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Averaging operators and set-valued maps

Vesko Valov [1]

Presented by Petăr Kenderov

We investigate maps admitting, in general, non-linear averaging operators. Characterizations of maps admitting a normed, weakly additive averaging operator which preserves max (resp., min) and weakly preserves min (resp., max) is obtained. We also describe set-valued maps into completely metrizable spaces admitting lower semi-continuous selections. As a corollary, we obtain a description of surjective maps with a metrizable kernel and complete fibers which admit regular linear averaging operators.

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1 Introduction

All spaces in the paper are assumed to be Tychonoff. Continuous bounded real-valued functions on X are denoted by $C^*(X)$ (this space is denoted by $C(X)$ when X is compact).

Regular averaging operators were introduced by Pelczyński [13] (recall that a linear operator $u\colon C^*(S) \to C^*(K)$ is regular if u is of norm one and $u(1_S) = 1_K$, where $1_S, 1_K$ are the constant functions 1 on S and K). Since then regular averaging operators and maps admitting regular averaging operators (usually called Milyutin maps) were extensively studied, see [1], [3], [4], [5], [6], [7], [8], [9], [15], [17], [22]. To clarify the importance of regular averaging operators, let us mention that the classification result of Milyutin [11] (that the function spaces $C(K_1)$ and $C(K_2)$ of any uncountable metric compacta K_1, K_2

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are linearly homeomorphic) is based on the existence of a map from the Cantor set onto the unit interval admitting a regular averaging operator.

Regular extension operators and regular averaging operators were simultaneously introduced and investigated by Pelczyński [13]. Concerning the non-linear case, there are already some treatments and applications of general extension operators (see [2], [16], [19], [20]). In this paper we investigate mainly non-linear averaging operators. Let $f\colon X \to Y$ be a surjective map. We say that a map (not necessarily linear) $u\colon C^*(X) \to C^*(Y)$ is called *an averaging operator* for f if the support $S(\mu_y)$ of each μ_y, $y \in Y$, is contained in $f^{-1}(y)$. Here, $\mu_y\colon C^*(X) \to \mathbb{R}$, $y \in Y$, are the maps (we called them functionals), generated by u. Each μ_y is defined by $\mu_y(h) = u(h)(y)$, $h \in C^*(X)$.

The paper is organized as follows: Some definitions and properties of the support maps of general operators are given in Section 2. In Section 3 we consider normed, weakly additive averaging operators $u\colon C^*(X) \to C^*(Y)$ preserving max and weakly preserving min. An operator $u\colon C^*(X) \to C^*(Y)$, where X and Y are arbitrary spaces, is said to be (i) *normed*, (ii) *weakly additive*, (iii) *preserving* max, and (iv) *weakly preserving* min, if for every $f, g \in C^*(X)$ and every constant function c_X we have: (i) $u(1_X) = 1_Y$, (ii) $u(f + c_X) = u(f) + c_Y$, (iii) $u(\max\{f, g\}) = \max\{u(f), u(g)\}$, (iv) $u(\min\{f, c_X\}) = \min\{u(f), c_Y\}$. We say that u *preserves* min provided u satisfies equality (iii) with max replaced by min. Similarly, u *weakly preserves* max if u satisfies condition (iv) with min replaced by max. A functional $\mu\colon C^*(X) \to \mathbb{R}$ is normed, weakly additive, preserves max and weakly preserves min (resp., preserves min and weakly preserves max) provided μ satisfies the corresponding equalities above, where the constant functions c_Y are replaced by the constants c. This type of functionals were introduced by Radul [14]. A given operator has any of the above properties if and only if all functionals generated by this operator have the same property. Moreover, if $u : C^*(X) \to C^*(Y)$ (reps., $\mu\colon C^*(X) \to \mathbb{R}$) is normed, weakly additive, preserves max and weakly preserves min, then the operator $v\colon C^*(X) \to C^*(Y)$, $v(h) = -u(-h)$ (resp., the functional $\nu\colon C^*(X) \to \mathbb{R}$, $\nu(h) = -\mu(-h)$) is normed, weakly additive, preserves min and weakly preserves max.

We show (Theorem 3.3) that a surjective map $f\colon X \to Y$ admits a normed, weakly additive operator which preserves max (resp., min) and weakly preserves min (resp., max) if and only if there exists a continuous compact-valued map $\Phi\colon Y \to X$ such that $\Phi(y) \subset f^{-1}(y)$ for all $y \in Y$. This implies that if each map of a given family admits such an averaging operator, so is the product of all maps from the family (see Corollary 3.5). We also provide an external characterization of perfect surjective maps f such that f^{-1} admits a continuous compact-valued selection. This characterization is dual to Shirokov's description [18] of compact space X with the following property: for every embedding of X in another space Y there exists a compact-valued continuous retraction $r\colon Y \to X$ (i.e., a set-valued map r such that $r(x) = \{x\}$ for all $x \in X$).

In Section 4 we prove that if a map f with complete metrizable fibers admits a supportive averaging operator (this means that all functionals μ_y have the following property: $\mu_y(h) = \mu_y(g)$ provided f and g have the same restrictions on the support $S(\mu_y)$), then f admits a regular averaging operator (Corollary 4.2). This result is based on Proposition 4.1 stating that a map f with complete metrizable fibers and paracompact codomain admits a regular averaging operator iff f^{-1} has a lower semi-continuous selection. Because of that, it is interesting to have a description of maps f such that its inverse f^{-1} admits a lowed semi-continuous selection. Corollary 4.5 provides such a description and generalizes a similar result of Argiros-Arvanitakis [3].

Finally, in the last Section 5, we consider averaging operators of type $u\colon C^*(X) \to C^*_{lsc}(Y)$ or $u\colon C^*(X) \to C^*_{usc}(Y)$, where $C^*_{lsc}(Y)$ and $C^*_{usc}(Y)$ denote, respectively, bounded lower and upper semi-continuous functions on Y.

2 Preliminaries

The set of all normed, weakly additive functionals on $C^*(X)$ which preserve max (resp. min) and weakly preserve min (resp., max) is denoted by $\mathfrak{R}^*_{max}(X)$ (resp., $\mathfrak{R}^*_{min}(X)$). The topology of these two spaces is inherited from the product $\mathbb{R}^{C^*(X)}$. Identifying $C^*(X)$ with $C(\beta X)$, any functional μ on $C^*(X)$ can be considered as a function $\mu\colon C(\beta X) \to \mathbb{R}$. For any functional $\mu\colon C^*(X) \to \mathbb{R}$ we define its support $S(\mu)$ to be the following subset of the Čech-Stone compactification βX of X (see also [21] for a similar definition):

Definition 2.1 *[2] $S(\mu)$ is the set of all $x \in \beta X$ such that for every its neighborhood O_x in βX there exist $f, g \in C^*(X)$ with $\beta f|(\beta X\backslash O_x) = \beta g|(\beta X\backslash O_x)$ and $\mu(f) \neq \mu(g)$.*

Here, $\beta f\colon \beta X \to \mathbb{R}$ is the Čech-Stone extension of f and $\beta f|(\beta X\backslash O_x)$ denotes its restriction on the set $\beta X\backslash O_x$. Obviously, $S(\mu)$ is a closed subset of βX (possibly empty). If $\varnothing \neq S(\mu) \subset X$, we say that μ has a compact support. We consider the subspaces $\mathfrak{R}^*_{max}(X)_c \subset \mathfrak{R}^*_{max}(X)$ and $\mathfrak{R}^*_{min}(X)_c \subset \mathfrak{R}^*_{min}(X)$ consisting of functionals with compact supports.

We say that a functional μ on $C^*(X)$ is *supportive* if $\mu(h) = \mu(g)$ for any $h, g \in C(\beta X)$ with $h|S(\mu) = g|S(\mu)$. An operator $u\colon C^*(X) \to C^*(Y)$ is called supportive provided all functionals μ_y, $y \in Y$, are supportive.

The following property of the supports was established in [2, Corollary 2.3]:

Proposition 2.2 *Let μ be a weakly additive normed and monotone functional on $C^*(X)$. Then $S(\mu) \neq \varnothing$, and μ is supportive.*

Concerning the supports of normed weakly additive functionals which preserve max (resp., min) and weakly preserve min (resp., max), we have the following description (see Theorem 2.9 from [2]):

Proposition 2.3 *Let X be a Tychonoff space and μ a functional on $C^*(X)$. Then we have:*

(i) $\mu \in \mathfrak{R}^*_{min}(X)_c$ (resp., $\mu \in \mathfrak{R}^*_{min}(X)$) if and only if there exists a non-empty compact set $F \subset X$ (resp., $F \subset \beta X$) such that $F = S(\mu)$ and $\mu(f) = \inf\{f(x) : x \in F\}$ for all $f \in C(\beta X)$;

(ii) $\mu \in \mathfrak{R}^*_{max}(X)_c$ (resp., $\mu \in \mathfrak{R}^*_{max}(X)$) if and only if there exists a non-empty compact set $F \subset X$ (resp., $F \subset \beta X$) such that $F = S(\mu)$ and $\mu(f) = \sup\{f(x) : x \in F\}$ for all $f \in C(\beta X)$.

Let $\mu\colon C^*(X) \to \mathbb{R}$ be a functional and $f\colon X \to Y$ a map. Then f generates the functional $\mu^f : C^*(Y) \to \mathbb{R}$ defined by $\mu^f(h) = \mu(h\circ f)$, $h \in C^*(Y)$. We say that μ is *support-preserving* if $\beta f(S(\mu)) = S(\mu^f)$ for any space Y and any map $f\colon X \to Y$. When $u\colon C^*(X) \to C^*(Y)$ is an operator such that all μ_y are support-preserving functionals, then u is said to be support-preserving.

Corollary 2.4 *Every normed weakly additive functional which preserve* max (*resp.,* min) *and weakly preserve* min (*resp.,* max) *is support-preserving.*

P r o o f. Let μ be a normed weakly additive functional on $C^*(X)$ which preserves max and weakly preserves min, and $f\colon X \to Y$ is a map. Then μ^f is normed weakly additive functional on $C^*(Y)$ preserving max and weakly preserving min. Suppose there exists $y \in \beta f(S(\mu))\backslash S(\mu^f)$, and choose $h \in C(\beta Y)$ with $h(y) = 1$ and $h(S(\mu^f)) = 0$. Then by Proposition 2.3, $\mu^f(h) = 0$ and $\mu(h \circ \beta f) \geq 1$. But $\mu^f(h) = \mu(h \circ \beta f)$, a contradiction. So, $\beta f(S(\mu)) \subset S(\mu^f)$. Similarly, $S(\mu^f) \subset f(S(\mu))$.

Let μ be a normed weakly additive functional on $C^*(X)$ which preserves min and weakly preserves max. Then the equality $\nu_X(\mu)(g) = -\mu(-g)$ defines a functional $\nu_X(\mu)$ on $C^*(X)$, which is normed weakly additive, preserves max and weakly preserves min. Moreover, $S(\mu) = S(\nu_X(\mu))$. So, by the previous case, $S(\mu^f) = f(S(\mu))$.

Recall that a set-valued map $\Phi\colon Y \to X$ is lower (resp., upper) semi-continuous if for every open set $U \subset X$ (resp., for every closed set $F \subset X$) the set $\Phi^{-1}(U) = \{y \in Y : \Phi(y) \cap U \neq \varnothing\}$ is open (resp., the set $\Phi^{-1}(F)$ is closed) in Y.

Let $u\colon C^*(X) \to C^*(Y)$ be an operator. Then the support of every functional μ_y is a closed (possibly empty) subset of βX. We define the support map S_u of u to be the set-valued map $S_u\colon Y \to \beta X$, $S_u(y) = S(\mu_y)$. Proposition 2.2 and Proposition 2.3 easily imply continuity-type properties of the support map (see [2, Theorem 3.1] for the special case when u is an extender).

Proposition 2.5 *Let $u\colon C^*(X) \to C^*(Y)$ be a supportive operator. Then the support map $S_u\colon Y \to \beta X$ is lower semi-continuous. If u is normed, weakly additive operator which preserves* max *(resp.,* min*) and weakly preserves* min *(resp.,* max*), then S_u is both lower and upper semi-continuous.*

Proof. Suppose $S_u(y_0) \cap U \neq \varnothing$ for some $y_0 \in Y$ and open $U \subset X$. Then, according the definition of support, there exist $h_1, h_2 \in C(\beta X)$ such that $h_1|(\beta X\backslash U) = h_2|(\beta X\backslash U)$ and $u(h_1)(y_0) \neq u(h_2)(y_0)$. Let $V = \{y \in Y : u(h_1)(y) \neq u(h_2)(y)\}$. Obviously, V is a neighborhood of y_0 in Y. Since u is supportive, the existence of $y \in V$ with $S_u(y) \subset \beta X\backslash U$ yields $u(h_1)(y) = u(h_2)(y)$, a contradiction. So, S_u is lower semi-continuous.

Suppose u is a normed, weakly additive operator which preserves max and weakly preserves min. Since u is supportive (Proposition 2.2), S_u is lower semi-continuous. So, we need to show that S_u is upper semi-continuous. To this end, let $S_u(y^*) \subset W$ with $W \subset X$ open. Choose $h \in C(\beta X)$ such that $h(\beta X\backslash W) = 0$ and $h(S_u(y^*)) = 1$. The last equality implies $u(h)(y^*) = 1$. Hence, $O = u(h)^{-1}(0, \infty)$ is a neighborhood of y^* and $S_u(y) \subset W$ for all $y \in O$. Indeed, if $S_u(y)\backslash W \neq \varnothing$, then by Proposition 2.3(i), $u(h)(y) \leq 0$, a contradiction. Therefore, S_u is lower semi-continuous. The case u is normed, weakly additive, preserves max and weakly preserves min is similar.

3 Averaging operators with continuous values

In this section we consider operators between spaces of continuous functions.

Let $f\colon X \to Y$ be a surjective map. We say that f admits an *averaging operator* $u\colon C^*(X) \to C^*(Y)$ if the support of any functional μ_y, $y \in Y$, is contained in $f^{-1}(y)$. Since all $S(\mu_y)$, $y \in Y$, are compact, any averaging operator has compact supports.

The notion "averaging" is borrowed from the classical linear averaging operators, see [13]. It means that $\inf\{h(x) : x \in f^{-1}(y)\} \leq u(h)(y) \leq \sup\{h(x) : x \in f^{-1}(y)\}$ for all $h \in C^*(X)$ and $y \in Y$. Next proposition shows that every averaging supportive operator has this property.

Proposition 3.1 *Let $f: X \to Y$ be a surjective map and $u: C^*(X) \to C^*(Y)$ be a monotone, normed and supportive operator. Consider the following conditions:*

(i) *u is an averaging operator for f;*

(ii) $\inf\{h(x) : x \in f^{-1}(y)\} \leq u(h)(y) \leq \sup\{h(x) : x \in f^{-1}(y)\}$ *for all $h \in C^*(X)$ and $y \in Y$;*

(iii) *$u(g \circ f) = g$ for all $g \in C^*(Y)$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. Suppose u is a supportive averaging operator for f. Since u is monotone and normed, so are the functionals μ_y, $y \in Y$. Moreover, each $S(\mu_y)$ is contained in $f^{-1}(y)$ and has the following property: $\mu_y(h_1) = \mu_y(h_2)$ provided $h_1|S(\mu_y) = h_2|S(\mu_y)$, $h_1, h_2 \in C^*(X)$. Consequently, $\inf\{h(x) : x \in f^{-1}(y)\} \leq u(h)(y) \leq \sup\{h(x) : x \in f^{-1}(y)\}$ for all $h \in C^*(X)$ and all $y \in Y$. Indeed, consider the set-valued map $\Phi_y: \beta X \to \mathbb{R}$ defined by $\Phi_y(x) = \beta h(x)$ if $x \in \overline{f^{-1}(y)}^{\beta X}$ and $\Phi_y(x) = [a, b]$ if $x \notin \overline{f^{-1}(y)}^{\beta X}$, where $a = \inf\{h(x) : x \in f^{-1}(y)\}$ and $b = \sup\{h(x) : x \in f^{-1}(y)\}$. This map is lower semi-continuous and convex-valued. So, according to Michael's selection theorem [10], there exists a selection h' for Φ_y. Obviously, $a \leq h'(x) \leq b$ for all $x \in X$. Since $h'|f^{-1}(y) = h|f^{-1}(y)$ and $S(\mu_y) \subset f^{-1}(y)$, $\mu_y(h) = \mu_y(h')$. On the other hand, by monotonicity of μ_y, $a = \mu_y(a) \leq \mu_y(h') \leq \mu_y(b) = b$. This provides the implication $(i) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (iii)$ is trivial. If $g \in C^*(Y)$ and $y \in Y$, then $(g \circ f)|f^{-1}(y)$ is the constant $g(y)$. Hence, $u(g \circ f) = g$.

Obviously, if $f: X \to Y$ is a surjective map and u satisfies condition (iii) from Proposition 3.1, then $\mu_y^f = \delta_y$ for all $y \in Y$. This implies that $S(\mu_y) \subset f^{-1}(y)$, $y \in Y$, provided u is support-preserving. Hence, by Proposition 3.1, we have the following corollary.

Corollary 3.2 *Let $f: X \to Y$ be a surjective map and $u: C^*(X) \to C^*(Y)$ be a monotone, normed supportive and support-preserving operator with compact supports. Then conditions (i), (ii) and (iii) from Proposition 3.1 are equivalent.*

Next, we characterize maps admitting normed weakly additive averaging operators which preserve min (resp. max) and weakly preserve max (resp., min).

Theorem 3.3 *For any surjective map $f\colon X \to Y$ the following conditions are equivalent:*

(i) *f admits a normed weakly additive averaging operator which preserves* min (*resp.,* max) *and weakly preserves* max (*resp.,* min)*;*

(ii) *There exists an embedding $g\colon Y \to \mathfrak{R}^*_{min}(X)_c$ (resp., $g\colon Y \to \mathfrak{R}^*_{max}(X)_c$) with $S(g(y)) \subset f^{-1}(y)$ for all $y \in Y$;*

(iii) *There exists a continuous compact-valued map $\Phi\colon Y \to X$ such that $\Phi(y) \subset f^{-1}(y)$ for all $y \in Y$.*

P r o o f. We are going to prove the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$ in the case of normed weakly additive operators (or functionals) which preserve min and weakly preserve max.

If $u\colon C^*(X) \to C^*(Y)$ is a normed weakly additive averaging operator for f which preserves min and weakly preserves max, then we define $g\colon Y \to \mathfrak{R}^*_{min}(X)_c$ by $g(y)(h) = u(h)(y)$, $h \in C^*(X)$. Obviously, g is continuous and $S(g(y)) \subset f^{-1}(y)$ for all $y \in Y$. The last inclusions imply that g is one-to-one. Let us show that g is an embedding. Suppose $\{g(y_\alpha)\}$ is a net in $g(Y)$ converging to some $g(y)$. Then, $\varphi(y_\alpha) = g(y_\alpha)(\varphi \circ f)$ converges to $\varphi(y) = g(y)(\varphi \circ f)$ for every $\varphi \in C^*(Y)$. Hence, the net $\{y_\alpha\}$ converges to y. This completes the proof of $(i) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (iii)$ follows from the observation that the compact-valued map $\Phi(y) = S(g(y)$ is both upper and lower semi-continuous (see the proof of [2, Theorem 3.1]), and $\Phi(y) \subset f^{-1}(y)$ for all $y \in Y$.

For the final implication $(iii) \Rightarrow (i)$, let $h \in C^*(X)$ and consider the function $u(h)\colon Y \to \mathbb{R}$, $u(h)(y) = \inf\{h(x) : x \in \Phi(y)\}$. Since Φ is compact-valued and continuous, $u(h) \in C^*(Y)$. Obviously, the support $S(\mu_y)$ of any functional μ_y generated by u is the set $\Phi(y)$, $y \in Y$. So, u is an averaging operator for f. According to Proposition 2.3(i), all μ_y belong to $\mathfrak{R}^*_{min}(X)_c$. Therefore, u is a normed weakly additive operator preserving min and weakly preserving max.

Next corollary follows from Theorem 3.3 and the following result of Pasynkov [12]: For every paracompact space Y of positive dimension there exists a one-dimensional space X with $\dim X = 1$ and a perfect open surjection from X onto Y.

Corollary 3.4 *For every paracompact space Y of positive dimension there exists a space X with* $\dim X = 1$ *and a map $f\colon X \to Y$ admitting a normed weakly additive averaging operator preserving* min (*resp.,* max) *and weakly preserving* max (*resp.,* min).

Corollary 3.5 *Let $f_\alpha\colon X_\alpha \to Y_\alpha$, $\alpha \in \Lambda$, be a family of maps each of them admitting a normed weakly additive averaging operator preserving* min (*resp.,* max) *and weakly preserving* max (*resp.,* min). *Then the product map $f = \prod_{\alpha\in\Lambda} f_\alpha\colon \prod_{\alpha\in\Lambda} X_\alpha \to \prod_{\alpha\in\Lambda} Y_\alpha$ also admits such an averaging operator.*

Proof. By Theorem 3.3 there exist continuous compact-valued maps $\Phi_\alpha\colon Y_\alpha \to X_\alpha$, $\alpha \in \Lambda$, such that $\Phi_\alpha(y_\alpha) \subset f^{-1}(y_\alpha)$, $y_\alpha \in Y_\alpha$. Then the map $\Phi\colon \prod_{\alpha\in\Lambda} Y_\alpha \to \prod_{\alpha\in\Lambda} X_\alpha$, $\Phi(y) = \prod_{\alpha\in\Lambda} \Phi_\alpha(y_\alpha)$, is compact-valued and continuous. Moreover, $\Phi(y) \subset f^{-1}(y)$ for all $y \in \prod_{\alpha\in\Lambda} Y_\alpha$. Then, we can apply again Theorem 3.3 to conclude that f admits a normed weakly additive averaging operator preserving min (resp., max) and weakly preserving max (resp., min).

We say that a map $f\colon X \to Y$ is said to be *co-exponential* if there exists a function $\mathrm{e} : \mathcal{T}_X \to \mathcal{T}_Y$ between the topologies of X and Y satisfying the following conditions:

(1) $\mathrm{e}(X) = Y$ and $\mathrm{e}(\varnothing) = \varnothing$;

(2) $\mathrm{e}(U \cap V) = \mathrm{e}(U) \cap \mathrm{e}(V)$ for any $U, V \in \mathcal{T}_X$;

(3) $\overline{\mathrm{e}(U)}^Y \subset \mathrm{e}(V)$ provided $U, V \in \mathcal{T}_X$ with $\overline{U}^X \subset V$;

(4) $\varnothing \neq \mathrm{e}(U) \subset f(U)$ for all $U \in \mathcal{T}_X$ containing a fiber of f.

If f is an embedding and condition (4) is replaced by $\mathrm{e}(U) \cap X = U$, $U \in \mathcal{T}_X$, we obtain the Shirokov's notion [18] exponential embedding. Shirokov [18, Theorem 1] proved that a compactum X is exponentially embedded in another compactum Y iff there exists a continuous compact-valued retraction from Y into X. Concerning maps admitting averaging operators, we have the following proposition.

Proposition 3.6 *Let $f\colon X \to Y$ be a perfect surjective map. Then f admits a normed weakly additive averaging operator which preserves* min *and weakly preserves* max *if and only if f is co-exponential.*

Proof. Suppose f is co-exponential. We define the compact-valued map $\Phi\colon Y \to \beta X$ by $\Phi(y) = \bigcap\{\overline{U}^{\beta X} : U \in \gamma_y\}$, where $\gamma_y = \{U \in \mathcal{T}_{\beta X} : y \in \mathrm{e}(U \cap X)\}$. According to condition (2) all families γ_y, $y \in Y$, are closed with respect to finite-intersections. This implies that each $\Phi(y)$ is a non-empty compact subset of βX and the map Φ is upper semi-continuous.

To show that Φ is lower semi-continuous, suppose $\Phi(y_0) \cap U_1 \neq \varnothing$ for some $y_0 \in Y$ and $U_1 \in \mathcal{T}_{\beta X}$. Let $U_2 \subset \beta X$ be an open set containing $\Phi(y_0) \cup \overline{U_1}$, and consider the set

$$G = \mathrm{e}(U_2 \cap X) \backslash \bigcap \{e(V \cap X) : V \in \mathcal{A}\},$$

where $\mathcal{A}$ consists of all open $V \subset \beta X$ with $\beta X \backslash U_1 \subset V$.

Claim 1. *G is a neighborhood of y_0*

Indeed, since $\Phi(y_0) \subset U_2$, there are finitely many open sets $V_i \subset \beta X$, $i = 1, .., k$, such that $\Phi(y_0) \subset \bigcap_{i=1}^{i=k} \overline{V_i} \subset U_2$ and $y_0 \in \bigcap_{i=1}^{i=k} \mathrm{e}(V_i \cap X) = \mathrm{e}(\bigcap_{i=1}^{i=k} V_i \cap X)$. So, we have $\Phi(y_0) \subset \overline{W} \subset U_2$ and $y_0 \in \mathrm{e}(W \cap X)$, where $W = \bigcap_{i=1}^{i=k} V_i$. Because the function e is monotone (by condition (2)), we have $y_0 \in \mathrm{e}(U_2 \cap X)$. To show that $y_0 \notin H = \bigcap \{e(V \cap X) : V \in \mathcal{A}\}$, let $x_0 \in \Phi(y_0) \cap U_1$ and $V_0 = \beta X \backslash O(x_0)$, where $O(x_0)$ is a neighborhood of x_0 in βX with $\overline{O(x_0)} \subset U_1$. Obviously, $V_0 \in \mathcal{A}$ and $\overline{V}_0$ does not contain $\Phi(y_0)$. So, $y_0 \notin \mathrm{e}(V_0)$. Finally, let us prove that H is closed in Y. To this end take a net $\{y_\alpha\} \subset H$ converging to some $y^* \in Y$. For any $V \in \mathcal{A}$ fix an open set $W_V \subset \beta X$ such that $\beta X \backslash U_1 \subset W_V \subset \overline{W_V} \subset V$. Then, by (3), $\overline{\mathrm{e}(W_V \cap X)} \subset \mathrm{e}(V \cap X)$. But $H \subset \mathrm{e}(W_V \cap X)$ because $W_V \in \mathcal{A}$. Hence, $H \subset \overline{\mathrm{e}(W_V \cap X)}$, which implies that $y^* \in \mathrm{e}(V \cap X)$ for all $V \in \mathcal{A}$. Therefore, $H \subset Y$ is closed. Consequently, G is a neighborhood of y_0 in Y.

Suppose $\Phi(y) \cap U_1 = \varnothing$ for some $y \in G$. Then there exist $V \in \mathcal{A}$ with $\Phi(y) \subset \beta X \backslash U_1 \subset V$ and $y \notin \mathrm{e}(V \cap X)$. As above, we can find an open set $V_1 \subset \beta X$ such that $\Phi(y) \subset V_1 \subset \overline{V_1} \subset V$ and $y \in \mathrm{e}(V_1 \cap X)$. Then, $y \in \mathrm{e}(V_1 \cap X) \subset \mathrm{e}(V \cap X)$, a contradiction. Therefore, $\Phi(y) \cap U_1 \neq \varnothing$ for all $y \in G$. So, Φ is lower semi-continuous.

Finally, we are going to prove that $\Phi(y) \subset f^{-1}(y)$ for any $y \in Y$. Indeed, otherwise for some $y_0 \in Y$ there exists $x_0 \in \Phi(y_0) \backslash f^{-1}(y_0)$. Choose $W \in \mathcal{T}_{\beta X}$ containing x_0 with $\overline{W} \cap f^{-1}(y_0) = \varnothing$ and a neighborhood $O(y_0) \subset Y$ of y_0 such that $f^{-1}(O(y_0)) \cap \overline{W} = \varnothing$ (this is possible because f is perfect). Since $\Phi(y_0)$ meets W, we can assume that $\Phi(y) \cap W \neq \varnothing$ for all $y \in O(y_0)$ (recall that Φ is lower semi-continuous). By condition (4), $\varnothing \neq \mathrm{e}(U) \subset f(U) \subset O(y_0)$, where $U = f^{-1}(O(y_0))$. Hence, for every $y \in \mathrm{e}(U)$ we have $\Phi(y) \cap W \neq \varnothing$ and $\Phi(y) \subset \overline{U}^{\beta X}$, a contradiction.

So, we have a continuous compact-valued map $\Phi \colon Y \to X$ with $\Phi(y) \subset f^{-1}(y)$ for all $y \in Y$. Therefore, by Theorem 3.3, f admits a normed weakly additive averaging operator with compact supports preserving min (resp., max) and weakly preserving max (resp., min).

For the converse implication, suppose f admits a normed weakly additive averaging operator with compact supports preserving min (resp., max) and weakly preserving max (resp., min). Then, by Theorem 3.3, there exists a compact-valued continuous map $\Phi\colon Y \to X$ with $\Phi(y) \subset f^{-1}(y)$, $y \in Y$. We define $\mathrm{e}(U) = \{y \in Y : \Phi(y) \subset U\}$ for every $U \in \mathcal{T}_X$. Since Φ is upper semi-continuous, each $\mathrm{e}(U)$ is open in Y. Obviously, e satisfies conditions (1), (2) and (4). To show that condition (2) also holds, let $\overline{U} \subset V$ for some open $U, V \subset X$. Then, for every $y \in \overline{\mathrm{e}(U)}$ there exists a net $\{y_\alpha\} \subset \mathrm{e}(U)$ converging to y. So, $\Phi(y_\alpha) \subset U$ for all α. This yields $\Phi(y) \subset \overline{U}$. Indeed, otherwise there would be a neighborhood $O(y)$ of y in Y with $\Phi(z) \cap X \backslash \overline{U} \neq \varnothing$ for all $z \in O(y)$ (because Φ is lower semi-continuous). But that would imply the existence of α with $\Phi(y_\alpha) \cap X \backslash \overline{U} \neq \varnothing$, a contradiction. Hence, $\Phi(y) \subset \overline{U} \subset V$, i.e., $y \in \mathrm{e}(V)$. Consequently, $\overline{\mathrm{e}(U)} \subset \mathrm{e}(V)$.

4 Linear averaging operators

In this section we provide a characterization of surjective maps between metric spaces with complete fibers. We say that an operator $u\colon C^*(X) \to C^*(Y)$ is a *regular averaging* for a given surjection $f\colon X \to Y$ if u is linear, monotone, normed and $u(g \circ f) = g$ for all $g \in C^*(Y)$. A map $f\colon X \to Y$ is said to have a *metrizable kernel* if there exists a metric space M and a map $q\colon X \to M$ such that the diagonal map $f \triangle q\colon X \to Y \times M$ is an embedding. If each $q(f^{-1}(y))$, $y \in Y$, is a complete subspace of M (with respect to a given metric on M), then we say that f has complete fibers.

Proposition 4.1 *Let $f\colon X \to Y$ be a surjective map with complete metrizable fibers, where Y is paracompact. Then f admits a regular averaging operator with compact supports if and only if there exists a lower semi-continuous map $\varphi\colon Y \to X$ with $\varphi(y) \subset f^{-1}(y)$ for all $y \in Y$.*

P r o o f. We fix a metric space M and a map $q\colon X \to M$ such that $f \triangle q$ is an embedding and all sets $q(f^{-1}(y))$, $y \in Y$, are complete.

Suppose f admits a regular averaging operator u with compact supports. Then $S_u(y) \subset f^{-1}(y)$ for every $y \in Y$, where S_u is the support map of u. Since, by Proposition 2.2, every regular averaging operator is supportive, S_u is lower semi-continuous (see Proposition 2.5).

For the converse implication, suppose $\varphi\colon Y \to X$ is a lower semi-continuous map with $\varphi(y) \subset f^{-1}(y)$ for all $y \in Y$. Considering the closures of all $\varphi(y)$ in X, we may assume that φ is closed-valued. By [15], there exists a zero-dimensional paracompact space Z and a perfect surjection $g\colon Z \to Y$ admitting a regular averaging operator $v\colon C^*(Z) \to C^*(Y)$. Since all functionals ν_y, $y \in Y$, generated by v are probability measures, v is support-preserving. Hence, according to Corollary 3.2, $S(\nu_y) \subset g^{-1}(y)$ for all $y \in Y$. Consider the lower semi-continuous map $\Phi = q \circ \varphi \circ g\colon Z \to M$. Each value $\Phi(y)$ is closed in $q(f^{-1}(y))$, $y \in Y$. Hence, all values of Φ are complete. By Michael's zero-dimensional selection theorem, Φ admits a continuous selection k. Then the map $\bar{g} = k \triangle g\colon Z \to X$ is a continuous selection for the map $f^{-1} \circ g$. Now, define $u\colon C^*(X) \to C^*(Y)$ by $u(h)(y) = v(h \circ \bar{g})(y)$. Obviously, u is linear, normed and monotone. Moreover, it is easily seen that $S(supp(\mu_y)) \subset f^{-1}(y)$ for any functional μ_y generated by u. So, according to Proposition 3.1, u is averaging for f.

Proposition 2.5 and Proposition 4.1 imply next corollary.

Corollary 4.2 *Let Y be a paracompact space and $f\colon X \to Y$ a surjective map with complete metrizable fibers admitting a supportive averaging operator with compact supports. Then f admits also a regular averaging operator with compact supports.*

We say that a set-valued map $\Phi\colon Y \to X$ is *weakly lower semi-continuous* (br., wlsc) if there exists a function $\theta\colon \mathcal{T}_X \to \mathcal{T}_Y$ such that:

(5) $\theta(X) = Y$;

(6) $\theta(U) \subset \Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \varnothing\}$;

(7) If $\{U_\alpha : \alpha \in \Lambda\} \subset \mathcal{T}_X$ and $U \subset \bigcup_{\alpha\in\Lambda} U_\alpha$, then $\theta(U) \subset \bigcup_{\alpha\in\Lambda} \theta(U_\alpha)$.

Obviously, conditions (5) and (6) imply that $\Phi(y) \neq \varnothing$ for all $y \in Y$.

Next theorem provides a characterization of wlsc maps in terms of selections.

Theorem 4.3 *Let (X, d) be a metric space and $\Phi\colon Y \to X$ a set-valued map such that each $\Phi(y)$, $y \in Y$, is complete in X. Then Φ is wlsc if and only if Φ admits a lower semi-continuous selection.*

Proof. Suppose Φ is wlsc and $\theta\colon \mathcal{T}_X \to \mathcal{T}_Y$ is a function satisfying the above conditions. For every $y \in Y$ let $\mathcal{B}_y = \{U \in \mathcal{T}_X : y \in \theta(U)\}$. Obviously, $X \in \mathcal{B}_y \neq \varnothing$ for all $y \in Y$. Define $\phi(y)$, $y \in Y$, to be the set of all $x \in X$ such that $x = \lim x_n$, where $x_n \in U_n$ and $\{U\}_{n\geq 1} \subset \mathcal{B}_y$ is a sequence with $diam(U_n) \leq 2^{-n}$, $n \geq 1$. Since $\theta(U) \subset \Phi^{-1}(U)$,

$$\Phi(y) \cap U \neq \varnothing \text{ for any } y \in \theta(U). \tag{8}$$

Claim 2. If $y \in \theta(U)$, then $\phi(y) \cap U \neq \varnothing$.

Indeed, let $\overline{U} \subset \bigcup\{V_\alpha : \alpha \in \Lambda_1\}$ with $U \cap V_\alpha \neq \varnothing$ and $diam(V_\alpha) \leq 2^{-1}$ for all $\alpha \in \Lambda_1$. By condition (7), $y \in \theta(V_{\alpha(1)})$ for some $\alpha_1 \in \Lambda_1$. We put $U_1 = V_{\alpha(1)}$. Continuing in this way, we construct by induction a sequence $\{U_n\} \subset \mathcal{B}_y$ such that $diam(U_n) \leq 2^{-n}$ and $U_n \cap U_{n+1} \neq \varnothing$ for all n. Then, by (8), we can choose points $x_n \in \Phi(y) \cap U_n$, $n \geq 1$. Since U_n meets U_{n+1}, we have $d(x_n, x_{n+1}) \leq 2^{n-1}$. Consequently, $\{x_n\}$ is a Cauchy sequence in $\Phi(y)$. Because $\Phi(y)$ is complete, there exists a point $x \in \Phi(y)$ which the limit of $\{x_n\}$. Obviously, x belongs to $\phi(y) \cap U$.

Claim 3. *For every $y \in Y$ we have $\varnothing \neq \phi(y) \subset \Phi(y)$.*

Claim 2 implies $\phi(y) \neq \varnothing$ for any y because $\theta(X) = Y$. Suppose there exists $x \in \phi(y) \backslash \Phi(y)$ for some $y \in Y$. Then the distance between x and $\Phi(y)$ is positive (recall that $\Phi(y) \subset X$ is closed). So, according to the definition of $\phi(y)$, x is contained in some $W \in \mathcal{B}_y$ with $W \cap \Phi(y) = \varnothing$. Hence, $y \in \theta(W)$ and W is disjoint with $\Phi(y)$, which contradicts condition (8). This completes the proof of Claim 3.

Claim 4. *ϕ is lower semi-continuous.*

Let $x_0 \in \phi(y_0) \cap U \neq \varnothing$, where $y_0 \in Y$ and $U \subset X$ is open. Using the definition of $\phi(y_0)$, we can find an open set $V \subset X$ containing x_0 such that $V \subset U$ and $y_0 \in \theta(V)$. Then, according to Claim 2, $\phi(y) \cap U \neq \varnothing$ for all $y \in \theta(V)$. Therefore, ϕ is lower semi-continuous selection for Φ.

To prove the sufficiency in Theorem 4.3, suppose Φ admits a lower semi-continuous selection ϕ. Then $\theta(U) = \phi^{-1}(U)$ is open in Y for any $U \in \mathcal{T}_X$. Conditions (5) and (7) are obviously satisfied. Condition (6) also holds because $\phi(y) \subset \Phi(y)$ for all $y \in Y$. So, Φ is wlsc.

Next remark follows from the proof of Theorem 4.3 (see the proof of Claim 2).

Remark If X is a compact metric space, then Theorem 4.3 remains true provided Φ satisfies conditions (4), (5) and the following one:

$$\text{if } U \subset \bigcup_{i=1}^{i=k} U_i, \text{ then } \theta(U) \subset \bigcup_{i=1}^{i=k} \theta(U_i). \tag{7'}$$

Corollary 4.4 *Let Y be a paracompact space and $f\colon X \to Y$ a surjective map with complete metrizable fibers. Then f admits a regular averaging operator with compact supports if and only if there exists a function $\theta\colon \mathcal{T}_X \to \mathcal{T}_Y$ such that $\theta(U) \subset f(U)$ for all $U \in \mathcal{T}_X$ and θ satisfies conditions* (5) *and* (7).

Proof. Let M be a metric space and $g\colon X \to M$ a map such that $f \triangle g$ embeds X into $Y \times M$. Suppose there exists a function $\theta\colon \mathcal{T}_X \to \mathcal{T}_Y$ satisfying the conditions from Corollary 4.5. Consider the set-valued map $\Phi\colon Y \to M$, $\Phi(y) = g(f^{-1}(y))$, and define the function $\theta_1\colon \mathcal{T}_M \to \mathcal{T}_Y$ defined by $\theta_1(V) = \theta(g^{-1}(V))$. Then θ_1 satisfies conditions (5) - (7). So, by Theorem 4.3, Φ admits a lower semi-continuous selection ϕ_1. It is easily seen that the map $\phi\colon Y \to X$, $\phi(y) = (f\triangle g)^{-1}(y \times \phi_1(y))$, is lower semi-continuous and $\phi(y) \subset f^{-1}(y)$ for all $y \in Y$. Therefore, according to Proposition 4.1, f admits a regular averaging operator with compact supports.

If f admits a regular averaging operator u with compact supports, the support map $S_u\colon Y \to X$ is a lower semi-continuous selection for the map f^{-1}. Then the function $\theta\colon \mathcal{T}_X \to \mathcal{T}_Y$, $\theta(U) = S_u^{-1}(U)$, satisfies conditions (5) and (7), and $\theta(U) \subset f(U)$ for all $U \in \mathcal{T}_X$.

The case of Corollary 4.5 when X is a metric compactum and f satisfies conditions (5), (6) and (7′) was established in [3, Theorem 10]. Another description of surjective maps between compacta (not necessarily metrizable) admitting lower semi-continuous selections, which is quite different from the above one, was obtained in [7, Corollary 4.3].

5 Averaging operators with semi-continuous values

Suppose $f\colon X \to Y$ is a surjective map. In this section we consider operators $u\colon C^*(X) \to C^*_{lsc}(Y)$ or $u\colon C^*(X) \to C^*_{usc}(Y)$, where $C^*_{lsc}(X)$ (resp., $C^*_{usc}(X)$) is the set of all bounded lower (resp., upper) semi-continuous functions on X. As above, any such an operator is said to be averaging for f if $S(\mu_y) \subset f^{-1}(y)$ for all $y \in Y$, where μ_y are the functionals on $C^*(X)$ generated by u.

Here is a result analogical to Theorem 3.3.

Theorem 5.1 *For any surjective map $f\colon X \to Y$ the following conditions are equivalent:*

(i) The map f admits s a normed weakly additive averaging operator $u\colon C^*(X) \to C^*_{usc}(Y)$ with compact supports such that u preserves min *and weakly preserves* max*;*

(ii) The map f admits a normed weakly additive averaging operator $u\colon C^*(X) \to C^*_{lsc}(Y)$ with compact supports such that preserves max *and weakly preserves* min*;*

(iii) There exists a lower semi-continuous map $\Phi\colon Y \to X$ with compact nonempty values such that $\Phi(y) \subset f^{-1}(y)$ for all $y \in Y$.

P r o o f. First, let us observed that conditions (i) and (ii) are equivalent. Indeed, if u satisfies (i), then the operator v, $v(h) = -u(-h)$, satisfies (ii). Similarly, (ii) implies (i). So, it suffices to prove that (i) is equivalent to (iii). Suppose $u\colon C^*(X) \to C^*_{usc}(Y)$ is a normed, weakly additive averaging operator of f with compact supports such that u preserves min and weakly preserves max. Then each functional μ_y, $y \in Y$, is normed, weakly additive preserving min and weakly preserving max. Moreover $S(\mu_y) \subset f^{-1}(y)$. By Proposition 2.2 and Proposition 2.5, the support map S_u is lower semi-continuous. This implies $(i) \Rightarrow (iii)$. To prove the implication $(iii) \Rightarrow (i)$, we define $u(h)(y) = \min\{h(x) : x \in \Phi(y)\}$, $h \in C^*(X)$, where $\Phi\colon Y \to X$ is a lower semi-continuous selection for the map f^{-1} with nonempty compact values. It is easily seen that $u(h) \in C^*_{usc}(Y)$ for any $h \in C^*(X)$, u is normed, weakly additive, preserves min and weakly preserves max. It also follows that $S(\mu_y) = \Phi(y)$, $y \in Y$. Hence, u is an averaging operator for f.

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