



Functional extenders and set-valued retractions[☆]

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ABSTRACT

We describe the supports of a class of real-valued maps on $C^*(X)$ introduced by Radul (2009) [6]. Using this description, a characterization of compact-valued retracts of a given space in terms of functional extenders is obtained. For example, if $X \subset Y$, then there exists a continuous compact-valued retraction from Y onto X if and only if there exists a normed weakly additive extender $u: C^*(X) \rightarrow C^*(Y)$ with compact supports preserving \min (resp., \max) and weakly preserving \max (resp., \min). Similar characterizations are obtained for upper (resp., lower) semi-continuous compact-valued retractions. These results provide characterizations of (not necessarily compact) absolute extenders for zero-dimensional spaces, as well as absolute extenders for one-dimensional spaces, involving non-linear functional extenders.

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1. Introduction

All spaces in the paper are assumed to be Tychonoff. Continuous (and bounded) real-valued functions on X are denoted, respectively, by $C(X)$ and $C^*(X)$.

Some purely topological properties have been characterized using functional extenders. For example, *Dugundji spaces* were defined by Pelczynski [1] in terms of linear extension operators between function spaces. Later, Haydon [2] proved that a compactum X is a Dugundji space iff it is an absolute extensor for 0-dimensional spaces, notation $AE(0)$ -spaces. Other results of this type are Shapiro's characterization [3] of compact absolute extenders for one-dimensional spaces (br., $AE(1)$) in terms of extenders between non-negative function spaces, as well as the second authors's characterization [4] of (not necessarily compact) absolute extenders for 0-dimensional spaces. Following this line, the second author [5] obtained recently a characterization of κ -metrizable compacta involving special function extenders.

In this paper we provide another result in this direction by characterizing set-valued retracts of a given space in terms of functional extenders. Recall that a map $u: C^*(X) \rightarrow C^*(Y)$, where X is a subspace of Y , is called an *extender* if $u(f)$ extends f for all $f \in C^*(X)$. Every map $u: C^*(X) \rightarrow C^*(Y)$ generates the maps (called *functionals*) $\mu_y: C^*(X) \rightarrow \mathbb{R}$, $y \in Y$, defined by $\mu_y(f) = u(f)(y)$. We consider functionals which are *normed*, *weakly additive*, *preserving max or min* and *weakly preserved min or max*. This class of functionals was introduced by Radul [6]. A functional $\mu: C^*(X) \rightarrow \mathbb{R}$ is said to be (i) normed, (ii) weakly additive, (iii) preserving max, and (iv) weakly preserving min, if for every $f, g \in C^*(X)$ and every constant function c_X we have: (i) $\mu(1_X) = 1$, (ii) $\mu(f + c_X) = \mu(f) + c$, (iii) $\mu(\max\{f, g\}) = \max\{\mu(f), \mu(g)\}$, and (iv) $\mu(\min\{f, c_X\}) = \min\{\mu(f), c\}$. We say that μ *preserves min* provided μ satisfies equality (iii) with max replaced by min. Similarly, μ *weakly preserves max* if μ satisfies condition (iv) with min replaced by max.

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A map $u : C^*(X) \rightarrow C^*(Y)$ is normed, weakly additive, preserves max and weakly preserves min (resp., preserves min and weakly preserves max) provided u satisfies the corresponding equalities above with the constants c replaced by the constant functions c_Y on Y . Obviously, u has each of these properties if and only if all functionals $\mu_y, y \in Y$, have the same property.

The set of all normed, weakly additive functionals on $C^*(X)$ which preserve max (resp. min) and weakly preserve min (resp., max) is denoted by $\mathfrak{R}_{\max}(X)$ (resp., $\mathfrak{R}_{\min}(X)$). The topology of these two spaces is inherited from the product $\mathbb{R}^{C^*(X)}$. We describe the supports of the functionals from $\mathfrak{R}_{\max}(X) \cup \mathfrak{R}_{\min}(X)$ and introduce the subspaces $\mathfrak{R}_{\max}(X)_c \subset \mathfrak{R}_{\max}(X)$ and $\mathfrak{R}_{\min}(X)_c \subset \mathfrak{R}_{\min}(X)$ consisting of functionals with compact supports. As a result of this description, we obtain a characterization of the functionals from $\mathfrak{R}_{\min}(X)_c$ and $\mathfrak{R}_{\min}(X)$ (Theorem 2.9): $\mu \in \mathfrak{R}_{\min}(X)_c$ (resp., $\mu \in \mathfrak{R}_{\min}(X)$) if and only if there exists a non-empty compact subset $F \subset X$ (resp., $F \subset \beta X$) such that $\mu(f) = \inf\{f(x) : x \in F\}$ (resp., $\mu(f) = \inf\{\beta f(x) : x \in F\}$). A similar characterization holds for the functionals from $\mathfrak{R}_{\max}(X)_c$ and $\mathfrak{R}_{\max}(X)$. Actually, there exists a homeomorphism $\nu_X : \mathfrak{R}_{\max}(X) \rightarrow \mathfrak{R}_{\min}(X)$ such that $\nu_X(\mathfrak{R}_{\max}(X)_c) = \mathfrak{R}_{\min}(X)_c$. For any $\mu \in \mathfrak{R}_{\max}(X)$ the functional $\nu_X(\mu) \in \mathfrak{R}_{\min}(X)$ is defined by $\nu_X(\mu)(f) = -\mu(-f), f \in C^*(X)$.

We also establish that for any Tychonoff space X each of the spaces $\mathfrak{R}_{\max}(X)_c$ and $\mathfrak{R}_{\min}(X)_c$ is homeomorphic to the hyperspace $\text{exp}_c X$ of the non-empty compact subsets of X (see Theorem 3.1) with the Vietoris topology. Proposition 3.2 shows that similar results hold for $\text{exp}_c X$ equipped with the upper or the lower Vietoris topology. When X is compact, $\mathfrak{R}_{\max}(X)_c = \mathfrak{R}_{\max}(X)$ and $\mathfrak{R}_{\min}(X)_c = \mathfrak{R}_{\min}(X)$, so we have a characterization of the hyperspace $\text{exp} X$ which was earlier established by Radul [6].

We also prove (see Theorem 3.3) that if $X \subset Y$, then there exists a continuous compact-valued retraction from Y onto X iff there exists a normed weakly additive extender $u : C^*(X) \rightarrow C^*(Y)$ with compact supports preserving min (resp., max) and weakly preserving max (resp., min). Based on Theorem 3.3, we show (Theorem 3.4) that for any Tychonoff space X the following conditions are equivalent to $X \in AE(1)$: (i) for any C -embedding of X into a space Y there exists an extender $u : C^*(X) \rightarrow C^*(Y)$ with compact supports such that u is normed, weakly additive, preserves min and weakly preserves max; (ii) for any C -embedding of X into a space Y there exists an extender $u : C^*(X) \rightarrow C^*(Y)$ with compact supports such that u is normed, weakly additive, preserves max and weakly preserves min; (iii) for any C -embedding of X into a space Y there exists a map $\theta : Y \rightarrow \mathfrak{R}_{\min}(X)_c$ such that $\theta(x) = \delta_x$ for all x in X ; and (iv) for any C -embedding of X into a space Y there exists a map $\theta : Y \rightarrow \mathfrak{R}_{\max}(X)_c$ such that $\theta(x) = \delta_x$ for all x in X .

In Section 4 we establish an analog of Theorem 3.3 concerning upper (resp., lower) semi-continuous compact-valued retracts. For example, Theorem 4.1 states that the existence of an upper semi-continuous compact-valued retraction $r : Y \rightarrow X$ is equivalent to each of the following conditions: (i) there exists a normed weakly additive extender $u : C^*(X) \rightarrow C_{lsc}^*(Y)$ with compact supports preserving min and weakly preserving max; and (ii) there exists a normed weakly additive extender $u : C^*(X) \rightarrow C_{usc}^*(Y)$ with compact supports preserving max and weakly preserving min. Here, $C_{lsc}^*(Y)$ (resp., $C_{usc}^*(Y)$) denotes all bounded lower (resp., upper) semi-continuous real-valued functions on Y . Theorem 4.1 implies another characterization of $AE(0)$ -spaces in terms of non-linear extenders. In the last section we introduce the class of *Zarichnyi spaces* and raise some questions.

Finally, let us mention that the results in Section 2 are taken from the first author's M.Sc. thesis [7] which was written under the supervision of the second author.

2. Functionals from $\mathfrak{R}_{\max}(X)$ and $\mathfrak{R}_{\min}(X)$ and their supports

Let $\mathfrak{R}_{\max}(X)$ (resp., $\mathfrak{R}_{\min}(X)$) be the space of all normed, weakly additive functionals on $C^*(X)$ which preserve max and weakly preserve min (resp., preserve min and weakly preserve max).

In this section we describe the supports of the functionals from sets $\mathfrak{R}_{\max}(X)$ and $\mathfrak{R}_{\min}(X)$. For any functional $\mu : C^*(X) \rightarrow \mathbb{R}$ we define its support $S(\mu)$ to be the following subset of the Čech–Stone compactification βX of X (see [8] for a similar definition):

Definition 2.1. $S(\mu)$ is the set of all $x \in \beta X$ such that for every neighborhood O_x of x in βX there exist $f, g \in C^*(X)$ with $\beta f|(\beta X \setminus O_x) = \beta g|(\beta X \setminus O_x)$ and $\mu(f) \neq \mu(g)$.

Here, $\beta f : \beta X \rightarrow \mathbb{R}$ is the Čech–Stone extension of f and $\beta f|(\beta X \setminus O_x)$ denotes its restriction on the set $\beta X \setminus O_x$. Obviously, $S(\mu)$ is a closed subset of βX (possibly empty). If $\emptyset \neq S(\mu) \subset X$, we say that μ has a compact support. Identifying $C^*(X)$ with $C(\beta X)$, any functional μ on $C^*(X)$ can be considered as a function $\mu : C(\beta X) \rightarrow \mathbb{R}$.

For any μ let \mathcal{A}_μ be the family of all closed non-empty sets $A \subset \beta X$ such that for any $f, g \in C(\beta X)$ we have $\mu(f) = \mu(g)$ provided $f|A = g|A$. It is clear that $\beta X \in \mathcal{A}_\mu$, so $\mathcal{A}_\mu \neq \emptyset$.

A functional μ on $C(\beta X)$ is called *monotone* if $f \leq g$ implies $\mu(f) \leq \mu(g)$. Obviously, every functional preserving max or min is monotone.

Lemma 2.2. Suppose that $\mu : C(\beta X) \rightarrow \mathbb{R}$ is a normed functional with $\mu(0_X) = 0$. Then \mathcal{A}_μ is closed with respect to finite intersections and $S(\mu) = \bigcap \{A : A \in \mathcal{A}_\mu\}$. Moreover, $S(\mu) \in \mathcal{A}_\mu$ provided μ is weakly additive and monotone.

Proof. Suppose $A, B \in \mathcal{A}_\mu$ with $A \cap B = \emptyset$. There exists $f \in C(\beta X)$ such that $f(A) = 1$ and $f(B) = 0$. So, $f|_A = 1_{\beta X}|_A$ and $f|_B = 0_{\beta X}|_B$. This implies $\mu(f) = \mu(1_{\beta X}) = 1$ and $\mu(f) = \mu(0_{\beta X}) = 0$, a contradiction. So, $A \cap B \neq \emptyset$ for any two elements of \mathcal{A}_μ .

We claim that $A \cap B \in \mathcal{A}_\mu$. Indeed, suppose f and g are two functions from $C(\beta X)$ such that $f|(A \cap B) = g|(A \cap B)$. Consider the function $h \in C(A \cup B)$ defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ provided $x \in B$. Next, extend h to a function $\bar{h} \in C(\beta X)$. Since $f|_A = \bar{h}|_A$, $\mu(f) = \mu(\bar{h})$. Similarly, $g|_B = \bar{h}|_B$ implies $\mu(f) = \mu(\bar{h})$. Hence, $\mu(f) = \mu(g)$.

Therefore, by induction, $\bigcap_{i=1}^{i=k} A_i \in \mathcal{A}_\mu$ for any finitely many $A_1, \dots, A_k \in \mathcal{A}_\mu$.

To prove the equality $S(\mu) = \bigcap \{A : A \in \mathcal{A}_\mu\}$, suppose $x \notin S(\mu)$. Then, there exists a neighborhood $O_x \subset \beta X$ of x such that $\mu(f) = \mu(g)$ for every $f, g \in C(\beta X)$ with $f|(\beta X \setminus O_x) = g|(\beta X \setminus O_x)$ (we can assume that $\beta X \setminus O_x \neq \emptyset$ by choosing a smaller O_x). Consequently, $\beta X \setminus O_x \in \mathcal{A}_\mu$ and $x \notin A_\mu = \bigcap \{A : A \in \mathcal{A}_\mu\}$. If $x \notin A_\mu$, there exists $A \in \mathcal{A}_\mu$ with $x \notin A$. Then $O_x = \beta X \setminus A$ is a neighborhood of x such that $\mu(f) = \mu(g)$ for all $f, g \in C(\beta X)$ with $f|(\beta X \setminus O_x) = g|(\beta X \setminus O_x)$. Hence, $x \notin S(\mu)$.

Finally, suppose μ is weakly additive, and let $f|S(\mu) = g|S(\mu)$ for some $f, g \in C(\beta X)$. Then, for every $\epsilon > 0$ the set $U_\epsilon = \{x \in \beta X : |f(x) - g(x)| < \epsilon\}$ is a neighborhood of $S(\mu)$. So, we can find finitely many $B_1, \dots, B_j \in \mathcal{A}_\mu$ such that $S(\mu) \subset B_0 = \bigcap_{i=1}^{i=j} B_i \subset U_\epsilon$. Next, there exists a function $h \in C(\beta X)$ with $h|B_0 = f|B_0$ and $g(x) - \epsilon \leq h(x) \leq g(x) + \epsilon$ for all $x \in \beta X$. Indeed, consider the lower semi-continuous convex-valued map $\Phi : \beta X \rightarrow \mathbb{R}$, defined by $\Phi(x) = f(x)$ for $x \in B_0$ and $\Phi(x)$ to be the interval $[g(x) - \epsilon, g(x) + \epsilon]$ for $x \notin B_0$. According to Michael's selection theorem [9], Φ admits a continuous selection $h \in C(\beta X)$. Since $B_0 \in \mathcal{A}_\mu$, $\mu(f) = \mu(h)$. On the other hand, the inequalities $g - \epsilon \leq h \leq g + \epsilon$ imply $\mu(g) - \epsilon \leq \mu(h) \leq \mu(g) + \epsilon$ (recall that μ is weakly additive and monotone). Hence, $|\mu(f) - \mu(g)| < \epsilon$ for every $\epsilon > 0$ which yields $\mu(f) = \mu(g)$. \square

Corollary 2.3. $S(\mu) \in \mathcal{A}_\mu$ for any normed and weakly additive monotone functional μ .

Proof. This follows from Lemma 2.2 because $1 = \mu(0_X + 1_X) = \mu(0_X) + 1$ implies $\mu(0_X) = 0$. \square

For any functional μ on $C(\beta X)$ we denote by Λ_μ the family of all closed subsets $A \subset \beta X$ satisfying the following condition: if $B \subset \beta X$ is a closed disjoint set from A , then there exists $g \in C(\beta X)$ such that $\mu(g) = 0$, $g(A) \subset (-\infty, 0]$ and $g(B) \subset (0, \infty)$. Without loss of generality, we may assume that $\beta X \in \Lambda_\mu$.

Lemma 2.4. Let μ be a normed, monotone, weakly additive functional weakly preserving max and min. Then, $\mu(f) = \inf_{A \in \Lambda_\mu} \sup_{x \in A} f(x)$ for any $f \in C(\beta X)$.

Proof. This lemma follows from [10, Theorem 4.2] and the proof is omitted. \square

Lemma 2.5. Let μ be a normed, monotone, weakly additive functional weakly preserving max and min. Then $A \in \Lambda_\mu$ if and only if $\mu(f) \neq 0$ for all $f \in C(O_A)$ and all O_A .

Proof. Suppose $A_0 \in \Lambda_\mu$ and $f \in C(O_{A_0})$ for some O_{A_0} . Since $\mu(f) = \inf_{A \in \Lambda_\mu} \sup_{x \in A} f(x)$ (see Lemma 2.4) and $f(A_0) = -1$, $\mu(f) \neq 0$.

Now, suppose $A \subset \beta X$ is closed such that $\mu(f) \neq 0$ for all $f \in C(O_A)$ and all O_A . To show that $A \in \Lambda_\mu$, take $B \subset \beta X$ to be a closed set disjoint with A . Let $O_A = X \setminus B$ and $g \in C(O_A)$. Then $g(A) = -1$, $g(B) = 0$ and $\mu(g) \neq 0$. Since $-1_{\beta X} \leq g \leq 0_{\beta X}$, we have $-1 \leq \mu(g) < 0$. Hence, $h(x) = -\mu(g) > 0$ for all $x \in B$ and $h(x) = g(x) - \mu(g) = -1 - \mu(g) \leq 0$ for all $x \in A$, where $h = g - \mu(g)$. Therefore, $A \in \Lambda_\mu$. \square

Lemma 2.6. Let μ be a normed, monotone, weakly additive functional weakly preserving max and min. Then $s(\mu) = \{x \in \beta X : \{x\} \in \Lambda_\mu\}$ is a closed subset of $S(\mu)$. Moreover, $A \cap s(\mu) \neq \emptyset$ for all $A \in \Lambda_\mu$ if μ preserves min.

Proof. Let us show first that $s(\mu) \subset S(\mu)$. Indeed, otherwise there exists $x \in s(\mu) \setminus S(\mu)$, and take any $f \in C(O_x)$, where $O_x = \beta X \setminus S(\mu)$. Since $\{x\} \in \Lambda_\mu$, by Lemma 2.5, $\mu(f) \neq 0$. On the other hand, $f \in C(O_x)$ implies $f(x) = 0$ for all $x \in S(\mu)$. So, $f|S(\mu) = 0_{\beta X}|S(\mu)$ which yields $\mu(f) = 0$ (see Corollary 2.3).

Next, let $x \notin s(\mu)$. According to Lemma 2.5, there exists a neighborhood O_x of x and $f \in C(O_x)$ with $\mu(f) = 0$. Hence, $f(\bar{U}) = -1$ and $f(\beta X \setminus O_x) = 0$ for some open $U \subset \beta X$ satisfying $x \in U \subset \bar{U} \subset O_x$. Consequently, $U \cap s(\mu) = \emptyset$ because $f \in C(O_y)$ for all $y \in U$, where $O_y = O_x$.

Finally, let μ preserve min, and suppose $A \cap s(\mu) = \emptyset$ for some $A \in \Lambda_\mu$. Then, for each $x \in A$ there exist neighborhoods O_x and U_x of x and $f \in C(O_x)$ such that $x \in U_x \subset \bar{U}_x \subset O_x$, $f_x \in C(O_x)$, $f_x(\bar{U}_x) = -1$, $f_x(\beta X \setminus O_x) = 0$ and $\mu(f_x) = 0$. Take finitely many points $x_1, \dots, x_k \in A$ with $A \subset U = \bigcup_{i=1}^{i=k} U_{x_i}$. Let $f = \min\{f_{x_i} : i \leq k\}$ and $O_A = \bigcup_{i=1}^{i=k} O_{x_i}$. Then $\mu(f) = \min\{\mu(f_{x_i}) : i \leq k\} = 0$. On the other hand, we have $A \subset U \subset \bar{U} \subset O_A$, $f(\bar{U}) = -1$ and $f(x) = 0$ for all $x \notin O_A$. So, $f \in C(O_A)$ which, according to Lemma 2.5, implies $\mu(f) \neq 0$. This contradiction completes the proof. \square

Corollary 2.7. Let μ be a normed, weakly additive functional weakly preserving max and preserving min. Then $s(\mu) = S(\mu)$ and $\mu(f) = \inf\{f(x) : x \in S(\mu)\}$ for all $f \in C(\beta X)$.

Proof. We show first that $\inf_{A \in \Lambda_\mu} \sup_{x \in A} f(x) = \inf\{f(x) : x \in s(\mu)\}$ for any $f \in C(\beta X)$. Indeed, since every $x \in s(\mu)$ belongs to Λ_μ , we have $\inf_{A \in \Lambda_\mu} \sup_{x \in A} f(x) \leq \inf\{f(x) : x \in s(\mu)\}$. The reverse inequality follows from the fact that every $A \in \Lambda_\mu$ intersect $s(\mu)$ (Lemma 2.6). Hence, by Lemma 2.4, $\mu(f) = \inf_{A \in \Lambda_\mu} \sup_{x \in A} f(x) = \inf\{f(x) : x \in s(\mu)\}$.

To complete the proof, it suffices to show that $s(\mu) = S(\mu)$. Suppose $f|_{s(\mu)} = g|_{s(\mu)}$ for some $f, g \in C(\beta X)$. Then, $\inf\{f(x) : x \in s(\mu)\} = \inf\{g(x) : x \in s(\mu)\}$, so $\mu(f) = \mu(g)$. This means that $s(\mu) \in \mathcal{A}_\mu$. But, $S(\mu)$ is the smallest element of \mathcal{A}_μ (Lemma 2.2). Therefore, $s(\mu) = S(\mu)$. \square

Concerning the functionals $\mu \in \mathfrak{R}_{\max}(X)$, their supports have the following property.

Proposition 2.8. *Let μ be a normed, weakly additive functional weakly preserving min and preserving max. Then $\mu(f) = \sup\{f(x) : x \in S(\mu)\}$ for all $f \in C(\beta X)$.*

Proof. The proof follows from the fact that the map $\nu_X: \mathfrak{R}_{\max}(X) \rightarrow \mathfrak{R}_{\min}(X)$, $\nu_X(\mu)(f) = -\mu(-f)$, is a homeomorphism. Indeed, if $\mu \in \mathfrak{R}_{\max}(X)$, then the functional $\nu = \nu_X(\mu) \in \mathfrak{R}_{\min}(X)$ and, according to Corollary 2.7, $\nu(f) = \inf\{f(x) : x \in S(\nu)\}$ for any $f \in C(\beta X)$. Consequently, $\mu(f) = \sup\{f(x) : x \in S(\nu)\}$. The last equality implies that $S(\nu)$ is the support of μ , which completes the proof. \square

We complete this section with the following characterization of the functionals from $\mathfrak{R}_{\min}(X) \cup \mathfrak{R}_{\max}(X)$.

Theorem 2.9. *Let X be a Tychonoff space and μ a functional on $C^*(X)$. Then we have:*

- (i) $\mu \in \mathfrak{R}_{\min}(X)_c$ (resp., $\mu \in \mathfrak{R}_{\min}(X)$) if and only if there exists a non-empty compact set $F \subset X$ (resp., $F \subset \beta X$) such that $F = S(\mu)$ and $\mu(f) = \inf\{f(x) : x \in F\}$ for all $f \in C(\beta X)$;
- (ii) $\mu \in \mathfrak{R}_{\max}(X)_c$ (resp., $\mu \in \mathfrak{R}_{\max}(X)$) if and only if there exists a non-empty compact set $F \subset X$ (resp., $F \subset \beta X$) such that $F = S(\mu)$ and $\mu(f) = \sup\{f(x) : x \in F\}$ for all $f \in C(\beta X)$.

Proof. We are going to prove the first item only, the proof of the second one is similar. If $\mu \in \mathfrak{R}_{\min}(X)_c$ (resp., $\mu \in \mathfrak{R}_{\min}(X)$), then $F = S(\mu)$ is a non-empty compact subset of X (resp., βX) and, by Corollary 2.7, $\mu(f) = \inf\{f(x) : x \in F\}$, $f \in C(\beta X)$. Suppose there exists a compact $F \subset X$ (resp., $F \subset \beta X$) with $\mu(f) = \inf\{f(x) : x \in F\}$ for all $f \in C(\beta X)$. It is easily seen that $\mu \in \mathfrak{R}_{\min}(X)$ and Corollary 2.7 implies $F = S(\mu)$. Moreover, $\mu \in \mathfrak{R}_{\min}(X)_c$ provided $F \subset X$. \square

3. Set-valued continuous retractions and AE(1)-spaces

Below, by $\exp \beta X$ we denote all closed non-empty subsets of βX with the Vietoris topology, and by $\exp_c X$ the subspace of $\exp \beta X$ consisting of all compact subsets of X .

Recall that a set-valued map $\varphi: X \rightarrow Y$ between two spaces is called lower (resp., upper) semi-continuous if the set $\{x \in X : r(x) \cap U \neq \emptyset\}$ (resp., $\{x \in X : r(x) \subset U\}$) is open in X for every open $U \subset Y$. When φ is both lower and upper semi-continuous, then it is called continuous. We also say that φ is compact-valued if $\varphi(x)$ is a non-empty compactum for each $x \in X$.

Theorem 3.1. *Let X be a Tychonoff space. Then*

- (i) Each of the spaces $\mathfrak{R}_{\min}(X)$ and $\mathfrak{R}_{\max}(X)$ is homeomorphic to $\exp \beta X$;
- (ii) Each of the spaces $\mathfrak{R}_{\min}(X)_c$ and $\mathfrak{R}_{\max}(X)_c$ is homeomorphic to $\exp_c X$.

Proof. We are going to prove only item (i), the proof of (ii) is similar. First, observe that $\mathfrak{R}_{\min}(X)$ is a compact subspace of the product $\mathbb{R}^{C^*(X)}$. Indeed, Theorem 2.9 implies that $\mathfrak{R}_{\min}(X)$ is a subset of the compact product $K = \prod \{[a_f, b_f] : f \in C^*(X)\}$, where $a_f = \inf\{f(x) : x \in X\}$ and $b_f = \sup\{f(x) : x \in X\}$. Moreover, if $\{\mu_\alpha\}$ is a net in $\mathfrak{R}_{\min}(X)$ converging to some $\mu \in K$, then $\{\mu_\alpha(f)\}$ converges to $\mu(f)$ for all $f \in C^*(X)$. This yields that $\mu \in \mathfrak{R}_{\min}(X)$.

Consider the set-valued map $\Phi: \mathfrak{R}_{\min}(X) \rightarrow \beta X$, $\Phi(\mu) = S(\mu)$. Obviously, $\Phi(\delta_x) = \{x\}$ for all $x \in \beta X$. Next, we are going to show that the map Φ is lower semi-continuous. Suppose $S(\mu_0) \cap U \neq \emptyset$ for some $\mu_0 \in \mathfrak{R}_{\min}(X)$ and open $U \subset \beta X$. Take $x_0 \in S(\mu_0) \cap U$ and a function $g \in C(\beta X)$ with $g(x_0) = -1$ and $g(\beta X \setminus U) = 1$. Then, by Corollary 2.7, $\mu_0(g) = \inf\{g(x) : x \in S(\mu_0)\} \leq -1$. Hence, the set $V = \{\mu \in \mathfrak{R}_{\min}(X) : \mu(g) < 0\}$ is a neighborhood of μ_0 in $\mathfrak{R}_{\min}(X)$. For every $\mu \in V$ we have $S(\mu) \cap U \neq \emptyset$ (otherwise $S(\mu) \subset \beta X \setminus U$ and $\mu(g) = \inf\{g(x) : x \in S(\mu)\} = 1$, a contradiction). Therefore, Φ is lower semi-continuous.

Assume now that $\mu_0 \in \mathfrak{R}_{\min}(X)$ and $S(\mu_0) \subset U$ with $U \subset \beta X$ being open. Choose $h \in C(\beta X)$ such that $h(S(\mu_0)) = 0$ and $h(\beta X \setminus U) = -1$. Then $W = \{\mu \in \mathfrak{R}_{\min}(X) : \mu(h) > -1/2\}$ is a neighborhood of μ_0 and $S(\mu) \subset U$ for all $\mu \in W$. So, Φ is upper semi-continuous.

Since Φ is both lower semi-continuous and upper semi-continuous, it is continuous considered as a single-valued map from $\mathfrak{R}_{\min}(X)$ into $\exp \beta X$. Φ is also one-to-one. Indeed, if $\Phi(\mu_1) = \Phi(\mu_2)$, then $\mu_1(f) = \mu_2(f)$ (see Corollary 2.7) for every $f \in C(\beta X)$. So, $\mu_1 = \mu_2$.

Finally, let us show that Φ is surjective. For every $F \in \exp \beta X$ we define the functional $\mu_F: C^*(X) \rightarrow \mathbb{R}$, $\mu_F(f) = \inf\{f(x) : x \in F\}$. It is easily seen that $\mu_F \in \mathfrak{R}_{\min}(X)$. It suffices to prove that $S(\mu_F) = F$. If there exists $a \in S(\mu_F) \setminus F$, we take $g \in C(\beta X)$ such that $g(a) = 0$ and $g(F) = 1$. The last equality implies $\mu_F(g) = 1$. On the other hand, by Corollary 2.7, $\mu_F(g) \leq 0$. Similarly, we can obtain a contradiction if $F \setminus S(\mu_F) \neq \emptyset$. Hence, Φ is a homeomorphism between $\mathfrak{R}_{\min}(X)$ and $\exp \beta X$. Since $S(\mu) \subset X$ for all $\mu \in \mathfrak{R}_{\min}(X)_c$, it also follows that Φ is a homeomorphism from $\mathfrak{R}_{\min}(X)_c$ onto $\exp_c X$. \square

We denote by $\mathfrak{R}_{\min}^{lsc}(X)_c$ (resp., $\mathfrak{R}_{\max}^{lsc}(X)_c$) the set $\mathfrak{R}_{\min}(X)_c$ (resp., $\mathfrak{R}_{\max}(X)_c$) with the topology generated by the family $\{\mu : \mu(f_i) > a_i, i = 1, \dots, k\}$, where $f_i \in C^*(X)$ and $a_i \in \mathbb{R}$. Similarly, $\mathfrak{R}_{\min}^{usc}(X)_c$ (resp., $\mathfrak{R}_{\max}^{usc}(X)_c$) is the set $\mathfrak{R}_{\min}(X)_c$ (resp., $\mathfrak{R}_{\max}(X)_c$) with the topology generated by the family $\{\mu : \mu(f_i) < a_i, i = 1, \dots, k\}$. Moreover, $\exp_c^+ X$ and $\exp_c^- X$ denote the set $\exp_c X$ with the upper (resp., lower) Vietoris topology. Recall that the upper (resp., lower) Vietoris topology on $\exp_c X$ is the topology generated by the families $\{F \in \exp_c X : F \subset U\}$ (resp., $\{F \in \exp_c X : F \cap U \neq \emptyset\}$), where $U \subset X$ is open.

Following the proof of **Theorem 3.1**, one can establish the following proposition.

Proposition 3.2. *Let X be a Tychonoff space. Then*

- (i) *Each of the spaces $\mathfrak{R}_{\min}^{lsc}(X)_c$ and $\mathfrak{R}_{\max}^{usc}(X)_c$ is homeomorphic to $\exp_c^+ X$.*
- (ii) *Each of the spaces $\mathfrak{R}_{\min}^{usc}(X)_c$ and $\mathfrak{R}_{\max}^{lsc}(X)_c$ is homeomorphic to $\exp_c^- X$.*

The next results provide a connection between continuous set-valued retractions and extenders (recall that a continuous set-valued map means a set-valued map which is both lower and upper semi-continuous).

Theorem 3.3. *Let X be a subspace of Y . Then the following conditions are equivalent:*

- (i) *There exists a continuous compact-valued map $r : Y \rightarrow \beta X$ with $r(x) = \{x\}$ for all $x \in X$;*
- (ii) *There exists an extender $u : C^*(X) \rightarrow C^*(Y)$ which is normed, weakly additive, preserves min and weakly preserves max;*
- (iii) *There exists an extender $u : C^*(X) \rightarrow C^*(Y)$ which is normed, weakly additive, preserves max and weakly preserves min.*

Moreover, there exists a continuous compact-valued retraction $r : Y \rightarrow X$ iff the extenders from (ii) and (iii) have compact supports.

Proof. Suppose $r : Y \rightarrow \beta X$ is a continuous compact-valued map with $r(x) = \{x\}$ for all $x \in X$. Then, for every $f \in C^*(X)$ the equality $u(f)(y) = \inf\{\beta f(x) : x \in r(y)\}$, $y \in Y$, defines a function $u(f) : Y \rightarrow \mathbb{R}$. Since r is both lower and upper semi-continuous, each $u(f)$, $f \in C^*(X)$, is continuous. Moreover, $u(f)(x) = f(x)$ provided $x \in X$. So, u is an extender, and one can check that it is normed, weakly additive, preserves min and weakly preserves max. Hence, (i) implies (ii). The implication (i) \Rightarrow (iii) is similar, we define the extender u by $u(f)(y) = \sup\{\beta f(x) : x \in r(y)\}$, where $f \in C^*(X)$ and $y \in Y$.

It is easily seen that if $r : Y \rightarrow X$ is a continuous compact-valued retraction, then $u(f)(y) = \inf\{f(x) : x \in r(y)\}$ (resp., $u(f)(y) = \sup\{f(x) : x \in r(y)\}$) defines a normed and weakly additive extender $u : C^*(X) \rightarrow C^*(Y)$ with compact supports such that u preserves min (resp., max) and weakly preserves max (resp., min).

To prove the implication (ii) \Rightarrow (i) (resp., (iii) \Rightarrow (i)), let $\theta : Y \rightarrow \mathfrak{R}_{\min}(X)$ (resp., $\theta : Y \rightarrow \mathfrak{R}_{\max}(X)$) be the map $\theta(y) = \mu_y$. Here $\mu_y : C^*(X) \rightarrow \mathbb{R}$ are the functionals generated by the extender u , i.e., $\mu_y(f) = u(f)(y)$ for all $f \in C^*(X)$ and $y \in Y$. It follows from the last equality that θ is continuous. Moreover, the compact-valued map assigning to each μ_y its support $S(\mu_y)$ is lower and upper semi-continuous (see the proof of **Theorem 3.1**). So, $r(y) = S(\mu_y)$ defines a continuous compact-valued map from Y into βX . Since $\mu_x = \delta_x$ for any $x \in X$, $r(x)$ is the point-set x .

If the extender u from items (ii) and (iii) has compact supports, then $S(\mu_y) \subset X$ for all $y \in Y$. Hence, in this case r is a continuous compact-valued retraction from Y onto X . \square

We are now in a position to prove the characterization of $AE(1)$ -spaces mentioned in the introduction. We recall the definition of absolute extendors for n -dimensional spaces (br., $AE(n)$) in the class of Tychonoff spaces (see [11]): $X \in AE(n)$ if any map $g : Z_0 \rightarrow X$, where Z_0 is a subset of a space Z with $\dim Z \leq n$ and $C(g)(C(X)) \subset C(Z)|Z_0$, can be extended to a map $\bar{g} : Z \rightarrow X$. Here, $C(g)(C(X)) \subset C(Z)|Z_0$ means that for every function $h \in C(X)$ the composition $h \circ g$ is extendible over Z . In particular, this is true if Z is normal and $Z_0 \subset Z$ closed.

We say that a set $A \subset X$ is C -embedded in X if every function f from $C(A)$ can be extended to a function $h \in C(X)$.

Theorem 3.4. *For any space X the following conditions are equivalent:*

- (i) $X \in AE(1)$;
- (ii) *For any C -embedding of X into a space Y there exists an extender $u : C^*(X) \rightarrow C^*(Y)$ with compact supports such that u is normed, weakly additive, preserves min and weakly preserves max;*
- (iii) *For any C -embedding of X into a space Y there exists an extender $u : C^*(X) \rightarrow C^*(Y)$ with compact supports such that u is normed, weakly additive, preserves max and weakly preserves min;*
- (iv) *For any C -embedding of X into a space Y there exists a continuous map $\theta : Y \rightarrow \mathfrak{R}_{\min}(X)_c$ such that $\theta(x) = \delta_x$ for all x in X ;*
- (v) *For any C -embedding of X into a space Y there exists a continuous map $\theta : Y \rightarrow \mathfrak{R}_{\max}(X)_c$ such that $\theta(x) = \delta_x$ for all x in X .*

Proof. Observe that (ii) \Leftrightarrow (iii) and (iv) \Leftrightarrow (v). The first equivalence follows from the fact that an operator $u : C^*(X) \rightarrow C^*(Y)$ is a normed, weakly additive extender which preserves max and weakly preserves min iff the operator $v : C^*(X) \rightarrow C^*(Y)$, $v(f) = -u(-f)$, is a normed, weakly additive extender which preserves min and weakly preserves max. Concerning the second equivalence, observe that a map $\theta : Y \rightarrow \mathfrak{R}_{\max}(X)_c$ is continuous with $\theta(x) = \delta_x$ for all x in X if and only if the map $\theta' : Y \rightarrow \mathfrak{R}_{\min}(X)_c$, $\theta'(y) = \nu_X(\theta(y))$, is continuous and $\theta'(x) = \delta_x$ for all x in X . Here, $\nu_X : \mathfrak{R}_{\max}(X)_c \rightarrow \mathfrak{R}_{\min}(X)_c$ is the homeomorphism considered above.

So, it suffices to prove the implications (i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (i). Suppose $X \in AE(1)$ and X is C -embedded in a space Y . Considering Y as a C -embedded subset of the product \mathbb{R}^τ for some cardinal τ , we may assume that $Y = \mathbb{R}^\tau$. Following the proof of implication (i) \Rightarrow (ii) of Theorem 3.9 from [12], we embed \mathbb{R}^τ as a dense subset of \mathbb{I}^τ ($\mathbb{I} = [0, 1]$) and let $g: T \rightarrow \mathbb{I}^\tau$ be an open monotone surjection with T being an $AE(0)$ -compactum of dimension one (such T exists by [13, Theorem 9]). Since g is open, $K = g^{-1}(\mathbb{R}^\tau)$ is dense in T . Hence, by [14, Corollary 7], $\dim K = \dim T = 1$. Let $K_0 = g^{-1}(X)$ and $g_0 = g|_{K_0}$. Because X is C -embedded in \mathbb{R}^τ , it is easily seen that $C(g_0)(C(X)) \subset C(K)|_{K_0}$. Therefore, the map g_0 can be extended to a map $h: K \rightarrow X$ (recall that $X \in AE(1)$). Then the compact-valued map $r: \mathbb{R}^\tau \rightarrow X$, defined by $r(y) = h(g^{-1}(y))$, is both lower semi-continuous (because g is open) and upper semi-continuous (because g is closed). Moreover, since each fiber $g^{-1}(y)$, $y \in \mathbb{R}^\tau$, is a continuum, so are the values of r . Finally, observe that $r(x) = \{x\}$ for all $x \in X$. Then, according to Theorem 3.3, there exists a normed weakly additive extender $u: C^*(X) \rightarrow C^*(\mathbb{R}^\tau)$ with compact supports such that u preserves min and weakly preserves max. This completes the proof of the implication (i) \Rightarrow (ii).

The implication (ii) \Rightarrow (iv) follows from the proof of Theorem 3.3. Indeed, the map $\theta: Y \rightarrow \mathfrak{R}_{\min}(X)_c$, $\theta(y) = \mu_y$, is the required one.

To prove the last implication (iv) \Rightarrow (i), consider X as a C -embedded subset of some \mathbb{R}^τ , and let $\theta: \mathbb{R}^\tau \rightarrow \mathfrak{R}_{\min}(X)_c$ be a continuous map with $\theta(x) = \delta_x$, $x \in X$. It was established in the proof of Theorem 3.3 that the equality $r(y) = S(\theta(y))$, $y \in \mathbb{R}^\tau$, defines a continuous compact-valued retraction from \mathbb{R}^τ onto X . For every $y \in \mathbb{R}^\tau$, let $F(y)$ be the closure in \mathbb{R}^τ of the convex hull $\text{conv}(r(y))$. Since $r(y)$ is compact, $F(y)$ is a convex compact subset of \mathbb{R}^τ . Finally, let $\varphi(y) = r(F(y))$, $y \in \mathbb{R}^\tau$. It is easily seen that $\varphi: \mathbb{R}^\tau \rightarrow X$ is upper semi-continuous. Since r is continuous and compact-valued, each $\varphi(y)$ is a continuum. Hence, φ is an upper semi-continuous continuum-valued retraction from \mathbb{R}^τ onto X . Therefore, by [12, Theorem 3.9(ii)], $X \in AE(1)$. \square

Corollary 3.5. *A space X is an $AE(1)$ if and only if for every C -embedding of X into a space Y there exists an extender $u: C(X) \rightarrow C(Y)$ with compact supports such that u is normed, weakly additive, preserves min and weakly preserves max (resp., preserves max and weakly preserves min).*

Proof. Suppose $X \in AE(1)$. As in the proof of Theorem 3.4 we can assume that Y is a subset of \mathbb{R}^τ for some τ and there exists a continuous compact-valued retraction $r: \mathbb{R}^\tau \rightarrow X$. Then $u(f)(y) = \inf\{f(x) : x \in r(y)\}$ (resp., $u(f)(y) = \sup\{f(x) : x \in r(y)\}$) defines the required extender $u: C(X) \rightarrow C(\mathbb{R}^\tau)$. The other direction follows directly from Theorem 3.4 because for any monotone normed and weakly additive extender $u: C(X) \rightarrow C(\mathbb{R}^\tau)$ we have $u(C^*(X)) \subset C^*(\mathbb{R}^\tau)$. \square

4. Upper and lower semi-continuous retractions

In this section we describe a connection between upper (resp., lower) semi-continuous retractions and functional extenders. Recall that a function $f: X \rightarrow \mathbb{R}$ is called lower (resp., upper) semi-continuous if $f^{-1}(a, \infty)$ (resp., $f^{-1}(-\infty, a)$) is open in X for every $a \in \mathbb{R}$. For any space X we denote by $C_{lsc}^*(X)$ (resp., $C_{usc}^*(X)$) the set of all bounded lower (resp., upper) semi-continuous functions on X .

The following theorem is an analog of Theorem 3.3.

Theorem 4.1. *Let X be a subspace of Y . Then the following conditions are equivalent:*

- (i) *There exists an upper semi-continuous compact-valued map $r: Y \rightarrow \beta X$ with $r(x) = \{x\}$ for all $x \in X$;*
- (ii) *There exists an extender $u: C^*(X) \rightarrow C_{lsc}^*(Y)$ which is normed, weakly additive, preserves min and weakly preserves max;*
- (iii) *There exists an extender $u: C^*(X) \rightarrow C_{usc}^*(Y)$ which is normed, weakly additive, preserves max and weakly preserves min.*

Moreover, the extenders from (ii) and (iii) have compact supports iff $r(y) \subset X$ for all $y \in Y$.

Proof. Let $r: Y \rightarrow \beta X$ be a compact-valued upper semi-continuous map with $r(x) = \{x\}$ for all $x \in X$. Then, for every $f \in C^*(X)$, the equality $u(f)(y) = \inf\{\beta f(x) : x \in r(y)\}$, $y \in Y$, defines a bounded function $u(f): Y \rightarrow \mathbb{R}$. Obviously, $u(f)(x) = f(x)$ provided $x \in X$. So, u is an extender, and it is easily seen that u is normed, weakly additive, preserves min and weakly preserves max. Using that r is upper semi-continuous, one can show that $u(f) \in C_{lsc}^*(Y)$. Indeed, suppose $u(f)(y_0) > a$ for some $y_0 \in Y$ and $a \in \mathbb{R}$. Then $r(y_0) \subset (a, \infty)$ and there exists a neighborhood $O(y_0) \subset Y$ of y_0 such that $r(y) \subset (a, \infty)$ for all $y \in O(y_0)$. Hence, $u(f)(y) > a$, $y \in O(y_0)$. Therefore, (i) implies (ii). The implication (i) \Rightarrow (iii) is similar, we define the extender u by $u(f)(y) = \sup\{\beta f(x) : x \in r(y)\}$, where $f \in C^*(X)$ and $y \in Y$. In this case $u(f) \in C_{usc}^*(Y)$.

Suppose $r: Y \rightarrow X$ is an upper semi-continuous compact-valued retraction. Then each of the extenders u defined above has compact supports. Indeed, by Theorem 2.9, the support of any functional μ_y , $\mu_y(f) = u(f)(y)$, is the set $r(y)$.

The implication (ii) \Rightarrow (i) follows from the observation that if $u: C^*(X) \rightarrow C_{lsc}^*(Y)$ is a normed, weakly additive extender which preserves min and weakly preserves max, then the functionals μ_y , $y \in Y$, belong to $\mathfrak{R}_{\min}(X)$. So, by Theorem 2.9, $S(\mu_y)$ is a non-empty compact subset of βX . Moreover, $S(\mu_x) = \{x\}$ for all $x \in X$ because u is an extender. Hence, the set-valued map $r: Y \rightarrow \beta X$, $r(y) = S(\mu_y)$, is compact-valued with $r(x) = \{x\}$ for $x \in X$. Let us show that r is upper semi-continuous. Suppose $r(y_0) \subset U$ for some $y_0 \in Y$ and open $U \subset \beta X$. Take a function $f \in C(\beta X)$ with $f(x) = 1$ for all $x \in r(y_0)$ and $f(\beta X \setminus U) = 0$. Since $u(f)$ is lower semi-continuous, y_0 has a neighborhood $V(y_0)$ such that $u(f)(y) > 1/2$ for all $y \in V(y_0)$. This implies $r(y) \subset U$, $y \in V(y_0)$. Indeed, otherwise Corollary 2.7 would yield $\mu_y(f) = u(f)(y) = 0$ for some $y \in V(y_0)$. Obviously, $r(y) \subset X$ when u has compact supports. Similar arguments provide the proof of (iii) \Rightarrow (i). \square

We can establish now a characterization of $AE(0)$ -spaces in terms of normed weakly additive extenders with compact supports preserving min (resp., max) and weakly preserving max (resp., min).

Corollary 4.2. *The following conditions are equivalent for any space X :*

- (i) $X \in AE(0)$;
- (ii) For every C -embedding of X in a space Y there exists a normed weakly additive extender $u : C^*(X) \rightarrow C_{lsc}^*(Y)$ with compact supports which preserves min and weakly preserves max;
- (iii) For every C -embedding of X in a space Y there exists a normed weakly additive extender $u : C^*(X) \rightarrow C_{usc}^*(Y)$ with compact supports which preserves max and weakly preserves min.

Proof. The proof follows from [Theorem 4.1](#) and the following characterization of $AE(0)$ -spaces [4]: $X \in AE(0)$ if and only if for any C -embedding of X in a space Y there exists a compact-valued upper semi-continuous retraction $r : Y \rightarrow X$. \square

The next theorem shows that conditions (ii) and (iii) from [Theorem 4.1](#) can be weakened.

Theorem 4.3. *Let X be a subspace of Y . Then the following conditions are equivalent:*

- (i) There exists an upper semi-continuous compact-valued map $r : Y \rightarrow \beta X$ with $r(x) = \{x\}$ for all $x \in X$;
- (ii) There exists an extender $u : C^*(X) \rightarrow C_{lsc}^*(Y)$ preserving min with $u(1_X) = 1_Y$;
- (iii) There exists an extender $u : C^*(X) \rightarrow C_{usc}^*(Y)$ preserving max with $u(1_X) = 1_Y$.

Proof. According to [Theorem 4.1](#), it suffices to show the implications (ii) \Rightarrow (i) and (iii) \Rightarrow (i). We are going to prove first (iii) \Rightarrow (i). Suppose $u : C^*(X) \rightarrow C_{usc}^*(Y)$ is an extender preserving max and $u(1_X) = 1_Y$. For every open set $U \subset X$ let

$$e(U) = \bigcup \{u(h)^{-1}((-\infty, 1)) : h \in C_U\},$$

where C_U is the set of all $h \in C^*(X)$ such that $h(X) \subset (-\infty, 1]$ and $X \setminus U \subset h^{-1}(1)$. Since u preserves max, it is monotone. Hence, $u(h) \leq u(1_X) = 1_Y$ for all $h \in C_U$. Because each $u(h)$ is upper semi-continuous, $u(h)^{-1}((-\infty, 1))$ is open in Y , so is the set $e(U)$. Using that u is an extender, one can show that $e(U) \cap X = U$. Moreover, if $U \subset V$, then $C_U \subset C_V$ and we have $e(U) \subset e(V)$.

We claim that $e(U \cap V) = e(U) \cap e(V)$ for any two open sets $U, V \subset X$. Indeed, the inclusion $e(U \cap V) \subset e(U) \cap e(V)$ follows from monotonicity of the operator e . To prove the other inclusion, let $y \in e(U) \cap e(V)$. Then there exist $h_U \in C_U$ and $h_V \in C_V$ with $u(h_U)(y) < 1$ and $u(h_V)(y) < 1$. Obviously, $h_U^{-1}((-\infty, 1)) \cap h_V^{-1}((-\infty, 1)) = \emptyset$ implies $\max\{h_U, h_V\} = 1_X$. So, $u(\max\{h_U, h_V\})(y) = \max\{u(h_U)(y), u(h_V)(y)\} = 1$, which is a contradiction. Therefore, $h_U^{-1}((-\infty, 1)) \cap h_V^{-1}((-\infty, 1)) \neq \emptyset$. On the other hand, $g = \max\{h_U, h_V\}$ belongs to $C_{U \cap V}$ and $y \in g^{-1}((-\infty, 1))$. Thus, $y \in e(U \cap V)$.

Now, we define the set-valued map $r : Y \rightarrow \beta X$ by

$$r(y) = \bigcap \{\bar{U}^{\beta X} : y \in e(U)\} \quad \text{if } y \in \bigcup \{e(U) : U \in \mathcal{T}_X\}$$

and

$$r(y) = \beta X \quad \text{if } y \notin \bigcup \{e(U) : U \in \mathcal{T}_X\}.$$

Using that $\bigcap_{i=1}^{i=k} e(U_i) = e(\bigcap_{i=1}^{i=k} U_i)$ for any finitely many open sets $U_i \subset X$, one can show that r is an upper semi-continuous map with non-empty values. Since $e(U) \cap X = U$, $U \in \mathcal{T}_X$, we have $r(x) = \{x\}$ for all $x \in X$.

The proof of (ii) \Rightarrow (i) is similar. The only difference is the definition of the operator e . Now we define

$$e(U) = \bigcup \{u(h)^{-1}((1, \infty)) : h \in C_U\},$$

where C_U is the set of all $h \in C^*(X)$ such that $h(X) \subset [1, \infty)$ and $X \setminus U \subset h^{-1}(1)$. \square

The next corollary follows from [Theorem 4.3](#) and Dranishnikov’s characterization [15] of compact $AE(0)$ -spaces as upper semi-continuous retracts of Tychonoff cubes.

Corollary 4.4. *The following conditions are equivalent for a compact space X :*

- (i) $X \in AE(0)$;
- (ii) For every embedding of X in a space Y there exists an extender $u : C^*(X) \rightarrow C_{lsc}^*(Y)$ which preserves min and $u(1_X) = 1_Y$;
- (iii) For every embedding of X in a space Y there exists an extender $u : C^*(X) \rightarrow C_{usc}^*(Y)$ which preserves max and $u(1_X) = 1_Y$.

Observe that conditions (ii) and (iii) are equivalent. Indeed, if $u : C^*(X) \rightarrow C_{lsc}^*(Y)$ is an extender preserving min and $u(1_X) = 1_Y$, then the formula $v(h) = -u(-h)$ defines an extender $v : C^*(X) \rightarrow C_{usc}^*(Y)$ which preserves max and $v(1_X) = 1_Y$. Similarly, condition (iii) implies (ii).

Concerning lower semi-continuous retractions, one can establish the following analog of [Theorem 4.1](#).

Theorem 4.5. *Let X be a subspace of Y . Then the following conditions are equivalent:*

- (i) There exists a lower semi-continuous compact-valued map $r : Y \rightarrow \beta X$ with $r(x) = \{x\}$ for all $x \in X$;
- (ii) There exists an extender $u : C^*(X) \rightarrow C_{usc}^*(Y)$ which is normed, weakly additive, preserves min and weakly preserves max;
- (iii) There exists an extender $u : C^*(X) \rightarrow C_{lsc}^*(Y)$ which is normed, weakly additive, preserves max and weakly preserves min.

Moreover, the extenders from (ii) and (iii) have compact supports iff $r(y) \subset X$ for all $y \in Y$.

5. Concluding remarks

Considering extenders which preserve both max and min, we have the following proposition:

Proposition 5.1. *Let X be a subspace of Y . Then each of the following two conditions implies the existence of a neighborhood G of X in Y and an upper semi-continuous map $r: G \rightarrow \beta X$ with compact connected values such that $r(x) = \{x\}$ for all $x \in X$:*

- (i) *There exists an extender $u: C^*(X) \rightarrow C_{isc}^*(Y)$ with $u(1_X) = 1_Y$ such that u preserves both max and min;*
- (ii) *There exists an extender $u: C^*(X) \rightarrow C_{usc}^*(Y)$ with $u(1_X) = 1_Y$ such that u preserves both max and min.*

Proof. Suppose $u: C^*(X) \rightarrow C_{usc}^*(Y)$ is an extender satisfying condition (ii). We define the operator $e: \mathcal{T}_X \rightarrow \mathcal{T}_Y$ as in the proof of Theorem 4.3, implication (iii) \Rightarrow (i). Let $G = \bigcup \{e(U) : U \in \mathcal{T}_X\}$ and $r(y) = \bigcap \{\overline{U}^{\beta X} : y \in e(U)\}$ for all $y \in G$. We need to show that the values of r are connected.

Suppose $r(y_0)$ is not connected for some $y_0 \in G$. So, there are two non-empty open sets U_1, U_2 in βX with disjoint closures such that $r(y_0) \cap U_j \neq \emptyset, j = 1, 2$, and $r(y_0) \subset U_1 \cup U_2$. Fix finitely many open sets $W_i \subset \beta X, i = 1, \dots, k$, with $y_0 \in \bigcap_{i=1}^k e(W_i \cap X)$ and $r(y_0) \subset \bigcap_{i=1}^k \overline{W_i}^{\beta X} \subset U_1 \cup U_2$. Since $\bigcap_{i=1}^k e(W_i \cap X) = e((\bigcap_{i=1}^k W_i) \cap X)$, we can suppose that $y_0 \in e((U_1 \cup U_2) \cap X)$. Then, according to the definition of the operator e , there exists $h_0 \in C_{(U_1 \cup U_2) \cap X}$ with $y_0 \in u(h_0)^{-1}((-\infty, 1))$. Therefore, $h_0(X) \subset (-\infty, 1]$ and $h_0(x) = 1$ for all $x \in X \setminus (U_1 \cup U_2)$. Because U_1 and U_2 have disjoint closures, $h_0 = \min\{h_1, h_2\}$, where $h_j(x) = h_0(x)$ if $x \in U_j \cap X$ and $h_j(x) = 1$ if $x \notin U_j \cap X, j = 1, 2$. Hence, $u(h_0)(y_0) = \min\{u(h_1)(y_0), u(h_2)(y_0)\}$. So, $y_0 \in u(h_j)^{-1}((-\infty, 1))$ for some $j \in \{1, 2\}$. Since $h_j \in C_{U_j \cap X}$, we have $y_0 \in e(U_j \cap X)$. Consequently, $r(y_0) \subset \overline{U_j}^{\beta X}$ which contradicts the fact that $r(y_0)$ meets both $\overline{U_1}^{\beta X}$ and $\overline{U_2}^{\beta X}$. So, the map r has connected values.

Similar arguments work when u satisfies condition (i). \square

Corollary 5.2. *Let X be a compact connected subspace of a space Y and u is an extender satisfying one of the conditions (i) and (ii) from Proposition 5.1. Then there exists an upper semi-continuous map $r: Y \rightarrow X$ with compact connected values such that $r(x) = \{x\}$ for all $x \in X$.*

Proof. By Proposition 5.1, there exists an upper semi-continuous retraction $r_1: G \rightarrow X$ with non-empty compact connected values, where G is a neighborhood of X in Y . Then the map $r: Y \rightarrow X, r(x) = r_1(x)$ if $x \in G$ and $r(x) = X$ if $x \notin G$, is the required retraction. \square

According to [15], every compactum which is an upper semi-continuous compact and connected valued retract of a Tychonoff cube is an $AE(1)$. This result together with Corollary 5.2 yields the following one.

Corollary 5.3. *Let X be a compact connected space such that for any embedding of X in another space there exists an extender u satisfying one of the conditions (i) and (ii) from Proposition 5.1. Then $X \in AE(1)$.*

The last corollary leads to the following problem:

Question 5.4. *Is there any topological description of the class of compacta X such that for every embedding of X in another space Y there exists an extender satisfying one of the conditions (i) and (ii) from Proposition 5.1.*

Zarichnyi [16] investigated the functor of idempotent probability measures. For a compact space X a functional $\mu: C(X) \rightarrow \mathbb{R}$ is called an *idempotent measure* if μ is normed, weakly additive and preserves max. The space $I(X)$ of all idempotent probability measures on X is a compact subspace of $\mathbb{R}^{C(X)}$. We say that a compactum X is a *Zarichnyi space* if for every embedding of X in another space Y there exists a normed, weakly additive extender $u: C(X) \rightarrow C(Y)$ which preserves max. This is equivalent to the existence of a map $\theta: Y \rightarrow I(X)$ such that $\theta(x) = \delta_x$ for every $x \in X$. In particular, X is a Zarichnyi space provided $I(X)$ is an absolute retract. According to Corollary 3.5, every $AE(1)$ -compactum is a Zarichnyi space. But there exists a Zarichnyi space which is not an $AE(1)$. Indeed, let X be a metric infinite compactum which is not locally connected. Then, by [17, Theorem 5.3], $I(X)$ is homeomorphic to \mathbb{I}^ω . Consequently, X is a Zarichnyi space. Since X is not locally connected, $X \notin AE(1)$. On the other hand, by Corollary 4.4, any Zarichnyi space is an $AE(0)$.

Question 5.5. *Is there any $AE(0)$ -space which is not a Zarichnyi space?*

Let us note that every compact metric spaces is a Zarichnyi space. We already observed that for infinite metric compacta. In case X is a finite set of cardinality n , then $I(X)$ is homeomorphic to the $(n - 1)$ -dimensional simplex, see [16]. Therefore, if there exists an $AE(0)$ -space which is not a Zarichnyi space, it should be non-metrizable.

Note added in proof.

Recently Radul [18] proved that the class of Zarichnyi spaces coincides with $AE(0)$ -spaces. Therefore, the answer to Question 5.5 is no.

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