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ON A THEOREM OF ARVANITAKIS

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Abstract. Arvanitakis [A simultaneous selection theorem. *Preprint*] recently established a theorem which is a common generalization of Michael's convex selection theorem [Continuous selections I. *Ann. of Math.* (2) **63** (1956), 361–382] and Dugundji's extension theorem [An extension of Tietze's theorem, *Pacific J. Math.* **1** (1951), 353–367]. In this note we provide a short proof of a more general version of Arvanitakis's result.

§1. *Introduction.* Arvanitakis [1] recently established the following result extending both Michael's convex selection theorem [10] and Dugundji's simultaneous extension theorem [6].

THEOREM 1.1 [1]. *Let X be a space with property c, Y a complete metric space and $\Phi : X \to 2^Y$ a lower semi-continuous set-valued map with non-empty values. Then for every locally convex complete linear space E there exists a linear operator $S : C(Y, E) \to C(X, E)$ such that*

$$S(f)(x) \in \overline{\text{conv}}\, f(\Phi(x)) \quad \textit{for all } x \in X \textit{ and } f \in C(Y, E). \tag{1}$$

Furthermore, S is continuous when both $C(Y, E)$ and $C(X, E)$ are equipped with the uniform topology or the topology of uniform convergence on compact sets.

Here, $C(X, E)$ is the set of all continuous maps from X into E (if E is the real line, we write $C(X)$). We also denote by $C_b(X, E)$ the bounded functions from $C(X, E)$. Recall that a set-valued map $\Phi : X \to 2^Y$ is lower semi-continuous if the set $\{x \in X : \Phi(x) \cap U \neq \varnothing\}$ is open in X for any open $U \subset Y$. A space X is said to have property c [1] if X is paracompact and, for any space Y and a map $\phi : X \to Y$, ϕ is continuous if and only if it is continuous on every compact subspace of X. It is easily seen that the last condition is equivalent to X being a k-space (i.e., the topology of X is determined by its compact subsets; see [7]).

We provide a short proof of Theorem 1.1. Here is our slightly more general version of Theorem 1.1.

THEOREM 1.2. *Let X be a paracompact space, Y a complete metric space and $\Phi : X \to 2^Y$ a lower semi-continuous set-valued map with non-empty values. Then:*

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(i) *for every locally convex complete linear space E there exists a linear operator $S_b : C_b(Y, E) \to C_b(X, E)$ satisfying condition (1) such that S_b is continuous with respect to the uniform topology and the topology of uniform convergence on compact sets;*

(ii) *if X is a k-space or E is a Banach space, S_b can be continuously extended (with respect to both types of topologies) to a linear operator $S : C(Y, E) \to C(X, E)$ satisfying (1).*

Our proof of Theorem 1.2 is based on the idea from a result of Repovš *et al* [14] that Michael's zero-dimensional selection theorem yields the convex-valued selection theorem.

§2. *Proof of Theorem 1.2.* Let E be a locally convex linear space. We denote by E^* the set of all continuous linear functionals on E with the topology of uniform convergence on the weakly bounded subsets of E. The second dual E^{**} is the space of continuous functionals on E^* with the topology of uniform convergence on the equicontinuous subsets of E^*. It is well known that the canonical map $E \to E^{**}$ is an embedding; see [15].

We need Banakh's technique [3] concerning barycenters of some probability measures. First of all, for every compact space X let $P(X)$ be the space of all regular probability measures on X endowed with the w^*-topology. Each $\mu \in P(X)$ can also be considered as a continuous linear positive functional on $C(X)$ (the continuous real-valued functions on X with the uniform convergence topology) with $\mu(1_X) = 1$, where 1_X is the constant function on X having value one. Recall that for any $\mu \in P(X)$ there exists a closed non-empty set $\mathrm{supp}(\mu) \subset X$ such that $\mu(g) = \mu(f)$ for any $f, g \in C(X)$ with $f|\mathrm{supp}(\mu) = g|\mathrm{supp}(\mu)$, and $\mathrm{supp}(\mu)$ is the smallest closed subset of X with this property. If X is a Tychonoff space, we consider the following subsets of $P(\beta X)$, where βX is the Čech–Stone compactification of X:

$$P_\beta(X) = \{\mu \in P(\beta X) : \mathrm{supp}(\mu) \subset X\}$$

and

$$\hat{P}(X) = \{\mu \in P(\beta X) : \mu_*(X) = 1\}.$$

Here

$$\mu_*(X) = \sup\{\mu(B) : B \subset X \text{ is a Borel subset of } \beta X\}.$$

Every map $h : M \to E$ generates a map $P_\beta(h) : P_\beta(M) \to P_\beta(E)$ defined by $P_\beta(h)(\mu)(\phi) = \mu(\phi \circ h)$, where $\mu \in P_\beta(M)$ and $\phi \in C_b(E)$. In particular, if $i_M : M \hookrightarrow E$ is the inclusion of M in E, then $P_\beta(i_M)$ is one-to-one and $P_\beta(\delta_x) = \delta_x$ for all $x \in M$ (δ_x is the Dirac measure at the point x). The functors $\hat{P}$ and P_β were introduced in [2, 4], respectively.

Banakh [3] defined barycenters of measures from $\hat{P}(M)$, where M is a weakly bounded subset of some locally convex linear space E. For any such $M \subset E$ there exists an affine map (called a *barycenter map*) $b_M : \hat{P}(M) \to E^{**}$ which is continuous only when M is bounded in E; see [3, Theorem 3.2].

A convex subset $M \subset E$ is called *barycentric* if $b_M(\hat{P}(M)) \subset M$. It was established in [3, Proposition 3.10] that any complete bounded convex subset of E is barycentric. Since for any M we have $P_\beta(M) \subset \hat{P}(M)$, we can apply the Banakh arguments with $\hat{P}(M)$ replaced by $P_\beta(M)$, and this is done in the following proposition.

PROPOSITION 2.1. *Let E be a complete locally convex linear space. Then there exists a not necessarily continuous affine map $b_E : P_\beta(E) \to E$ such that $b_E(\mu) \in \overline{\mathrm{conv}}(\mathrm{supp}(\mu))$ for every $\mu \in P_\beta(E)$. Moreover, if $M \subset E$ is a bounded set then the map $b_E \circ P_\beta(i_M) : P_\beta(M) \to E$ is continuous.*

Proof. We follow the arguments from [3]. For every $\mu \in P_\beta(E)$ we consider the functional $b_E(\mu) : E^* \to \mathbb{R}$, defined by $b_E(\mu)(l) = \mu(l|\mathrm{supp}(\mu))$, $l \in E^*$.

Claim. $b_E(\mu)$ is continuous for all $\mu \in P_\beta(E)$.

Indeed, suppose $\{l_\alpha\} \subset E^*$ is a net in E^* converging to some $l_0 \in E^*$. This means that $\{l_\alpha\}$ is uniformly convergent to l_0 on every weakly bounded subset of E. In particular, $\{l_\alpha\}$ is uniformly convergent to l_0 on $\mathrm{supp}(\mu)$. Consequently, $\{\mu(l_\alpha)\}$ converges to $\mu(l_0)$.

Therefore, $b_E(\mu) \in E^{**}$ for any $\mu \in P_\beta(E)$. On the other hand, since $\mathrm{supp}(\mu) \subset E$ is compact and E is complete, $C(\mu) = \overline{\mathrm{conv}}(\mathrm{supp}(\mu))$ is a compact convex subset of E. Then, according to [3, Proposition 3.10], $C(\mu)$ is barycentric and contains $b_E(\mu)$. So, b_E maps $P_\beta(E)$ into E. The second half of Proposition 2.1 follows from the fact that E is embedded in E^{**} and Theorem 3.2 from [3], which (in our situation) states that the map $b_E \circ P_\beta(i_M) : P_\beta(M) \to E^{**}$ is continuous provided that M is bounded in E. □

The theory of maps between compact spaces admitting averaging operators was developed by Pelczyński [12]. For non-compact spaces we use the following definition [16]: a surjective continuous map $f : X \to Y$ admits an averaging operator with compact supports if there exists an embedding $g : Y \to P_\beta(X)$ such that $\mathrm{supp}(g(y)) \subset f^{-1}(y)$ for all $y \in Y$. Then the regular linear operator $u : C_b(X) \to C_b(Y)$, defined by

$$u(h)(y) = g(y)(h), \quad h \in C_b(X),\ y \in Y, \tag{2}$$

satisfies $u(\phi \circ f) = \phi$ for any $\phi \in C_b(Y)$. Such an operator u is called *averaging for f*.

PROPOSITION 2.2. *Let $f : X \to Y$ be a perfect map admitting an averaging operator with compact supports and E a complete locally convex linear space. Then there exists a linear operator $T_b : C_b(X, E) \to C_b(Y, E)$ such that:*

(i) *$T_b(h)(y) \in \overline{\mathrm{conv}}(h(f^{-1}(y)))$ for all $y \in Y$ and $h \in C_b(X, E)$;*
(ii) *$T_b(\phi \circ f) = \phi$ for any $\phi \in C_b(Y, E)$;*
(iii) *T_b is continuous when both $C_b(X, E)$ and $C_b(Y, E)$ are equipped with the uniform topology or the topology of uniform convergence on compact sets.*

Moreover, if Y is a k-space or E is a Banach space, T_b can be extended to a linear operator $T : C(X, E) \to C(Y, E)$ satisfying conditions (i)–(iii) *with $C_b(X, E)$ and $C_b(Y, E)$ replaced, respectively, by $C(X, E)$ and $C(Y, E)$.*

Proof. A similar statement to the first part was proved in [**16**, Proposition 3.1]. We fix an embedding $g : Y \to P_\beta(X)$ with $\operatorname{supp}(g(y)) \subset f^{-1}(y)$, $y \in Y$. For every $h \in C_b(X, E)$ consider the map

$$T_b(h) : Y \to E, \qquad T_b(h)(y) = b_E(P_\beta(i_{h(X)})(\nu_y)), \tag{3}$$

where $i_{h(X)} : h(X) \hookrightarrow E$ is the inclusion and $\nu_y \in P_\beta(h(X))$ is the measure $P_\beta(h)(g(y))$. According to Proposition 2.1, $T_b(h)$ is continuous (recall that $h(X) \subset E$ is bounded). It also follows from the definition of the map b_E that T_b is linear. Since

$$\operatorname{supp}(g(y)) \subset f^{-1}(y) \quad \text{and} \quad \operatorname{supp}(P_\beta(i_{h(X)})(\nu_y)) \subset h(f^{-1}(y)), \quad y \in Y,$$

it follows that

$$b_E(P_\beta(i_{h(X)})(\nu_y)) \subset \overline{\operatorname{conv}}(h(f^{-1}(y)))$$

(see Proposition 2.1). So, T_b satisfies condition (i). Moreover, $T_b(h)$ belongs to $C_b(Y, E)$ because $T_b(h)(y) \subset \overline{\operatorname{conv}}(h(X))$ for all $y \in Y$. It follows directly from (2) and (3) that T_b satisfies condition (ii). To prove (iii), assume that $K \subset Y$ is compact and let $W_1 = \{\phi \in C_b(Y, E) : \phi(K) \subset V_1\}$, where V_1 is a convex neighborhood of 0 in E. Obviously, W_1 is a neighborhood of the zero function in $C_b(Y, E)$. Take a convex neighborhood V_2 of 0 in E with $\overline{V}_2 \subset V_1$ and let $W_2 = \{h \in C_b(X, E) : h(H) \subset V_2\}$, $H = f^{-1}(K)$. Since H is compact (recall that f is a perfect map), W_2 is a neighborhood of 0 in $C_b(X, E)$. Moreover, for all $y \in Y$ and $h \in W_2$,

$$T_b(h)(y) \subset \overline{\operatorname{conv}}(h(H)) \subset \overline{V}_2 \subset V_1.$$

So, $T_b(W_2) \subset W_1$). This provides continuity of T_b with respect to the topology of uniform convergence on compact sets. Similarly, one can show that T_b is also continuous with respect to the uniform topology.

Assume that Y is a k-space and $h \in C(X, E)$. Then formula (3) provides a map $T(h) : Y \to E$ satisfying conditions (i) and (ii). We need to show that $T(h)$ is continuous on every compact set $L \subset Y$. And this follows from Proposition 2.1 because the set $h(f^{-1}(L)) \subset E$ is compact. So, $T(h)$ is continuous and, obviously, $T(h) = T_b(h)$ for all $h \in C_b(X, E)$. Continuity of T follows from the same arguments we used to prove continuity of T_b.

If E is a Banach space, then every $T(h)$, $h \in C(Y, E)$, is continuous without the requirement that Y be a k-space. Indeed, we fix $y_0 \in Y$ and $h \in C(X, E)$. Let V be a bounded closed neighborhood of $h(f^{-1}(y_0))$ in E. Then $h^{-1}(V)$ is a neighborhood of $f^{-1}(y_0)$ and, since f is a perfect map, there exists a closed neighborhood U of y_0 in Y with $W = f^{-1}(U) \subset h^{-1}(V)$. Then, according to Proposition 2.1, the map $b_E \circ P_\beta(i_V) : P_\beta(V) \to E$ is continuous. On the other hand $P_\beta(h)$ continuously maps $P_\beta(W)$ into $P_\beta(V)$ and $g(U) \subset P_\beta(W)$ is homeomorphic to U (recall that g is an embedding of Y into $P_\beta(X)$). Hence, $T(h)$ is continuous on U. Because U is a neighborhood of y_0 in Y, this implies continuity of $T(h)$ at y_0. □

Proof of Theorem 1.2. Suppose X, Y, Φ and E satisfy the hypotheses of Theorem 1.2. By [**14**] (see also [**13**]), there exists a zero-dimensional paracompact space X_0 and a perfect surjection $f : X_0 \to X$ admitting a regular averaging operator. By Proposition 2.2, there exists a linear operator $T_b : C_b(X_0, E) \to C_b(X, E)$ satisfying conditions (i)–(iii). The map $\tilde{\Phi} : X_0 \to 2^Y$, $\tilde{\Phi}(z) = \overline{\Phi(f(z))}$ is lower semi-continuous with closed non-empty values in Y. So, according to the Michael's zero-dimensional selection theorem [**11**], $\tilde{\Phi}$ has a continuous selection $\theta : X_0 \to Y$. Now, we define the linear operator $S_b : C_b(Y, E) \to C_b(X, E)$ by $S_b(h) = T_b(h \circ \theta)$, $h \in C_b(Y, E)$. Obviously, $\theta(f^{-1}(x)) \subset \overline{\Phi(x)}$ for every $x \in X$. Then, according to (i), for all $h \in C_b(Y, E)$ and $x \in X$ we obtain

$$S_b(h)(x) = T_b(h \circ \theta)(x) \subset \overline{\mathrm{conv}}((h \circ \theta)(f^{-1}(x))) \subset \overline{\mathrm{conv}}(h(\Phi(x))).$$

Continuity of S_b follows from continuity of T_b and the map θ.

If X is a k-space or E is a Banach space, the operator T_b can be extended to a linear operator $T : C(X_0, E) \to C(X, E)$ satisfying conditions (i)–(iii) in Proposition 2.2. Then $S : C(Y, E) \to C(X, E)$, $S(h) = T(h \circ \theta)$ is the required linear operator extending S_b. □

§3. *Remarks.* Let us show first that Theorem 1.2 implies Michael's selection theorem. Assume X is paracompact, Y is a Banach space and $\Phi : X \to 2^Y$ a lower semi-continuous map with closed convex values. Then, by Theorem 1.2 there exists a linear operator $S : C(Y, Y) \to C(X, Y)$ satisfying condition (1). Since the values of Φ are convex and closed, condition (1) yields that $S(\mathrm{id}_Y)(x) \in \Phi(x)$ for all $x \in X$, where id_Y is the identity on Y. Hence, $S(\mathrm{id}_Y)$ is a continuous selection for Φ.

The original Dugundji theorem [**6**] states that if X is a metric space, $A \subset X$ its closed subset and E a locally convex linear space, then there exists a linear operator $S : C(A, E) \to C(X, E)$ such that $S(f)$ extends f for any $f \in C(A, E)$. When both E and A are complete, the Dugundji theorem can be derived from Theorem 1.2. Indeed, let A be a completely metrizable closed subset of a paracompact k-space X and E a complete locally convex linear space. Consider the set-valued map $\Phi : X \to 2^A$, $\Phi(x) = \{x\}$ if $x \in A$ and $\Phi(x) = A$ if $x \notin A$. Let $S : C(A, E) \to C(X, E)$ be a linear operator satisfying (1). Then $S(f)(x) = f(x)$ for all $f \in C(A, E)$ and $x \in A$. So, S is an extension operator. If X is not necessarily a k-space, there exists an extension linear operator $S_b : C_b(A, E) \to C_b(X, E)$.

Heath and Lutzer [**9**, Example 3.3] provided an example of a paracompact X and a closed set $A \subset X$ homeomorphic to the rational numbers such that there is no extension operator from $C(A)$ to $C(X)$. This space is Michael's line, i.e., the real line with topology consisting of all sets of the form $U \cup V$, where U is an open subset of the rational numbers and V is a subset of the irrational numbers.

It is easily seen that this a k-space. So, the assumption in the above result that A be completely metrizable is essential.

The original Dugundji theorem with E complete can be derived from Proposition 2.2 and the well-known fact that every closed subset of a zero-dimensional metric space X is a retract of X; see, for example, [8, Problem 4.1.G]. Indeed, assume X is a metric space and $A \subset X$ its closed subset. By [5], there exist a zero-dimensional metric space X_0 and a perfect surjection $f : X_0 \to X$ admitting an averaging operator. Let $A_0 = f^{-1}(A)$ and $r : X_0 \to A_0$ be a retraction. Define the linear operator $S : C(A, E) \to C(X, E)$ by $S(h) = T(h \circ f \circ r)$, where E is a complete locally convex linear space, $h \in C(A, E)$ and $T : C(X_0, E) \to C(X, E)$ is the operator from Proposition 2.2. It follows from Proposition 2.2(i) that S is an extension operator.

The proof of Theorem 1.2 is based on two main facts: the zero-dimensional Michael selection theorem and the Repovš–Semenov–Shchepin result [14] that each paracompactum is a continuous image of a zero-dimensional one under a perfect map admitting an averaging operator. So, the zero-dimensional Michael selection theorem implies not only the convex-valued selection theorem but also the Dugundji extension theorem. Actually we have the following corollary from Proposition 2.2 (Sel(Φ) denotes all continuous selections for Φ).

COROLLARY 3.1. *Let $f : X \to Y$ be a perfect map admitting an averaging operator with compact supports and E a Banach space. Suppose $\Phi : Y \to 2^E$ is a lower semi-continuous set-valued map with closed convex non-empty values. Then there exists an affine map from* Sel($\Phi \circ$ f) *to* Sel(Φ) *which is continuous when both* Sel($\Phi \circ$ f) *and* Sel(Φ) *are equipped with the uniform topology or the topology of uniform convergence on compact sets.*

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