



# Homogeneous ANR-spaces and Alexandroff manifolds



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## ABSTRACT

We specify a result of Yokoi [18] by proving that if  $G$  is an abelian group and  $X$  is a homogeneous metric ANR compactum with  $\dim_G X = n$  and  $\check{H}^n(X; G) \neq 0$ , then  $X$  is an  $(n, G)$ -bubble. This implies that any such space  $X$  has the following properties:  $\check{H}^{n-1}(A; G) \neq 0$  for every closed separator  $A$  of  $X$ , and  $X$  is an Alexandroff manifold with respect to the class  $D_G^{n-2}$  of all spaces of dimension  $\dim_G \leq n - 2$ . We also prove that if  $X$  is a homogeneous metric continuum with  $\check{H}^n(X; G) \neq 0$ , then  $\check{H}^{n-1}(C; G) \neq 0$  for any partition  $C$  of  $X$  such that  $\dim_G C \leq n - 1$ . The last provides a partial answer to a question of Kallipoliti and Papasoglu [8].

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## 1. Introduction

In this paper we establish some properties of homogeneous metric compacta. One of the main problems concerning homogeneous compacta is the Bing and Borsuk [2] question whether any closed separator of an  $n$ -dimensional homogeneous metric ANR-space is cyclic in dimension  $n - 1$ . Yokoi's result [18, Theorem 3.3] provides a partial answer to this question. Our first result is a clarification of [18, Theorem 3.3], we omit the requirement  $G$  to be a principal ideal domain.

**Theorem 1.1.** *Let  $X$  be a homogeneous metric ANR-continuum with cohomological dimension  $\dim_G X = n$  and  $\check{H}^n(X; G) \neq 0$ , where  $G$  is an abelian group. Then  $X$  is an  $(n, G)$ -bubble.*

Following Yokoi [18], a compactum  $X$  is called an  $(n, G)$ -bubble if  $\check{H}^n(X; G) \neq 0$  and  $\check{H}^n(A; G) = 0$  for every closed proper set  $A \subset X$ . This is a reformulation of the notion of an  $n$ -bubble introduced by Kuperberg [13] and Choi [4], see also Karimov and Repovš [9] for the stronger notion of an  $\check{H}^n$ -bubble.

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**Corollary 1.2.** *Let  $X$  be a homogeneous metric ANR-continuum with  $\check{H}^n(X; G) \neq 0$  and  $\dim_G X = n$ . Then*

- (i)  $X$  is a strong  $V_G^n$ -continuum;
- (ii)  $X$  is an Alexandroff manifold with respect to the class  $D_G^{n-2}$  of all spaces of dimension  $\dim_G \leq n - 2$ ;
- (iii) If  $A$  is a closed separator of  $X$ , then  $\check{H}^{n-1}(A; G) \neq 0$ .

Item (ii) from [Corollary 1.2](#) was proved in [\[11\]](#) under the additional requirement that  $G$  is a principal ideal domain. Here,  $\check{H}^n(X; G)$  denotes the reduced  $n$ -th Čech cohomology group of  $X$  with coefficients from  $G$ . We say that a set  $A \subset X$  is massive if  $A$  has a non-empty interior in  $X$ .

Recall that a space  $X$  is an *Alexandroff manifold with respect to a class  $\mathcal{C}$* , see [\[10\]](#) and [\[16\]](#), if for every two disjoint, closed massive sets  $X_0, X_1 \subset X$  there exists an open cover  $\omega$  of  $X$  such that there is no partition  $P$  in  $X$  between  $X_0$  and  $X_1$  admitting an  $\omega$ -map onto a space  $Y \in \mathcal{C}$ . This definition is inspired by the Alexandroff notion of  $V^n$ -continua [\[1\]](#) which is obtained when  $\mathcal{C}$  is the class of all compacta whose covering dimension is  $\leq n - 2$ .

A compactum  $X$  is said to be a  $V_G^n$ -continuum [\[15\]](#) if for every two open, disjoint subsets  $U_1, U_2$  of  $X$  there exists an open cover  $\omega$  of  $X_0 = X \setminus (U_1 \cup U_2)$  such that any partition  $P$  in  $X$  between  $U_1$  and  $U_2$  does not admit an  $\omega$ -map  $g$  of  $P$  onto a space  $Y$  with  $g^*: \check{H}^{n-1}(Y; G) \rightarrow \check{H}^{n-1}(P; G)$  being trivial. If, in addition, there exists also an element  $e \in \check{H}^{n-1}(X_0; G)$  such that for any partition  $P$  between  $U_1$  and  $U_2$  and any  $\omega$ -map  $g$  of  $P$  onto a space  $Y$  we have  $0 \neq i_P^*(e) \in g^*(\check{H}^{n-1}(Y; G))$ , where  $i_P$  is the embedding  $P \hookrightarrow X_0$ ,  $X$  is called a *strong  $V_G^n$ -continuum*. For example, every  $(n, G)$ -bubble is a strong  $V_G^n$ -continuum, see [\[11\]](#).

It follows directly from the above definitions that  $V_G^n$ -continua are Alexandroff manifolds with respect to the class  $D_G^{n-2}$  of all spaces of dimension  $\dim_G \leq n - 2$ . The converse is not true, for example the Menger  $n$ -dimensional compactum is a  $V^n$ -continuum, but it is not a  $V_G^n$ -continuum for any group  $G$ , see [\[11\]](#).

Homogeneous metric compacta (not necessary ANR) are also interesting class of spaces. Krupski [\[12\]](#) has shown that any such an  $n$ -dimensional space is a Cantor  $n$ -manifold. One of the ingredients of Krupski's proof is the classical result established by Hurewicz and Menger [\[6\]](#) and Tumarkin [\[17\]](#) that any  $n$ -dimensional compactum contains an  $n$ -dimensional Cantor  $n$ -manifold. Kuz'minov [\[14\]](#) provided a cohomological counterpart of this fact about  $V^n$ -continua (see [\[10\]](#) for more general results). Concerning  $V_G^n$ -continua we have the following statement which provides a positive answer of Question 4.3 from [\[11\]](#):

**Theorem 1.3.** *Any compactum  $X$  with  $\check{H}^n(X; G) \neq 0$  contains a strong  $V_G^n$ -continuum.*

[Theorem 1.3](#) could be also compared with the cohomological version of Kuperberg's result [\[13, Theorem 5.5\]](#) that any  $n$ -dimensional  $n$ -cyclic metric compactum  $X$  for which  $\check{H}^n(X; \mathbb{Z})$  is finitely generated (in particular, any  $n$ -dimensional  $n$ -cyclic ANR) contains an  $n$ -bubble.

The condition  $\check{H}^n(X; G) \neq 0$  in the above theorem is essential. For example, let  $X$  be the square  $\mathbb{I}^2$  and suppose  $X$  contains a (strong)  $V_{\mathbb{Z}}^2$ -continuum  $K$ , where  $\mathbb{Z}$  is the group of all integers. Then  $\dim K = 2$ , so  $K$  contains a non-empty interior  $U$  in  $X$ . Now, take a segment  $\mathbb{I}$  joining two opposite sides of  $\mathbb{I}^2$  and intersecting  $U$ . Obviously  $\mathbb{I} \cap K$  is a partition of  $K$ . Since  $\check{H}^{n-1}(P; G) \neq 0$  for every partition  $P$  of a  $V_G^n$ -continuum,  $\check{H}^1(\mathbb{I} \cap K; \mathbb{Z}) \neq 0$ . On the other hand, because  $\mathbb{I} \cap K$  is an one-dimensional subset of  $\mathbb{I}$ ,  $\check{H}^1(\mathbb{I} \cap K; \mathbb{Z}) = 0$ .

Kuperberg [\[13\]](#) asked whether any  $n$ -dimensional metric compactum contains an  $(n - 1)$ -bubble. This question is still open, but the following corollary provides a result in this direction.

**Corollary 1.4.** *Any compactum  $X$  with  $\dim_G X = n$  contains a strong  $V_G^{n-1}$ -continuum.*

For finite-dimensional metric compacta and  $V_G^n$ -continua [Theorem 1.3](#) and [Corollary 1.4](#) were established in [\[15\]](#).

**Proposition 1.5.** *Let  $X$  be a homogeneous metric continuum with  $\check{H}^n(X; G) \neq 0$ . Then for any partition  $C$  of  $X$  there exists an open cover  $\omega$  of  $X$  such that  $C$  does not admit any  $\omega$ -map  $g: C \rightarrow Y$  onto a space of dimension  $\dim_G Y \leq n - 1$  with  $g^* : \check{H}^{n-1}(Y; G) \rightarrow \check{H}^{n-1}(C; G)$  being a trivial homomorphism.*

Let us note that the  $n$ -dimensional universal Menger compactum  $\mu^n$ , which is a homogeneous continuum cyclic in dimension  $n$ , contains a separator  $C$  such that  $\dim C = n$  and  $\check{H}^{n-1}(C; G) = 0$ , see [11, Corollary 2.6]. Therefore, the restriction  $\dim_G Y \leq n - 1$  in Proposition 1.5 and the condition  $\dim_G P \leq n - 1$  in next corollary are essential.

**Corollary 1.6.** *Let  $X$  be a homogeneous metric continuum such that  $\check{H}^n(X; G) \neq 0$ . Then  $\check{H}^{n-1}(P; G) \neq 0$  for every partition  $P$  of  $X$  with  $\dim_G P \leq n - 1$ .*

Kallipoliti and Papasoglu [8] have shown that every 2-dimensional locally connected, simply connected homogeneous metric continuum cannot be separated by an arc, and asked if the simple connectedness can be dropped from this result. Corollary 1.6 provides a partial answer to the Kallipoliti–Papasoglu question.

## 2. Cohomological carriers

In this section we consider cohomological carriers of non-trivial elements of  $\check{H}^n(X; G)$  and establish some properties of them. We fix an abelian group  $G$ , an integer  $n$  and a metric compactum  $X$  with  $\check{H}^n(X; G) \neq 0$ . A closed non-empty set  $A \subset X$  is said to be a *cohomological carrier* (shortly, a carrier) of a non-zero element  $\alpha \in \check{H}^n(X; G)$  if  $i_A^*(\alpha) \neq 0$  and  $i_B^*(\alpha) = 0$  for every proper closed subset  $B \subset A$ , where  $i_A$  denotes the inclusion map  $A \hookrightarrow X$ .

**Lemma 2.1.** *For every non-zero element  $\alpha \in \check{H}^n(X; G)$  there exists a carrier. Moreover, if  $A$  is a carrier of  $\alpha$ , then  $\check{H}^{n-1}(P; G) \neq 0$  for any closed partition  $P$  of  $A$ .*

**Proof.** The first part of Lemma 2.1 follows from Zorn’s lemma and the continuity of Čech cohomology. For the second part, suppose  $A$  is a carrier of  $\alpha$  and  $P$  a partition of  $A$ . Then there exist two closed proper subsets  $A_1$  and  $A_2$  of  $A$  such that  $A = A_1 \cup A_2$  and  $P = A_1 \cap A_2$ . Consider the Mayer–Vietoris exact sequence

$$\check{H}^{n-1}(P; G) \rightarrow \check{H}^n(A; G) \rightarrow \check{H}^n(A_1; G) \oplus \check{H}^n(A_2; G).$$

For every  $i = 1, 2$  let  $\partial_i: \check{H}^n(A; G) \rightarrow \check{H}^n(A_i; G)$  be generated by the inclusion  $A_i \hookrightarrow A$ . Denote also by  $\Delta$  and  $\varphi$ , respectively, the left and right homomorphism of the above sequence. Since each  $A_i$  is a proper subset of  $A$  we have  $\varphi(\beta) = (\partial_1(\beta), \partial_2(\beta)) = 0$ , where  $\beta = i_A^*(\alpha)$ . So, there exists  $\gamma \in \check{H}^{n-1}(P; G)$  with  $\Delta(\gamma) = \beta$ . Because  $\beta$  is a non-trivial element of  $\check{H}^n(A; G)$ , so is  $\gamma$ . Hence,  $\check{H}^{n-1}(P; G) \neq 0$ .  $\square$

Everywhere below, if  $B \subset A$ , then  $i_{A,B}$  denotes the inclusion  $B \hookrightarrow A$ . The next lemma is an analogue of Lemma 4 from [4].

**Lemma 2.2.** *Let  $A \subset X$  be a carrier of a non-trivial element  $\alpha \in \check{H}^n(X; G)$  and  $B$  a closed subset of  $X$ . Then  $A \subset B$  if and only if  $\text{Ker}(j_B^*) \subset \text{Ker}(j_A^*)$ , where  $j_A = i_{A \cup B, A}$  and  $j_B = i_{A \cup B, B}$  are the corresponding inclusions.*

**Proof.** Obviously  $A \subset B$  implies  $\text{Ker}(j_B^*) \subset \text{Ker}(j_A^*)$ . Suppose that  $\text{Ker}(j_B^*) \subset \text{Ker}(j_A^*)$ , but  $B$  does not contain  $A$ . Then  $A \cap B$  is a proper closed subset of  $A$  (possibly empty). The left homomorphism in the Mayer–Vietoris exact sequence

$$\check{H}^n(A \cup B; G) \rightarrow \check{H}^n(A; G) \oplus \check{H}^n(B; G) \rightarrow \check{H}^n(A \cap B; G)$$

is defined by  $(j_A^*, j_B^*)$ , while the right one  $i^*$  assigns to each  $(\beta_1, \beta_2) \in \check{H}^n(A; G) \oplus \check{H}^n(B; G)$  the difference  $i_{A, A \cap B}^*(\beta_1) - i_{B, A \cap B}^*(\beta_2)$ . Since  $A \cap B$  is a proper subset of  $A$ ,  $i_{A \cap B}^*(\alpha) = 0$ . Then  $i^*((i_A^*(\alpha), 0)) = 0$ . Consequently, there exists  $\beta \in \check{H}^n(A \cup B; G)$  with  $(j_A^*(\beta), j_B^*(\beta)) = (i_A^*(\alpha), 0)$ . So,  $\beta \in \text{Ker}(j_B^*)$  and, according to our assumption,  $\beta \in \text{Ker}(j_A^*)$ . The last relation contradicts  $i_A^*(\alpha) \neq 0$ . Therefore,  $A \subset B$ .  $\square$

The next proposition is actually Theorem 5 from [4]. We provide a different proof of that theorem.

**Proposition 2.3.** *Let  $A \subset X$  be a carrier for a non-trivial element of  $\check{H}^n(X; G)$  and  $f: X \rightarrow X$  a map homotopic to the identity on  $X$ . If  $\dim_G X \leq n$ , then  $A \subset fA$ .*

**Proof.** By [7], we can identify the cohomological group  $\check{H}^n(A \cup fA; G)$  with the group  $[A \cup fA, K(G, n)]$  of homotopy classes from  $A \cup fA$  to  $K(G, n)$ , where  $n > 0$  and  $K(G, n)$  denotes an Eilenberg–MacLane complex. Similarly,  $\check{H}^n(A; G)$  and  $\check{H}^n(fA; G)$  are identified with the groups  $[A, K(G, n)]$  and  $[fA, K(G, n)]$ , respectively.

By Lemma 2.2, it suffices to prove that if  $\alpha \in \text{Ker}(j_{fA}^*)$  then  $\alpha \in \text{Ker}(j_A^*)$ . So, we fix  $\alpha \in \check{H}^n(A \cup fA; G)$  with  $j_{fA}^*(\alpha) = 0$ . According to the above identifications, there exists a map  $g: A \cup fA \rightarrow K(G, n)$  such that  $\alpha = [g]$ . Since  $\dim_G X \leq n$ ,  $g$  can be extended to a map  $\tilde{g}: X \rightarrow K(G, n)$ . Because  $j_{fA}^*(\alpha) = 0$ , we can find a homotopy  $H_1: fA \times [0, 1] \rightarrow K(G, n)$  with  $H_1(x, 0) = g(x)$  and  $H_1(x, 1) = *$  for all  $x \in fA$ , where  $*$  is a point from  $K(G, n)$ . Then the homotopy  $H_2: A \times [0, 1] \rightarrow K(G, n)$ ,  $H_2(x, t) = H_1(f(x), t)$ , connects the constant map  $\kappa: A \hookrightarrow *$  and the map  $h: A \rightarrow K(G, n)$  defined by  $h(x) = g(f(x))$ . Next, consider a homotopy  $F: A \times [0, 1] \rightarrow X$  with  $F(x, 0) = f(x)$  and  $F(x, 1) = x$ , and define  $H: A \times [0, 1] \rightarrow K(G, n)$  by  $H(x, t) = \tilde{g}(F(x, t))$ . We have  $H(x, 0) = g(f(x))$  and  $H(x, 1) = g(x)$  for all  $x \in A$ . Hence,  $H$  is connecting the maps  $h$  and the restriction  $g|_A$  of  $g$  over  $A$ . Finally, combining  $H$  and  $H_2$ , we can produce a homotopy on  $A$  connecting the map  $g|_A$  and the constant map  $\kappa$ . Hence,  $j_A^*(\alpha) = [g|_A] = 0$ .  $\square$

Before proving the next property of carriers, we introduce some more notations. If  $\omega$  is a finite open cover of a closed set  $Z \subset X$ , we denote by  $|\omega|$  and  $p_\omega$ , respectively, the nerve of  $\omega$  and a map from  $Z$  onto  $|\omega|$  generated by a partition of unity subordinated to  $\omega$ . Furthermore, if  $C \subset Z$  and  $\omega(C) = \{W \cap C : W \in \omega\}$ , then  $p_{\omega(C)}: C \rightarrow |\omega(C)|$  is the restriction  $p_\omega|_C$ . Recall also that  $p_\omega$  generates maps  $p_\omega^*: \check{H}^k(|\omega|; G) \rightarrow \check{H}^k(Z; G)$  for  $k \geq 0$ . Moreover, if  $q_\omega: Z \rightarrow |\omega|$  is a map generating by (another) partition of unity subordinated to  $\omega$ , then  $p_\omega$  and  $q_\omega$  are homotopic. So,  $p_\omega^* = q_\omega^*$ .

**Proposition 2.4.** *Let  $K$  be a carrier for a non-trivial element of  $\check{H}^n(X; G)$ . Then for any two open disjoint subsets  $U_1$  and  $U_2$  of  $K$  there exists an open cover  $\omega$  of  $K \setminus (U_1 \cap U_2)$  and an element  $\eta \in \check{H}^{n-1}(|\omega|; G)$  such that  $p_{\omega(C)}^*(i_{\omega(C)}^*(\eta)) \neq 0$  for every partition  $C$  of  $K$  between  $U_1$  and  $U_2$ , where  $i_{\omega(C)}$  is the inclusion  $|\omega(C)| \hookrightarrow |\omega|$ .*

**Proof.** Let  $K_1 = K \setminus U_1$ ,  $K_2 = K \setminus U_2$ ,  $C$  be a partition of  $K$  between  $U_1$  and  $U_2$ , and  $F_1, F_2$  closed subsets of  $K$  such that:  $F_1 \cap F_2 = C$ ,  $F_1 \cup F_2 = K$ ,  $F_1 \subset K_1$  and  $F_2 \subset K_2$ . Consider the commutative diagram whose rows are Mayer–Vietoris sequences:

$$\begin{CD} \check{H}^{n-1}(K_1 \cap K_2; G) @>\delta>> \check{H}^n(K; G) @>j>> \check{H}^n(K_1; G) \oplus \check{H}^n(K_2; G) \\ @VVi^*V @VVidV @VVi_1^* \oplus i_2^*V \\ \check{H}^{n-1}(C; G) @>\delta_1>> \check{H}^n(K; G) @>j_1>> \check{H}^n(F_1; G) \oplus \check{H}^n(F_2; G). \end{CD}$$

Since  $j(\beta) = 0$ , where  $\beta = i_K^*(\alpha)$ , there exists a non-zero element  $\gamma \in \check{H}^{n-1}(K_1 \cap K_2; G)$  with  $\delta(\gamma) = \beta$ . Consequently, we can find an open cover  $\omega$  of  $K_1 \cap K_2$  and a non-trivial element  $\eta \in \check{H}^{n-1}(|\omega|; G)$  such that  $p_\omega^*(\eta) = \gamma$ . It follows from the commutativity of the above diagram that  $i^*(\gamma) \neq 0$ . Then the equality  $p_{\omega(C)}^*(i_{\omega(C)}^*(\eta)) = i^*(\gamma)$  completes the proof.  $\square$

**Proposition 2.5.** *Every carrier for a non-trivial element of  $\check{H}^n(X; G)$  is a strong  $V_G^n$ -continuum.*

**Proof.** Indeed, suppose  $U_1$  and  $U_2$  are open subsets of  $K$  having disjoint closures. Let  $\omega$  be an open cover of  $K_0 = K \setminus (U_1 \cup U_2)$  and  $e \in \check{H}^{n-1}(|\omega|; G)$  a non-trivial element satisfying the hypotheses of Proposition 2.4. Assume  $C$  a partition of  $K$  between  $U_1$  and  $U_2$  admitting an  $\omega$ -map  $g$  onto a space  $T$ . Thus, we can find a finite open cover  $\tau$  of  $T$  such that  $\nu = g^{-1}(\tau)$  is refining  $\omega$ . Let  $p_\nu: C \rightarrow |\nu|$  be a map onto the nerve of  $\nu$  generated by a partition of unity subordinated to  $\nu$ . Obviously, the function  $V \in \tau \rightarrow g^{-1}(V) \in \nu$  provides a simplicial homeomorphism  $g_\tau^*: |\tau| \rightarrow |\nu|$ . Then the maps  $p_\nu$  and  $g_\tau = g_\nu^* \circ \pi_\tau \circ g$ , where  $\pi_\tau: T \rightarrow |\tau|$  is a map generated by a partition of unity subordinated to  $|\tau|$ , are homotopic. Hence,  $p_\nu^* = g^* \circ \pi_\tau^* \circ (g_\nu^*)^*$ .

On the other hand, since  $\nu$  refines  $\omega$ , we can find a map  $\varphi_\nu: |\nu| \rightarrow |\omega(C)|$  such that  $p_{\omega(C)}$  and  $\varphi_\nu \circ p_\nu$  are homotopic. Therefore,  $p_{\omega(C)}^* = p_\nu^* \circ \varphi_\nu^*$ . According to Proposition 2.4, there exists  $\eta \in \check{H}^{n-1}(|\omega|; G)$  with  $p_{\omega(C)}^*(i_{\omega(C)}^*(\eta)) \neq 0$ . Since  $i_C^*(p_\nu^*(\eta)) = p_{\omega(C)}^*(i_{\omega(C)}^*(\eta))$ ,  $e = p_\omega^*(\eta)$  is a non-zero element of  $\check{H}^{n-1}(K_0; G)$ . Here  $p_\omega: K_0 \rightarrow |\omega|$  is a map generated by a partition of unity subordinated to  $\omega$  and  $i_C: C \hookrightarrow K_0$  is the inclusion map. Moreover, both equalities  $p_\nu^* = g^* \circ \pi_\tau^* \circ (g_\nu^*)^*$  and  $p_{\omega(C)}^* = p_\nu^* \circ \varphi_\nu^*$  yield that  $i_C^*(e)$  is a non-trivial element of  $g^*(\check{H}^{n-1}(T; G))$ .  $\square$

### 3. Proof of Theorem 1.1 and Corollary 1.2

**Proof of Theorem 1.1.** Suppose  $G$  is an abelian group,  $X$  is a non-trivial homogeneous metric ANR-continuum with  $\dim_G X = n$  and  $\check{H}^n(X; G) \neq 0$ . Since  $X$  is an ANR,  $n \geq 1$  and there exists a positive  $\epsilon$  such that any two  $\epsilon$ -close maps from  $X$  into  $X$  are homotopic (we say that two maps  $f_1, f_2: X \rightarrow X$  are  $\epsilon$ -close if  $\text{dist}(f_1(x), f_2(x)) < \epsilon$  for each  $x \in X$ ).

First, we are going to prove that any carrier of a non-trivial element of  $\check{H}^n(X; G)$  is the whole space  $X$ . Indeed, suppose  $A \subset X$  is a carrier of a non-trivial  $\alpha \in \check{H}^n(X; G)$ , but  $A$  is a proper subset of  $X$ . According to the Effros theorem [5], there corresponds a positive number  $\delta$  with the following property: whenever  $x$  and  $y$  are points from  $X$  and  $\text{dist}(x, y) < \delta$ , there is a homeomorphism  $h: X \rightarrow X$  such that  $h(x) = y$  and  $h$  is  $\epsilon$ -close to the identity  $id_X$  on  $X$ . Because  $A \neq X$ , we can choose points  $a \in A$  and  $b \notin A$  with  $\text{dist}(a, b) < \delta$ . Consequently, there would be a homeomorphism  $f: X \rightarrow X$  such that  $f(a) = b$  and  $\text{dist}(f, id_X) < \epsilon$ . Obviously,  $g = f^{-1}: (X, fA) \rightarrow (X, A)$  generates the isomorphism  $g^*: \check{H}^n(X; G) \rightarrow \check{H}^n(X; G)$  and  $B = f(A)$  is a carrier for the element  $g^*(\alpha)$ . Moreover, the homeomorphism  $g$  is also  $\epsilon$ -close to  $id_X$ . Hence,  $g$  is homotopic to  $id_X$ . Applying Proposition 2.3 to the carrier  $B$  and the homeomorphism  $g$ , we obtain that  $B \subset g(B) = A$  which contradicts  $b \in B \setminus A$ . Hence,  $A$  should be the whole space  $X$ .

Now we can finish the proof. Assume there exists a proper subset  $B \subset X$  with  $\check{H}^n(B; G) \neq 0$ , and choose a non-trivial element  $\beta \in \check{H}^n(B; G)$ . Since  $\dim_G X = n$ , there exists  $\alpha \in \check{H}^n(X; G)$  with  $\beta = i_B(\alpha)$ . Because the carrier of  $\alpha$  is  $X$  and  $B$  is a proper subset of  $X$ ,  $i_B(\alpha) = 0$ , a contradiction.  $\square$

**Proof of Corollary 1.2.** Item (i) follows from Proposition 2.5. Item (ii) follows from the simple observation that any  $V_G^n$ -continuum is an Alexandroff manifold with respect to the class  $D_G^{n-2}$ . Since  $X$  is a carrier for every non-trivial  $\alpha \in \check{H}^n(X; G)$ , item (iii) follows from Lemma 2.1.  $\square$

### 4. $V_G^n$ -continua

In this section we provide the proofs of Theorem 1.3, Corollary 1.4 and Proposition 1.5 and Corollary 1.6.

**Proof of Theorem 1.3.** Since  $\check{H}^n(X; G) \neq 0$ ,  $X$  contains a carrier  $A$  of a non-trivial element of  $\check{H}^n(X; G) \neq 0$ . Then, according to Proposition 2.5,  $A$  is a strong  $V_G^n$ -continuum.  $\square$

**Proof of Corollary 1.4.** This corollary follows directly from Theorem 1.3 because every compactum  $X$  with  $\dim_G X = n$  contains a closed subset  $F$  such that  $\check{H}^{n-1}(F; G) \neq 0$  (see, for example, [14]).  $\square$

**Proof of Proposition 1.5.** Suppose there exists a partition  $C$  of  $X$  such that for every open cover  $\omega$  of  $X$ ,  $C$  admits an  $\omega$ -map  $g_\omega: C \rightarrow Y_\omega$  onto a space of dimension  $\dim_G Y_\omega \leq n - 1$  with  $g_\omega^*: \check{H}^{n-1}(Y_\omega; G) \rightarrow \check{H}^{n-1}(C; G)$  being a trivial homomorphism. Then, according to [3, Theorem 2.4],  $\dim_G C \leq n - 1$ . Obviously, the boundary  $B$  of  $C$  in  $X$  is also a partition of  $X$  and  $\dim_G B \leq n - 1$ . Moreover, we have the commutative diagram below, where  $g_\omega|_B: B \rightarrow g_\omega(B)$  is the restriction of  $g_\omega$

$$\begin{array}{ccc} \check{H}^{n-1}(Y_\omega; G) & \xrightarrow{g_\omega^*} & \check{H}^{n-1}(C; G) \\ \downarrow i_{g(B)}^* & & \downarrow i_B^* \\ \check{H}^{n-1}(g_\omega(B); G) & \xrightarrow{(g_\omega|_B)^*} & \check{H}^{n-1}(B; G) \end{array}$$

Since  $\dim_G Y_\omega \leq n - 1$ ,  $i_{g(B)}^*$  is a surjection. This implies that  $(g_\omega|_B)^*$  is the trivial homomorphism because so is  $g_\omega^*$ . Therefore, considering  $B$  instead of  $C$ , we may assume that  $C$  does not have interior points in  $X$ . The above diagram also shows that for every closed subset  $A \subset C$  and every  $\omega$  the restriction  $g_\omega|_A$  is an  $\omega$ -map onto  $g_\omega(A)$  such that  $(g_\omega|_A)^*: \check{H}^{n-1}(g_\omega(A); G) \rightarrow \check{H}^{n-1}(A; G)$  is the trivial homomorphism.

By Theorem 1.3, there exists a strong  $V_G^n$ -continuum  $K \subset X$ . Since  $X$  is homogeneous, we may also assume that  $K \cap C \neq \emptyset$ . Observe that  $z \in K \setminus C$  for some  $z$ . Indeed, the inclusion  $K \subset C$  would imply that if  $P$  is a partition of  $K$  and  $\gamma$  any open cover of  $K$ , then  $P$  admits a  $\gamma$ -map  $h_\gamma$  onto a space  $T$  such that  $(h_\gamma)^*: \check{H}^{n-1}(T_\gamma; G) \rightarrow \check{H}^{n-1}(P; G)$  is trivial. This would contradict the fact that  $K$  is a strong  $V_G^n$ -continuum. Let  $X \setminus C = U \cup V$  and  $z \in V$ , where  $U$  and  $V$  are nonempty, open and disjoint sets in  $X$ . Then the Effros theorem [5] allows us to push  $K$  towards  $U$  by a small homeomorphism  $h: X \rightarrow X$  so that the image  $h(K)$  meets both  $U$  and  $V$  (see the proof of Lemma 2 from [12] for a similar application of Effros' theorem). Therefore,  $S = h(K) \cap C$  is a partition of  $h(K)$  such that for any  $\omega$  the restriction  $g_\omega|_S$  is an  $\omega$ -map generating a trivial homomorphism  $(g_\omega|_S)^*$ , a contradiction.  $\square$

**Proof of Corollary 1.6.** It follows directly from Proposition 1.5.  $\square$

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