold;*
- (3) *If $X \notin \mathbf{C}$, then it contains a compact Mazurkiewicz \mathbf{C} -manifold;*
- (4) *If X is metrizable and strongly infinite-dimensional, then it contains a compact Mazurkiewicz \mathcal{WID} -manifold.*

Some particular cases of Theorem 2.2 were established by different authors, see [1], [12], [15], [21], [29], [30].

Here is the main questions arising from the above results.

Question 2.4. *Let X be a homogeneous compact metric continuum and $\mathcal{C} \in \{\mathcal{WID}, \mathbf{C}, \mathcal{D}_{\mathcal{K}}^{n-2}, \mathcal{D}_{\mathcal{K}}^{<\infty}\}$, where $n \geq 1$ and $D_{\mathcal{K}}(U) = n$ in the case $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{n-2}$. Is X an Alexandroff \mathcal{C} -manifold? What is the answer of the above question if, in addition, X is locally connected?*

Krupski [18] conjectured that any n -dimensional, homogeneous metric ANR-continuum is a V^n -continuum. Next result, which is still unpublished, provides a partial solution to Krupski's conjecture and partial answer of Question 2.4.

Theorem 2.5. *Every homogeneous, metric n -dimensional ANR-continuum X is a V^n -continuum provided $H^n(X, \mathbb{Z}) \neq 0$.*

As it was mentioned above, if \mathcal{C} is the class of all spaces of covering dimension $\leq n - 2$, then every Alexandroff \mathcal{C} -manifold is a Mazurkiewicz \mathcal{C} -manifold. So, we have next question.

Question 2.6. *Let \mathcal{C} be one of the above four classes. Is there any Alexandroff \mathcal{C} -manifold which is not a Mazurkiewicz \mathcal{C} -manifold?*

Question 2.7. *Let X and \mathcal{C} be as in Question 2.4. Is X a Mazurkiewicz \mathcal{C} -manifold? More generally, is it true that X satisfy the conclusion in Theorem 2.2 with $U = X$?*

Having in mind Theorem 2.3, next question is also interesting.

Question 2.8. *Let $\mathcal{C} \in \{\mathbf{C}, \mathcal{WID}\}$ and X be a compact metric space with $X \notin \mathcal{C}$. Does X contain a compact Alexandroff \mathcal{C} -manifold?*

The last question in this section is inspired by the following result of the first author [28]: If $M \subset \mathbb{I}^n$ is a set of dimension $\dim M \leq k$, where $k \leq n-2$, then every two different points $x, y \in \mathbb{I}^n \setminus M$ can be joined by a V^{n-k-1} -continuum $K \subset \mathbb{I}^n \setminus M$.

Question 2.9. *Let X be a homogeneous metric (locally connected) continuum and $M \subset X$ an F_σ -set with $\dim F \leq k \leq n-2$. Is it true that any two massive closed disjoint sets $A, B \subset X$ can be joined in $X \setminus M$ by a V^{n-k-1} -continuum?*

We conclude this section with a few words about our definition of Alexandroff \mathcal{C} -manifolds. The Alexandroff definition of V^n -continua is based on the following property of the covering dimension: if X is a paracompact space and for every open cover ω of X there exists an ω -map X onto a paracompact space Y with $\dim Y \leq n$, then $\dim X \leq n$. It follows from [6, Theorem 2.4] that the dimension $D_{\mathcal{K}}$ has a similar property for spaces X admitting perfect maps onto metrizable spaces. Unfortunately, this is not true for the classes \mathbf{C} and \mathcal{WID} . For example, for any open cover ω of the Hilbert cube Q there exists an ω -map onto a finite-dimensional space, but Q is strongly infinite-dimensional.

3. INDECOMPOSABLE CONTINUA AND CANTOR MANIFOLDS

Recall that a continuum is indecomposable if it is not the union of two proper sub-continua. The results in this section came out from the observation [20] that any indecomposable continuum can not be separated by a proper connected subset. This means that indecomposable continua are Cantor manifolds with respect to the class \mathfrak{K} of all continua.

Theorem 3.1. *Any metric indecomposable continuum is both Mazurkiewicz $\mathfrak{K}(\aleph_0)$ -manifold and a strong Cantor $\mathfrak{K}(\aleph_0)$ -manifold, where $\mathfrak{K}(\aleph_0)$ is the class of all spaces having at most countably many components.*

Proof. Suppose X is a metric indecomposable continuum and let A and B be disjoint, closed, massive subsets of X . Since there are uncountably many disjoint composants of X , see [20], for any sequence $\{F_i\}_{i \geq 1}$ of proper closed subsets of X with $F_i \in \mathfrak{K}(\aleph_0)$, $i \geq 1$, there exists a component K of X disjoint from the

set $F = \bigcup_{i=1}^{\infty} F_i$. Because each component is dense in X [20], there exist points $a \in A \cap K$ and $b \in B \cap K$. Let $p \in X$ be the point generating K , i.e. K is the set of all $x \in X$ such that there exists a proper sub-continuum P_x of X containing p and x . Then $P = P_a \cup P_b \subset K$ is a continuum in $X \setminus F$ containing a, b . Therefore, X is a Mazurkiewicz $\mathfrak{K}(\aleph_0)$ -manifold.

Assume that X is not a strong Cantor $\mathfrak{K}(\aleph_0)$ -manifold. So, $X = \bigcup_{i=1}^{\infty} H_i$ for some sequence $\{H_i\}_{i \geq 1}$ of proper closed subset of X such that $H_i \cap H_j \in \mathfrak{K}(\aleph_0)$ for all $i \neq j$. According to the Baire theorem, there are two different integers n, m such that both H_m and H_n have non-empty interior. Let Q_n and Q_m be two massive disjoint subsets of X with $Q_n \subset H_n$ and $Q_m \subset H_m$. Since X is a Mazurkiewicz $\mathfrak{K}(\aleph_0)$ -manifold, there exists a continuum $C \subset X \setminus \bigcup_{i \neq j} H_i \cap H_j$ with $Q_n \cap C \neq \emptyset \neq Q_m \cap C$. This implies that C is a non-degenerate continuum covered by the disjoint family $\{C \cap H_i : i \geq 1\}$ of closed sets at least two of which are non-empty. This contradicts the Sierpiński theorem. Hence, X is a strong Cantor $\mathfrak{K}(\aleph_0)$ -manifold. \square

Corollary 3.2. *Any hereditarily indecomposable continuum is a weak Mazurkiewicz $\mathfrak{K}(\aleph_0)$ -manifold.*

Proof. According to Theorem 3.1, we may assume that X is non-metrizable. Then, by [13, Proposition 4.2], X is the limit space of an inverse system $S = \{X_\sigma, \pi_\sigma^\sigma, \Sigma\}$ consisting of metric hereditarily indecomposable continua X_σ . Since the limit space of each inverse sequence of hereditarily indecomposable continua is also hereditarily indecomposable, we may assume that S is σ -continuous (i.e., $\beta = \sup\{\alpha_n : n \geq 1\} \in \Sigma$ and $X_\beta = \varprojlim \{X_{\alpha_n}, \pi_{\alpha_n}^{\alpha_{n+1}}\}$ for every countable chain $\{\alpha_n : n \geq 1\}$ in Σ). Denote by $\pi_\sigma : X \rightarrow X_\sigma$, $\sigma \in \Sigma$, the limit projections. Let A and B be disjoint, closed and massive subsets of X , $\{F_i\}_{i \geq 1}$ a sequence of proper closed subsets of X with $F_i \in \mathfrak{K}(\aleph_0)$ for all i , and ω be an open cover of X . There exists $\alpha \in \Sigma$, an open cover ω_α of X_α and disjoint open sets U_A and U_B in X_α such that $\pi_\alpha^{-1}(U_A) \subset A$, $\pi_\alpha^{-1}(U_B) \subset B$, $\pi_\alpha^{-1}(\omega_\alpha)$ refines ω and $\pi_\alpha(F_i) \neq X_\alpha$ for all i . Such α exists because S is σ -continuous. Since X_α is a Mazurkiewicz $\mathfrak{K}(\aleph_0)$ -manifold, there is a continuum $P \subset X_\alpha \setminus \bigcup_{i=1}^{\infty} \pi_\alpha(F_i)$ joining U_A and U_B . Then the set $K = \pi_\alpha^{-1}(P) \subset X \setminus \bigcup_{i=1}^{\infty} F_i$ is joining A and B , and $\pi_\alpha : K \rightarrow P$ is an ω -map. This completes the proof. \square

Theorem 3.3. *Any metric indecomposable continuum is an Alexandroff \mathfrak{K} -manifold.*

Proof. Suppose there exists an indecomposable metric continuum (X, d) which is not an Alexandroff \mathfrak{K} -manifold. Consequently, we have two closed, disjoint, massive sets A and B in X satisfying the following condition: for every open cover ω of X there exists a partition P_ω in X between A and B , and a surjective ω -map $g_\omega : P_\omega \rightarrow Y_\omega$, where Y_ω is a continuum. Since X is a metric compactum, there is a sequence $\{\omega_n\}$ of open covers of X such that $\text{mesh}(\omega_n) < 1/n$ and the sequence $\{P_{\omega_n}\}$ converges to the compact set $P_0 \subset X$ with respect to the Hausdorff metric

generated by d . Let $a \in \text{Int}(A)$ and $b \in \text{Int}(B)$. Because each P_{ω_n} is disjoint from $A \cup B$, $a, b \notin P_0$.

Claim 1. P_0 is connected.

Indeed, otherwise let $P_0 = P_1 \cup P_2$ be the union of two disjoint non-empty closed subsets. Choose two open sets U_1, U_2 in X having disjoint closures such that $P_j \subset U_j$, $j = 1, 2$. Then there exists m such that $P_{\omega_m} \subset U_1 \cup U_2$ and $P_{\omega_m} \cap U_j \neq \emptyset$ for each $j = 1, 2$. We may assume that m is so big that the fibers of the map g_{ω_n} meets only one of the sets U_1, U_2 . The last condition implies that $g_{\omega_m}(P_{\omega_m} \cap \overline{U_1})$ and $g_{\omega_m}(P_{\omega_m} \cap \overline{U_2})$ are non-empty disjoint closed subsets of Y_{ω_m} whose union is Y_{ω_m} , a contradiction.

Claim 2. P_0 is a partition of X .

Assume P_0 is not a partition of X , and choose two closed sets A_1, B_1 in X both disjoint from P_0 and having non-empty interior such that $a \in A_1 \subset A$ and $b \in B_1 \subset B$. Since X is a Mazurkiewicz $\mathfrak{K}(\aleph_0)$ -manifold, there exists a continuum $K \subset X \setminus P_0$ joining A_1 and B_1 . So, $P_0 \subset X \setminus K$, which yields that $P_{\omega_s} \subset X \setminus K$ for some s . The last inclusion contradicts the fact that P_{ω_s} is a partition of X between A and B .

Hence, P_0 is a proper sub-continuum of X separating X , which is impossible because X is indecomposable. \square

Question 3.4. *Let X be an indecomposable continuum. Is it true that:*

- X is a strong Cantor $\mathfrak{K}(\aleph_0)$ -manifold;
- X is an Alexandroff \mathfrak{K} -manifold?

Next two propositions provide more examples of hereditarily indecomposable continua, and generalize the well known facts [3] that every $(n + 1)$ -dimensional (resp., strongly infinite-dimensional) metric compactum contains an n -dimensional (resp., strongly infinite-dimensional) hereditarily indecomposable continuum (see also [14] for another proofs).

Proposition 3.5. *Let X be a metric compactum of dimension $D_{\mathcal{K}}(X) = n + 1$, where $n \geq 0$ and $\mathcal{K} = \{K_0, K_1, \dots\}$ is a given stratum. Then X contains a hereditarily indecomposable continuum X_0 of dimension $D_{\mathcal{K}}(X_0) \in \{n, n + 1\}$.*

Proof. By [22], there exists a map $g: X \rightarrow \mathbb{I}$ such that all components of the fibers $g^{-1}(t)$, $t \in \mathbb{I}$, are hereditarily indecomposable. Let $g = h \circ p$ be the monotone-light decomposition of g with $p: X \rightarrow Y$ being monotone and $h: Y \rightarrow \mathbb{I}$ light. Then, by Hurewicz's theorem, $\dim Y \leq 1$. Assuming that $D_{\mathcal{K}}(p^{-1}(y)) \leq n - 1$ for all $y \in Y$, according to [5, Theorem 2.4], we obtain $D_{\mathcal{K}}(X) \leq n$. So, there exists $y_0 \in Y$ with $D_{\mathcal{K}}(p^{-1}(y_0)) \geq n$. Then $X_0 = p^{-1}(y_0)$ is the required continuum. \square

Similarly, one can prove the following propositions.

Proposition 3.6. *Let X be a strongly infinite-dimensional (resp., weakly infinite-dimensional with $X \notin \mathbf{C}$) metric compactum. Then X contains a strongly infinite-dimensional hereditarily indecomposable continuum X_0 (resp., weakly infinite-dimensional hereditarily indecomposable continuum $X_0 \notin \mathbf{C}$).*

Dranishnikov's example [8] of a strongly infinite-dimensional metric compactum having cohomological dimension $\dim_{\mathbb{Z}} = 3$ implies next corollary.

Corollary 3.7. *There exists a strongly infinite-dimensional hereditarily indecomposable continuum X with $\dim_{\mathbb{Z}}(X) \in \{2, 3\}$.*

Next proposition is analogue of Bing's partition theorems [3] (recall that a compactum X is called hereditarily indecomposable provided each continuum in X is indecomposable).

Proposition 3.8. *Let $f: X \rightarrow Y$ be a perfect map between metric spaces with X being connected, and let A, B be two closed disjoint subsets of X . Then there is a closed partition H of X between A and B with the following properties:*

- (i) *The intersection $f^{-1}(y) \cap H$ is hereditarily indecomposable for every $y \in Y$;*
- (ii) *If K is a continuum contained in some $f^{-1}(y)$ such that $K \cap (A \cup B) \neq \emptyset$, then K contains a component of the set $f^{-1}(y) \cap H$.*

Proof. Let $h: X \rightarrow \mathbb{I}$ be a continuous function with $h(A) = 0$ and $h(B) = 1$. According to [32], there exists a function $g: X \rightarrow \mathbb{I}$ such that $|g(x) - h(x)| < 1/4$ for all $x \in X$ and the restrictions $g_y = g|_{f^{-1}(y)} : f^{-1}(y) \rightarrow \mathbb{I}$, $y \in Y$, satisfy the following conditions: the fibers of g_y are hereditarily indecomposable, and any continuum $K \subset f^{-1}(y)$ either is contained in a fiber of g_y or contains a component of a fiber of g_y . Then $U_A = g^{-1}([0, 1/2))$ and $U_B = g^{-1}((1/2, 1])$ are disjoint neighborhoods of A and B , respectively. Moreover, $H = X \setminus (U_A \cup U_B) = g^{-1}(1/2)$ has the desired properties. \square

Below \mathfrak{In} denotes either the class of indecomposable continua or the class of hereditarily indecomposable compacta.

Proposition 3.9. *Let $X \in \mathfrak{In}$ and $f: X \rightarrow Y$ be a surjective map. Then there exists a compactum $Z \in \mathfrak{In}$ and maps $g: X \rightarrow Z$, $h: Z \rightarrow Y$ such that $w(Z) = w(Y)$. Moreover, we can have $D_{\mathcal{K}}(Z) \leq D_{\mathcal{K}}(X)$.*

Proof. K.P. Hart and E. Pol [13, Proposition 4.2], using elementary substructures and Löwenheim-Skolem theorem, proved a similar factorization theorem for hereditarily indecomposable continua and the covering dimension \dim . The first part of the proposition, concerning indecomposable continua, can be obtained applying Hart-Pol's arguments and the following characterization of indecomposable continua [24, Theorem 1.3]: A continuum X is indecomposable if and only if for any

nonempty open sets U and V in X , there exist closed sets A and B such that $X = A \cup B$, $A \cap B \subset U$, $A \cap V \neq \emptyset \neq B \cap V$. Using this fact and existence of a factorization theorem for extension dimension (see [23], or [26]), we construct an inverse sequence $S = \{Z_n, p_n^m\}$ of compacta and maps $g_n: X \rightarrow Z_n$, $h_n: Z_n \rightarrow Y$ satisfying the following conditions for all $k, n \in \mathbb{N}$:

- $w(Z_n) = w(Y)$;
- Z_{2k+1} is indecomposable (resp., hereditarily indecomposable);
- $D_{\mathcal{K}}(Z_{2k}) \leq D_{\mathcal{K}}(X)$;
- $h_1 \circ g_1 = f$, $g_n = p_n^{n+1} \circ g_{n+1}$ and $h_{n+1} = h_n \circ p_n^{n+1}$.

Then $Z = \lim_{\leftarrow} S$ is a compactum of weight $w(Z) = w(Y)$ and $D_{\mathcal{K}}(Z) \leq D_{\mathcal{K}}(X)$. Moreover, the maps g_n provide a map $g: X \rightarrow Z$ such that $f = h \circ g$, where $h = h_1 \circ p_1$ with p_1 being the projection from Z onto Z_1 . Finally, since Z is the limit of the inverse system $\{Z_{2k+1}, p_{2k+1}^{2m+1}\}$ consisting of compacta from \mathfrak{In} , Z is also from \mathfrak{In} . \square

Corollary 3.10. *A compactum X belongs to the class \mathfrak{In} if and only if X is the limit space of an inverse system of metric compacta from \mathfrak{In} .*

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