

ALEXANDROFF TYPE MANIFOLDS AND HOMOLOGY MANIFOLDS

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ABSTRACT. We introduce and investigate the notion of (strong) K_G^n -manifolds, where G is an abelian group. One of the result related to that notion (Theorem 3.3) implies the following partial answer to the Bing-Borsuk problem [6], whether any partition of a homogeneous metric ANR -space X of dimension n is cyclic in dimension $n-1$: If X is a homogeneous metric ANR -continuum with $\dim_G X = n$ and $H^n(X; G) \neq 0$, then $H^{n-1}(M; G) \neq 0$ for every set $M \subset X$, which cuts X between two disjoint open subsets of X . Another implication of Theorem 3.3 (Corollary 3.5) provides an analog of the classical result of Mazurkiewicz [26] that no region in \mathbb{R}^n can be cut by a subset of dimension $\leq n-2$. Concerning homology manifolds, it is shown that if X is arcwise connected complete metric space which is either a homology n -manifold over a group G or a product of at least n metric spaces, then X is a Mazurkiewicz arc n -manifold. We also introduce a property which guarantees that $H_k(X, X \setminus x; G) = 0$ for every $x \in X$ and $k \leq n-1$, where X is a homogeneous locally compact metric ANR .

1. INTRODUCTION

In this paper we investigate some properties of generalized Cantor manifolds. Cantor manifolds were introduced by Urysohn [33] as a generalization of Euclidean manifolds. It appeared that Euclidean manifolds have a richer structure and that was a motivation for the study of further specifications of Cantor manifolds. Another interesting fact is that homogeneous metric $ANRs$ have some common properties with generalized Cantor manifolds (see [19], [21], [22]), and the same is

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true for homology manifolds (see Corollary 4.2 below). This is not so surprising having in mind the Bing-Borsuk conjecture that every homogeneous separable locally compact metric ANR of dimension n is a homology n -manifold [6, Remark, pp.106-107].

Recall that a space X is a *Cantor n -manifold* if any partition of X is of dimension $\geq n - 1$ [33] (a partition of X is a closed set $P \subset X$ such that $X \setminus P$ is the union of two open nonempty disjoint sets). In other words, X cannot be the union of two proper closed sets whose intersection is of covering dimension $\leq n - 2$. *Strong Cantor manifolds* is another specification of Cantor manifolds which was considered by Hadžiivanov [13]. Hadžiivanov and Todorov [15] introduced the class of *Mazurkiewicz n -manifolds*, which is a proper sub-class of the strong Cantor n -manifolds. This notion has its roots in the classical Mazurkiewicz theorem [26] that no region X in the Euclidean n -space can be cut by a subset M with $\dim M \leq n - 2$ in following sense: any two points from $X \setminus M$ can be joined by a continuum $K \subset X \setminus M$.

But the strongest specification of Cantor manifolds is the notion of V^n -continua introduced by Alexandroff [1]: a continuum X is a *V^n -continuum* if for every two closed disjoint subsets X_0, X_1 of X , both having non-empty interiors, there exists an open cover ω of X such that there is no partition P in X between X_0 and X_1 admitting an ω -map into a space Y with $\dim Y \leq n - 2$ ($f: P \rightarrow Y$ is said to be an ω -map if there exists an open cover γ of Y such that $f^{-1}(\gamma)$ refines ω).

The above notions are related as follows: strong Cantor n -manifolds are Cantor n -manifolds, every V^n -continuum is a Mazurkiewicz n -manifold and Mazurkiewicz n -manifolds are strong Cantor n -manifolds [15]. None of the above inclusions is reversible [19].

More general concepts of the above notions were considered in [19] and [30]. In particular, we are going to use the following two, where \mathcal{C} is a class of topological spaces.

Definition 1. A connected space X is an *Alexandroff manifold with respect to \mathcal{C}* (br., *Alexandroff \mathcal{C} -manifold*) if for every two closed, disjoint subsets X_0, X_1 of X , both having non-empty interiors, there exists an open cover ω of X such that no partition P in X between X_0 and X_1 admits an ω -map onto a space $Y \in \mathcal{C}$.

Definition 2. A semi-continuum X is said to be a *Mazurkiewicz manifold with respect to \mathcal{C}* (br., *Mazurkiewicz \mathcal{C} -manifold*) provided for every two closed disjoint sets $X_0, X_1 \subset X$ with non-empty interiors, and every set $F = \bigcup_{i=0}^{\infty} F_i \subset X$ with each $F_i \in \mathcal{C}$ being proper closed subset of X , there exists a continuum K in $X \setminus F$

joining X_0 and X_1 . If, in the above definition, X_0 and X_1 can be joined by an arc in $X \setminus F$, X is called a *Mazurkiewicz arc \mathcal{C} -manifold*.

It is still unknown if every compact Alexandroff D_G^{n-2} -manifold is a Mazurkiewicz D_G^{n-2} -manifold, where D_G^{n-2} is the class of all spaces whose cohomological dimension \dim_G is $\leq n-2$. Searching for the answer of this question, we consider in Section 2 different types of connectedness between disjoint subsets of compacta. The notions of K_G^n -manifolds and strong K_G^n -manifolds, introduced in Section 2, play a crucial role in the paper. We also provide in Section 2 some examples of (strong) K_G^n -manifolds.

The main result in Section 3 is Theorem 3.3. A particular version of that theorem states that if (X, F) is a strong K_G^n -manifold, where X is a metric compactum, and $M \subset X$ is a set with $H^{n-1}(M, M \cap F; G) = 0$, then for any two disjoint nonempty open sets P and Q in X there exists a continuum $K \subset X \setminus M$ joining P and Q provided $M \subset X \setminus (P \cup Q)$. The last requirement can be avoided if $\dim_G M \leq n-1$. One corollary of this theorem providing a partial answer to the Bing-Borsuk question [6] was mentioned in the abstract. Another corollary is the following analog of the Mazurkiewicz theorem [26] cited above: If M is a bounded subset of \mathbb{R}^n with $\dim M \leq n-1$ and $H^{n-1}(M; \mathbb{Z}) = 0$, then every pair of nonempty disjoint open sets $P, Q \subset \mathbb{R}^n$ can be joined by a continuum in $\mathbb{R}^n \setminus M$.

The starting point for our considerations in Section 4 was the following result of Krupski [22, Proposition 1.7]: *Let X be a locally compact locally connected separable metric space such that for all $k < n$ and $x \in X$ the singular homology groups $H_k(X, X \setminus x)$ are trivial. Then every open connected subset of $U \subset X$ is a Cantor n -manifold.* We establish that, under the hypotheses of Krupski's result, every connected open subset of X is a Mazurkiewicz arc manifold with respect to the class of all spaces whose covering dimension is $\leq n-2$ (Theorem 4.1). The same conclusion also holds for arcwise connected open subsets of a complete metric space which is a product of n metric spaces. We also introduce the LS^n -property and show that LS^{n-2} implies $H_k(X, X \setminus x; G) = 0$ for every $x \in X$ and $k \leq n-1$ (see Theorem 4.5), where X is a homogeneous locally compact metric ANR. This improves a result of Mitchell [27].

Everywhere in this paper reduced Čech cohomology groups are considered. If it is not mentioned otherwise, we suppose that the coefficients of the cohomology groups are from a fixed group G and we omit the symbol for the group of coefficients.

Finally, let us raise the following questions (the second question was answered positively in [18] provided X is cyclic in dimension n):

Question 1.1. *Let X be a compact Alexandroff manifold with respect to the class D_G^{n-2} . Is it true that X is a Mazurkiewicz D_G^{n-2} -manifold? What about if (X, F) is a K_G^n -manifold or X is a V_G^n -continuum?*

Question 1.2. *Is it true that any homogeneous ANR-continuum X of dimension n is an Alexandroff $D_{\mathbb{Z}}^{n-2}$ -manifold?*

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2. ALEXANDROFF TYPES CONNECTEDNESS OF SPACES

In this section all spaces are assumed to be at least paracompact and G denotes an abelian group. For any space X the cohomological dimension $\dim_G X$ is the smallest integer n such that $H^{n+1}(X, A; G) = 0$ for all closed subsets $A \subset X$ (the reader is referred to [10] for basic facts on the cohomological dimension of compacta). If ω is an open cover of X and $Z \subset X$ is closed, we denote by $|\omega|$ and $|\omega_Z|$ the nerves of ω and the system $\omega_Z = \{U \cap Z : U \in \omega\}$. For any such ω and $Z \subset X$ let $p_{\omega_Z}^* : H^k(|\omega|, |\omega_Z|) \rightarrow H^k(X, Z)$, $k \geq 0$, be the projection between the k -th cohomology groups, where $p_{\omega_Z} : (X, Z) \rightarrow (|\omega|, |\omega_Z|)$ is a map generated by a partition of unity subordinated to ω (we call such a map p_{ω_Z} to be natural). Further, $i_A : (A, B) \rightarrow (X, Z)$ denotes the embedding of the pair (A, B) into the pair (X, Z) .

Definition 3. Let P and Q be disjoint nonempty open subsets of a continuum X and $F \subset X$ closed. We say that the pair (X, F) is K_G^n -connected between P and Q if there exist an open cover ω of $Y = X \setminus (P \cup Q)$ such that the following condition holds for every partition C of X between P and Q : any natural map $p_{\omega_C} : (C, C \cap F) \rightarrow (|\omega_C|, |\omega_{C \cap F}|)$ generates a non-trivial homomorphism

$$p_{\omega_C}^* : H^{n-1}(|\omega_C|, |\omega_{C \cap F}|) \rightarrow H^{n-1}(C, C \cap F).$$

If, in the above situation, there exists also $e \in H^{n-1}(|\omega|, |\omega_{F \cap Y}|)$ such that $p_{\omega_C}^*(i_{\omega_C}^*(e)) \neq 0$ for every partition C in X between P and Q , the pair (X, F) is called *strongly K_G^n -connected between P and Q* . Further, (X, F) is said to be a K_G^n -manifold (resp., *strong K_G^n -manifold*) if it is K_G^n -connected (resp., strongly K_G^n -connected) between any two nonempty open disjoint sets $P, Q \subset X$. If $F = \emptyset$, we will say that X is a (strong) K_G^n -manifold.

Definition 3 is justified by the following result which was actually established by Kuzminov [25]:

Theorem 2.1. *Every compactum X with $\dim_G X = n$ contains a pair (Y, F) of closed sets such that (Y, F) is a strong K_G^n -manifold.*

Similarly, the definition of Alexandroff \mathcal{C} -manifold yields the following one:

Definition 4. Let P and Q be open nonempty subsets of a continuum X with $\overline{P} \cap \overline{Q} = \emptyset$. We say that X is \mathcal{C} -connected in the sense of Alexandroff between P and Q (shortly, $A(\mathcal{C})$ -connected between P and Q) if there exists an open cover ω of $X \setminus (P \cup Q)$ such that no partition of X between P and Q admits an ω -map onto a space from the class \mathcal{C} .

Proposition 2.2. *Let P and Q be nonempty open subsets of a continuum X with $\overline{P} \cap \overline{Q} = \emptyset$ and $F \subset X$ closed. If the pair (X, F) is K_G^n -connected between P and Q , then X is $A(D_G^{n-2})$ -connected between P and Q , where D_G^{n-2} is the class of spaces of dimension $\dim_G \leq n - 2$.*

PROOF. According to Definition 3, we can find an open cover ω of $Y = X \setminus (P \cup Q)$ satisfying the following condition: for every partition C in X between P and Q and a natural map $p_{\omega_C} : (C, C \cap F) \rightarrow (|\omega_C|, |\omega_{C \cap F}|)$ there exists an element $e_C \in H^{n-1}(|\omega_C|, |\omega_{C \cap F}|)$ with $p_{\omega_C}^*(e_C) \neq 0$.

Suppose there exists a partition C of X between P and Q admitting an ω -map $g : C \rightarrow T$ onto a compactum T with $\dim_G T \leq n - 2$. Thus, we can find a finite open cover τ of T such that $\nu = g^{-1}(\tau)$ is refining ω . Let $p_\nu : (C, C \cap F) \rightarrow (|\nu|, |\nu_{C \cap F}|)$ and $p_\tau : (T, g(C \cap F)) \rightarrow (|\tau|, |\tau_{g(C \cap F)}|)$ be natural maps. Obviously, the function $V \in \tau \rightarrow g^{-1}(V) \in \nu$ provides a simplicial homeomorphism $g_\nu^T : (|\tau|, |\tau_{g(C \cap F)}|) \rightarrow (|\nu|, |\nu_{C \cap F}|)$. Then the maps p_ν and $g_\tau = g_\nu^T \circ p_\tau \circ g$ are homotopic. Hence,

$$(1) \quad p_\nu^* = g^* \circ p_\tau^* \circ (g_\nu^T)^*.$$

Because $H^{n-1}(T, g(C \cap F)) = 0$ (recall that $\dim_G T \leq n - 2$), the homomorphism $g^* : H^{n-1}(T, g(C \cap F)) \rightarrow H^{n-1}(C, C \cap F)$ is trivial. Then, by (1), so is the homomorphism $p_\nu^* : H^{n-1}(|\nu|, |\nu_{C \cap F}|) \rightarrow H^{n-1}(C, C \cap F)$. On the other hand, since ν refines ω , we can find a map $\varphi_\nu : (|\nu|, |\nu_{C \cap F}|) \rightarrow (|\omega_C|, |\omega_{C \cap F}|)$ such that p_{ω_C} and $\varphi_\nu \circ p_\nu$ are homotopic. Therefore, $p_{\omega_C}^* = p_\nu^* \circ \varphi_\nu^*$. Since $p_{\omega_C}^*(e_C) \neq 0$ for some $e_C \in H^{n-1}(|\omega_C|, |\omega_{C \cap F}|)$, $p_\nu^*(\varphi_\nu^*(e_C)) \neq 0$, which contradicts the triviality of p_ν^* .

Therefore, X is $A(D_G^{n-2})$ -connected between P and Q . □

Corollary 2.3. *A compactum X is an Alexandroff D_G^{n-2} -manifold provided (X, F) is a K_G^n -manifold for some closed set $F \subset X$.*

Definition 5. Let X , P , Q and F be as in Definition 3. We say that the pair (X, F) is V_G^n -connected between P and Q if there exists an open cover ω of $X \setminus (P \cup Q)$ such that $g^*: H^{n-1}(T, g(C \cap F)) \rightarrow H^{n-1}(C, C \cap F)$ is a non-trivial homomorphism for any partition C of X between P and Q and any surjective ω -map $g: C \rightarrow T$. The pair (X, F) is called a *relative V_G^n -continuum* provided (X, F) is V_G^n -connected between any two open sets $P, Q \subset X$ with disjoint closures.

When F is the empty set, the above definition provides V_G^n -continua introduced by Stefanov [29]. The proof of Proposition 2.2 yields the following stronger conclusion:

Corollary 2.4. *If a pair (X, F) is K_G^n -connected between two open sets $P, Q \subset X$ with disjoint closures, then (X, F) is V_G^n -connected between P and Q . In particular, every K_G^n -manifold (X, F) is a relative V_G^n -continuum.*

There is an interesting analogy between V_G^n -continua and relative V_G^n -continua. It follows from Theorem 2.1 and Corollary 2.4 that every compactum X with $\dim_G X = n$ contains a relative V_G^n -continuum. On the other hand, every compactum with $H^n(X) \neq 0$ contains a V_G^n -continuum (see [29] for finite-dimensional metric X , and [34] for any compact X).

We are going to provide more examples of K_G^n -manifolds and strong K_G^n -manifolds. A closed non-empty set $A \subset X$ is said to be a *cohomological carrier* of a non-zero element $\alpha \in H^n(X)$ if $i_A^*(\alpha) \neq 0$ and $i_B^*(\alpha) = 0$ for every proper closed subset $B \subset A$, where i_A denotes the inclusion map $A \hookrightarrow X$.

Next proposition was established by the second author using different terminology.

Proposition 2.5. [34, Proposition 2.5] *Every cohomological carrier of a non-zero element of $H^n(X)$ is a strong K_G^n -manifold.*

Following Yokoi [35], a compactum X is called an (n, G) -bubble if $H^n(X) \neq 0$ and $H^n(A) = 0$ for every closed proper set $A \subset X$. This is a reformulation of the notion of an n -bubble introduced by Kuperberg [23] and Choi [8], see also Karimov-Repovš [17] for the stronger notion of an H^n -bubble. A compactum X is said to be a *generalized (n, G) -bubble* [18] if there exists a surjective map $f: X \rightarrow Y$ such that the homomorphism $f^*: H^n(Y) \rightarrow H^n(X)$ is nontrivial, but $f_A^*(H^n(Y)) = 0$ for any proper closed subset A of X , where f_A is the restriction of f over A .

Proposition 2.6 below was actually established in [18, Theorem 2.3, Claim 1].

Proposition 2.6. *Any generalized (n, G) -bubble is a strong K_G^n -manifold.*

Proposition 2.7. [34, Theorem 1.1] *Every homogeneous metric ANR-continuum X with $\dim_G X = n$ and $H^n(X) \neq 0$ is an (n, G) -bubble, and hence a strong K_G^n -manifold.*

Proposition 2.8. *Let X be a continuum and $F \subset X$ a nonempty nowhere dense closed subset such that the quotient space X/F is a (strong) K_G^n -manifold. Then (X, F) is also a (strong) K_G^n -manifold.*

PROOF. Suppose X/F is a K_G^n -manifold, and U_1 and U_2 two nonempty disjoint open subsets of X . Since F is nowhere dense, there are disjoint open sets V_1 and V_2 in X such that $\bar{V}_j \subset U_j \setminus F, j = 1, 2$. Then $W_j = \pi(V_j), j = 1, 2$, are nonempty open and disjoint sets in X/F , where $\pi: X \rightarrow X/F$ is the quotient map. So, there exists a finite open cover γ of $(X/F) \setminus (W_1 \cup W_2)$ such that for any partition P of X/F between W_1 and W_2 any natural map $p_{\gamma_P}: P \rightarrow |\gamma_P|$ generates a non-trivial homomorphism $p_{\gamma_P}^*: H^{n-1}(|\gamma_P|) \rightarrow H^{n-1}(P)$. Hence, $\omega = \pi^{-1}(\gamma)$ is an open cover of $X \setminus (V_1 \cup V_2)$. If C is a partition of X between U_1 and U_2 , then $\pi(C)$ is a partition of X/F between W_1 and W_2 , and the function $\gamma \ni T \rightarrow \pi^{-1}(T) \in \omega$ generates a simplicial homeomorphism $\tau: (|\gamma_{\pi(C)}|, |\gamma_{\pi(C \cap F)}|) \rightarrow (|\omega_C|, |\omega_{C \cap F}|)$. Moreover, the homomorphism $p_{\gamma_{\pi(C)}}^*: H^{n-1}(|\gamma_{\pi(C)}|) \rightarrow H^{n-1}(\pi(C))$ is non-trivial. Since the maps $\tau \circ p_{\gamma_{\pi(C)}} \circ \pi$ and p_{ω_C} are homotopic (as maps from $(C, C \cap F)$ to $(|\omega_C|, |\omega_{C \cap F}|)$), $p_{\omega_C}^* = \pi^* \circ p_{\gamma_{\pi(C)}}^* \circ \tau^*$. In case $C \cap F = \emptyset$ the complex $|\gamma_{\pi(C \cap F)}|$ is also empty. So, $H^{n-1}(|\gamma_{\pi(C)}|, |\gamma_{\pi(C \cap F)}|) = H^{n-1}(|\gamma_{\pi(C)}|)$. Then the equality $p_{\omega_C}^* = \pi^* \circ p_{\gamma_{\pi(C)}}^* \circ \tau^*$ implies that $p_{\omega_C}^*$ is non-trivial (recall that τ^*, π^* are isomorphisms and $p_{\gamma_{\pi(C)}}^*$ is non-trivial). In case $C \cap F \neq \emptyset, \pi(C \cap F)$ is the point $\pi(F)$ of X/F and $|\gamma_{\pi(C \cap F)}|$ is a simplex of the nerve $|\gamma_{\pi(C)}|$. We have the following commutative diagram, where the vertical maps $p_{\gamma, k}^*$ are the dual maps generated by $p_{\gamma_{\pi(C)}}: \pi(C) \rightarrow |\gamma_{\pi(C)}|$.

$$\begin{array}{ccccc}
 H^{n-1}(|\gamma_{\pi(C)}|, |\gamma_{\pi(C \cap F)}|) & \xrightarrow{j_1^*} & H^{n-1}(|\gamma_{\pi(C)}|) & \xrightarrow{i_1^*} & H^{n-1}(|\gamma_{\pi(C \cap F)}|) \\
 \downarrow p_{\gamma, 1}^* & & \downarrow p_{\gamma, 2}^* & & \downarrow p_{\gamma, 3}^* \\
 H^{n-1}(\pi(C), \pi(C \cap F)) & \xrightarrow{j_2^*} & H^{n-1}(\pi(C)) & \xrightarrow{i_2^*} & H^{n-1}(\pi(C \cap F))
 \end{array}$$

Since $|\gamma_{\pi(C \cap F)}|$ is a simplex, $H^{n-1}(|\gamma_{\pi(C \cap F)}|) = 0$. Consequently, j_1^* is surjective, which implies that $p_{\gamma, 1}^*$ is non-trivial because so is $p_{\gamma, 2}^*$. Therefore, (X, F) is a K_G^n -manifold.

Suppose X/F is a strong K_G^n -manifold. Then there exists an element $e \in H^{n-1}(|\gamma_{\pi(C)}|)$ such that $p_{\gamma,2}^*(e) \in H^{n-1}(\pi(C))$ is non-trivial. Hence, so is the element $\pi^*(p_{\gamma,2}^*(e)) \in H^{n-1}(C)$ if $C \cap F = \emptyset$. But $\pi^*(p_{\gamma,2}^*(e)) = p_{\omega_C}^*((\tau^*)^{-1}(e))$. Therefore, $(\tau^*)^{-1}(e)$ is an element of $H^{n-1}(|\omega_C|)$ such that $p_{\omega_C}^*((\tau^*)^{-1}(e)) \in H^{n-1}(C)$ is non-trivial. This shows that (X, F) is a strong K_G^n -manifold. In case $C \cap F \neq \emptyset$, we have $H^{n-1}(|\gamma_{\pi(C \cap F)}|) = 0$ because $|\gamma_{\pi(C \cap F)}|$ is a simplex. Then it follows from the above diagram that there exists $e_1 \in H^{n-1}(|\gamma_{\pi(C)}|, |\gamma_{\pi(C \cap F)}|)$ with $j_1^*(e_1) = e$ and $p_{\gamma,1}^*(e_1) \neq 0$. As in the first case, we can show that $p_{\omega_C}^*((\tau^*)^{-1}(e_1)) \in H^{n-1}(C, C \cap F)$ is also non-trivial. Hence, (X, F) is a strong K_G^n -manifold. \square

Let us note that the implication " X/F is a (strong) K_G^n -manifold yields (X, F) is a (strong) K_G^n -manifold" in Proposition 2.8 can not be inverted. For example, (\mathbb{S}^1, F) is a strong $K_{\mathbb{Z}}^1$ -manifold but \mathbb{S}^1/F , where F is a two-point subset of \mathbb{S}^1 , is not a $K_{\mathbb{Z}}^1$ -manifold because the identification point $\pi(F)$ is a partition of \mathbb{S}^1/F .

Corollary 2.9. *The pair $(\mathbb{I}^n, \mathbb{S}^{n-1})$, $n \geq 1$, is a strong $K_{\mathbb{Z}}^n$ -manifold.*

PROOF. This follows from Proposition 2.8 because $\mathbb{I}^n/\mathbb{S}^{n-1} = \mathbb{S}^n$ is a strong $K_{\mathbb{Z}}^n$ -manifold according to Proposition 2.7. \square

The next proposition provides more examples of strong K_G^n -manifolds (recall that the Eilenberg-MacLane complexes $K(G, n)$ have the following property: $\dim_G X \leq n$ if and only if any map $g: A \rightarrow K(G, n)$ can be extended over X , where X is compact and $A \subset X$ is closed).

Proposition 2.10. *Let X be a continuum and $F \subset X$ a closed nowhere dense subset of X . If there exists a map $f: F \rightarrow K(G, n-1)$, which is not extendable over X but it is extendable over $Y \cup F$ for any proper closed subset Y of X , then (X, F) is a strong K_G^n -manifold. In particular, if $\dim X = n$ and $f: F \rightarrow \mathbb{S}^{n-1}$ is a map not extendable over X but extendable over any proper closed subset of X containing F , then (X, F) is a strong $K_{\mathbb{Z}}^n$ -manifold.*

PROOF. Since $H^{n-1}(F)$ is isomorphic to the group $[F; K(G, n-1)]$ of pointed homotopy classes of maps from F to $K(G, n-1)$, let $\alpha \in H^{n-1}(F)$ be the homotopy class $[f]$ of f . Let A be a proper closed subset of X . Consider the following diagram, where the vertical arrows are homomorphisms generated by

the corresponding inclusions

$$\begin{array}{ccccc} H^{n-1}(X) & \xrightarrow{i_1^*} & H^{n-1}(F) & \xrightarrow{\delta_1} & H^n(X, F) \\ \downarrow j_1^* & & \downarrow j_2^* & & \downarrow j_3^* \\ H^{n-1}(A) & \xrightarrow{i_2^*} & H^{n-1}(A \cap F) & \xrightarrow{\delta_2} & H^n(A, A \cap F). \end{array}$$

Because f is not extendable over X , $\alpha \notin i_1^*(H^{n-1}(X))$. Hence, $\beta = \delta_1(\alpha)$ is a non-trivial element of $H^n(X, F)$. On the other hand, since the interior of F in X is empty, $F \cup A \neq X$. So, f is extendable over $A \cup F$, which implies that the restriction $f|(A \cap F)$ is extendable over A . Hence, $[f|(A \cap F)] = j_2^*(\alpha)$ belongs to $i_2^*(H^{n-1}(A))$. So, $\delta_2(j_2^*(\alpha)) = j_3^*(\beta) = 0$. In this way we proved that β is a non-zero element of $H^n(X, F)$ whose image under the homomorphism $j_3^*: H^n(X, F) \rightarrow H^n(A, A \cap F)$ is trivial for every proper closed subset $A \subset X$.

Now, following the proof of Theorem 1 from [25], one can show that (X, F) is a strong K_G^n -manifold. Suppose U_1 and U_2 are non-empty open subsets of X with disjoint closures, and let $m_k: Y_k \hookrightarrow X$ be the inclusion of $Y_k = X \setminus U_k$ into X , $k = 1, 2$, and $Y = Y_1 \cap Y_2$. Consider the Mayer-Vietoris exact sequence

$$H^{n-1}(Y, Y \cap F) \xrightarrow{\delta} H^n(X, F) \xrightarrow{j} \bigoplus_{k=1}^2 H^n(Y_k, Y_k \cap F)$$

with $j = (m_1^*, m_2^*)$. Since $j(\beta) = 0$, there exists a non-trivial element $\gamma \in H^{n-1}(Y, Y \cap F)$ with $\delta(\gamma) = \beta$. Consequently, there exist an open cover ω of Y and $e \in H^{n-1}(|\omega|, |\omega_{Y \cap F}|)$ with $p_\omega^*(e) = \gamma$.

Let C be a partition of X between U_1 and U_2 . So, $X = P_1 \cup P_2$ and $C = P_1 \cap P_2$, where each P_k is a closed subset of X containing U_k , $k = 1, 2$. Denote by $i: C \hookrightarrow Y$, $i_{12}: P_1 \hookrightarrow Y_2$ and $i_{21}: P_2 \hookrightarrow Y_1$ the corresponding inclusions. Then we have the following commutative diagram, whose rows are Mayer-Vietoris sequences:

$$\begin{array}{ccccc} H^{n-1}(Y, Y \cap F) & \xrightarrow{\delta} & H^n(X, F) & \xrightarrow{j} & \bigoplus_{k=1}^2 H^n(Y_k, Y_k \cap F) \\ \downarrow i^* & & \downarrow id & & \downarrow i_{12}^* \oplus i_{21}^* \\ H^{n-1}(C, C \cap F) & \xrightarrow{\delta_C} & H^n(X, F) & \xrightarrow{j_C} & \bigoplus_{k=1}^2 H^n(P_k, P_k \cap F). \end{array}$$

Obviously,

$$(2) \quad \delta_C(i^*(\gamma)) = id(\delta(\gamma)) = \beta \neq 0.$$

On the other hand, the commutativity of the diagram

$$\begin{array}{ccc}
 H^{n-1}(|\omega|, |\omega_{Y \cap F}|) & \xrightarrow{p_\omega^*} & H^{n-1}(Y, Y \cap F) \\
 \downarrow i_C^* & & \downarrow i^* \\
 H^{n-1}(|\omega_C|, |\omega_{C \cap F}|) & \xrightarrow{p_{\omega_C}^*} & H^{n-1}(C, C \cap F)
 \end{array}$$

implies that $p_{\omega(C)}^*(i_C^*(e)) = i^*(p_\omega^*(e)) = i^*(\gamma)$. Therefore, according to (2), $p_{\omega(C)}^*(i_C^*(e)) \neq 0$.

Suppose now that $\dim X = n$ and $f: F \rightarrow \mathbb{S}^{n-1}$ is a map not extendable over X but extendable over any proper closed subset of X containing F . Since \mathbb{S}^{n-1} as an n -skeleton of $K(\mathbb{Z}, n - 1)$, f considered as a map to $K(\mathbb{Z}, n - 1)$ is not extendable over X because any such an extension would yield an extension of f (see the proof of [10, Theorem 1.4]). So, (X, F) is a strong $K_{\mathbb{Z}}^n$ -manifold. \square

We say that an n -system $S = \{(F_i^+, F_i^-) : i = 1, \dots, n\}$ of pairs with any pair consisting of closed disjoint subset of X is *essential* if for any sequence $\{C_i\}$ of partitions C_i of X between F_i^+ and F_i^- we have $\bigcap_{i=1}^n C_i \neq \emptyset$. A space X is said to be *spanned on S* if S is essential but the system $S_Y = \{(F_i^+ \cap Y, F_i^- \cap Y) : i = 1, \dots, n\}$ is not essential for every proper closed subset $Y \subset X$. Observe that if S spans X , then the set $F = \bigcup_{i=1}^n (F_i^+ \cup F_i^-)$ is nowhere dense in X .

Corollary 2.11. *Let X be an n -dimensional continuum and S be an n -system in X , $n \geq 2$, spanning X . Then (X, F) is a strong $K_{\mathbb{Z}}^n$ -manifold, where $F = \bigcup_{i=1}^n (F_i^+ \cup F_i^-)$.*

PROOF. Let $g_i: X \rightarrow J$, $J = [-1, 1]$, be a function with $g_i(F_i^+) = 1$ and $g_i(F_i^-) = -1$, $i = 1, \dots, n$. Denote by $g: X \rightarrow \mathbb{J}^n$ the diagonal product map $g(x) = (g_1(x), \dots, g_n(x))$, $x \in X$. Obviously, $f = g|_F$ is a map from F into \mathbb{S}^{n-1} (here \mathbb{S}^{n-1} is identified with the boundary of \mathbb{J}^n).

Since S is essential on X and inessential on every proper closed subset of X , the map f is not extendable over X but it is extendable over any proper closed set of X containing F . Indeed, suppose $\bar{f}: X \rightarrow \mathbb{S}^{n-1}$ is an extension of f . Then for each i the set $C_i = \bar{f}^{-1}(P_i)$, where $P_i = \{x \in \mathbb{I}^n : x_i = 0\}$, is a partition of X between F_i^+ and F_i^- and $\bigcap_{i=1}^n C_i = \emptyset$, a contradiction. If Y is a proper closed subset of X containing F , there exist partitions L_i of X between F_i^+ and F_i^- such that $\bigcap_{i=1}^n L_i \cap Y = \emptyset$. According to [2, Lemma 1, pp. 339], there are closed G_δ -sets D_i in Y , which are partitions between F_i^+ and F_i^- , such that such that $\bigcap_{i=1}^n D_i = \emptyset$ and $Y \cap L_i \subset D_i$ for each i . Then we can construct functions

$h_i: Y \rightarrow [-1, 1]$ with $h_i(F_i^+) = 1$, $h_i(F_i^-) = -1$ and $D_i = h_i^{-1}(0)$, $1 \leq i \leq n$. Obviously, the diagonal product h of all h_i is a map from Y into \mathbb{J}^n such that $a = (0, \dots, 0) \notin h(Y)$. Let $r: \mathbb{J}^n \setminus \{a\} \rightarrow \mathbb{S}^{n-1}$ be a retraction and $\tilde{f} = r \circ h$. It follows from the definition of f and \tilde{f} that $f(x)$ and $\tilde{f}(x)$ belong to the same face of \mathbb{J}^n for every $x \in F$. This implies that f and $\tilde{f}|_F$ are homotopic. So, by the homotopy extension theorem, f is extendable over Y .

Finally, according to Proposition 2.10, (X, F) is a strong $K_{\mathbb{Z}}^n$ -manifold. □

Corollary 2.11 extends a result proved independently by Hamamdzhiev [16] and the first author [31] that, under the hypotheses of this corollary, X is a V^n -continuum.

3. K_G^n -MANIFOLDS AND MAZURKIEWICZ MANIFOLDS

Another type of connectedness is inspired by Mazurkiewicz \mathcal{C} -manifolds, where \mathcal{C} is a given class of spaces.

Definition 6. Let $P, Q \subset X$ be nonempty open subsets of X with disjoint closures. The space X is said be $M(\mathcal{C})$ -connected between P and Q if for every set $F \subset X$ with $F \in \mathcal{C}$ there exists a continuum $K \subset X \setminus F$ connecting P and Q .

It is well known [15] that any continuum X which is $A(D^{n-2})$ -connected between two nonempty open sets $P, Q \subset X$ with disjoint closures is $M(\mathcal{C})$ -connected between P and Q with respect to the class \mathcal{C} of all normally placed $(n - 2)$ -dimensional subsets of X (here D^{n-2} is the class of all spaces of covering dimension $\dim \leq n - 2$). We don't know if this holds when the covering dimension is replaced by the cohomological dimension \dim_G . But this is true if instead of $A(D^{n-2})$ -connectedness the stronger K_G^n -connectedness of X is assumed (recall that a set M is a normally placed in X provided every two disjoint closed in M sets have disjoint open in X neighborhoods; for example, every F_σ - subset of a normal space is normally placed in that space).

Theorem 3.1. *Let the pair (X, F) be strongly K_G^n -connected between two open disjoint sets $P, Q \subset X$. Then for every Lindelöf normally placed subset $M \subset X$ with $\dim_G M \leq n - 2$ there exists a continuum $K \subset X \setminus M$ connecting P and Q .*

PROOF. Suppose there exists a Lindelöf normally placed subset $M \subset X$ with $\dim_G M \leq n - 2$ such that any continuum K connecting P and Q meets M . According to [19, Lemma 2.5], there are nonempty open sets $P_1, Q_1 \subset X$ such that $\bar{P}_1 \subset P$, $\bar{Q}_1 \subset Q$ and if K is a continuum in X joining \bar{P}_1 and \bar{Q}_1 , then

$K \cap M \setminus (P_1 \cup Q_1) \neq \emptyset$. We are going to show this is not true, which will complete the proof.

Let ω be a finite open cover of $Y = X \setminus (P_1 \cup Q_1)$ and e an element of $H^{n-1}(|\omega|, |\omega_{F \cap Y}|)$, both satisfying the requirements from Definition 3 with P and Q replaced by P_1 and Q_1 , respectively. Obviously, $M_1 = M \setminus (P_1 \cup Q_1)$ is closed in M . So, M_1 is also Lindelöf and normally placed in Y . Moreover, $\dim_G M_1 \leq n - 2$, which implies $H^{n-1}(M_1, M_1 \cap F) = 0$. Consequently, there exist an open cover γ of M_1 refining ω and a simplicial map

$$\pi_\omega^\gamma : (|\gamma|, |\gamma_{M_1 \cap F}|) \rightarrow (|\omega_{M_1}|, |\omega_{M_1 \cap F}|) \text{ such that } (\pi_\omega^\gamma)^*(i_{M_1}^*(e)) = 0,$$

where i_{M_1} is the embedding $(|\omega_{M_1}|, |\omega_{M_1 \cap F}|) \hookrightarrow (|\omega|, |\omega_F|)$. Since M_1 is Lindelöf, we may assume that γ is countable. Next, let $\gamma_1 = \{A_i : i = 1, 2, \dots\}$ be a system of closed in M_1 sets covering $M_1 \cap F$ such that $A_i \subset U_i \cap F$, where $U_i \in \gamma$, for all i . Using the proof of [11, Theorem 3.1.1], we find an open in M_1 system $\{L_i : i = 1, 2, \dots\}$ such that $\lambda_1 = \{\bar{L}_i^{M_1} : i = 1, 2, \dots\}$ is a swelling of γ_1 with $\bar{L}_i^{M_1} \subset U_i$, $i \geq 1$, where $\bar{L}_i^{M_1}$ is the closure of L_i in M_1 . Recall that λ_1 is a swelling of γ_1 if each $\bar{L}_i^{M_1}$ contains A_i and for any finite set of indices i_1, \dots, i_s we have $\bigcap_{k=1}^s \bar{L}_{i_k}^{M_1} \neq \emptyset$ iff $\bigcap_{k=1}^s A_{i_k} \neq \emptyset$. Consider the Lindelöf space $M_1 \setminus L$, where $L = \bigcup_{i \geq 1} L_i$. If $M_1 \setminus L$ is non-empty, there exists a countable system $\lambda_2 = \{B_i : i = 1, 2, \dots\}$ of closed subsets of M_1 such that λ_2 covers $M_1 \setminus L$, λ_2 refines γ and all B_i are contained in $M_1 \setminus F$. In case $M_1 \setminus L = \emptyset$, λ_2 is the empty family. Obviously, $\lambda = \lambda_1 \cup \lambda_2$ is a closed countable cover of M_1 refining γ . Let $\lambda = \{\Lambda_i : i = 1, 2, \dots\}$. Using again the proof of [11, Theorem 3.1.1], we find an open in M_1 swelling $\theta = \{W_i : i = 1, 2, \dots\}$ of λ such that θ refines γ . Moreover, we can insist that $W_i \cap F = \emptyset$ provided $\Lambda_i \cap F = \emptyset$.

Claim. *There exists an open in Y swelling $\tilde{\sigma} = \{V_i : i = 1, 2, \dots\}$ of λ refining ω such that $\Lambda_i \subset \bar{V}_i \cap M_1 \subset W_i$ for all i .*

We follow the proof of Lemma 4.1.2 from [31]. Obviously, it suffices to construct by induction open in Y swellings $\sigma_m = \{V_1, \dots, V_m\}$ of $\{\Lambda_1, \dots, \Lambda_m\}$, $m \geq 1$, such that

- $\sigma_{m+1} = \sigma_m \cup \{V_{m+1}\}$;
- $\Lambda_k \subset V_k \cap M_1 \subset \bar{V}_k \cap M_1 \subset W_k$, $k \geq 1$;
- each V_i is contained in an element of ω .

Suppose we already constructed σ_m for some m . Since Λ_{m+1} and $M \setminus W_{m+1}$ are closed disjoint subsets of M_1 and M_1 is normally placed in Y , there are disjoint open in Y sets V'_{m+1} and \tilde{V}_{m+1} with $\Lambda_{m+1} \subset V'_{m+1}$ and $M_1 \setminus W_{m+1} \subset \tilde{V}_{m+1}$. Let E be the union of all intersections $\bar{V}_{i_1} \cap \dots \cap \bar{V}_{i_s}$, $1 \leq i_k \leq m$, such that $\bigcap_{k=1}^s \Lambda_{i_k} \cap$

$\Lambda_{m+1} = \emptyset$. We claim that $E \cap \Lambda_{m+1} = \emptyset$. Indeed, otherwise $\bigcap_{k=1}^p \bar{V}_{j_k} \cap \Lambda_{m+1} \neq \emptyset$ for some indices j_1, \dots, j_p such that $j_k \leq m$ and $\bigcap_{k=1}^p \Lambda_{j_k} \cap \Lambda_{m+1} = \emptyset$. Then

$$\bigcap_{k=1}^p \bar{V}_{j_k} \cap \Lambda_{m+1} \subset \bigcap_{k=1}^p \bar{V}_{j_k} \cap M_1 \cap \Lambda_{m+1} \subset \bigcap_{k=1}^p W_{j_k} \cap W_{m+1} \neq \emptyset.$$

Since $\{W_i : i = 1, 2, \dots\}$ is a swelling of λ , this implies that $\bigcap_{k=1}^p \Lambda_{j_k} \cap \Lambda_{m+1} \neq \emptyset$, a contradiction. To finish the inductive step, observe that λ refines ω . So, there exists $O \in \omega$ containing Λ_{m+1} . We set $V_{m+1} = V'_{m+1} \cap O \setminus E$, and $\sigma_{m+1} = \sigma_m \cup \{V_{m+1}\}$. It is easily seen that σ_{m+1} is a swelling of $\{\Lambda_i : i = 1, 2, \dots, m + 1\}$, which completes the proof of the claim.

Recall that θ is a refining of γ . Moreover, any refining function between θ and γ provides a simplicial map

$$\pi_\gamma^\theta : (|\theta|, |\theta_{M_1 \cap F}|) \rightarrow (|\gamma_{M_1}|, |\gamma_{M_1 \cap F}|) \text{ such that } (\pi_\omega^\theta)^*(i_{M_1}^*(e)) = 0,$$

where π_ω^θ is the composition $\pi_\omega^\gamma \circ \pi_\gamma^\theta$. Let $\theta_j = \{W_i : \Lambda_i \in \lambda_j\}$, $j = 1, 2$. Obviously, $\theta = \theta_1 \cup \theta_2$. Consider the open in Y system

$$\sigma = \{V_i \setminus F : W_i \in \theta_2\} \cup \{V_i : W_i \in \theta_1\} \text{ and let } W = \bigcup_{\Sigma \in \sigma} \Sigma.$$

It follows from our construction that $\sigma = \{\Sigma_i : i = 1, 2, \dots\}$ has the following property: for any indices i_1, \dots, i_s we have

$$\bigcap_{k=1}^s \Sigma_{i_k} \neq \emptyset \text{ iff } \bigcap_{k=1}^s W_{i_k} \neq \emptyset \text{ and } \bigcap_{k=1}^s \Sigma_{i_k} \cap F \neq \emptyset \text{ iff } \bigcap_{k=1}^s W_{i_k} \cap F \neq \emptyset.$$

Hence, there exists a simplicial homeomorphism ξ between the nerves $(|\theta|, |\theta_{M_1 \cap F}|)$ and $(|\sigma|, |\sigma_{W \cap F}|)$. So, ξ^{-1} is also a simplicial homeomorphism from $(|\sigma|, |\sigma_{W \cap F}|)$ onto $(|\theta|, |\theta_{M_1 \cap F}|)$. Then for the simplicial map π below we have

$$\pi = i_{M_1} \circ \pi_\omega^\theta \circ \xi^{-1} : (|\sigma|, |\sigma_{W \cap F}|) \rightarrow (|\omega|, |\omega_{F \cap Y}|), \pi^*(e) = 0.$$

Suppose the set W contains a partition C of X between \bar{P}_1 and \bar{Q}_1 and consider the following diagram, where $i : (|\sigma_C|, |\sigma_{F \cap C}|) \rightarrow (|\sigma|, |\sigma_{W \cap F}|)$ and i_C are natural inclusions:

$$\begin{array}{ccc} (C, C \cap F) & \xrightarrow{i_C} & (Y, F \cap Y) \\ \downarrow p_{\sigma_C} & & \downarrow p_\omega \\ (|\sigma_C|, |\sigma_{F \cap C}|) & \xrightarrow{\pi \circ i} & (|\omega|, |\omega_{F \cap Y}|) \end{array}$$

It follows from our construction that for every $x \in C$ (resp., $x \in C \cap F$) both points $\pi(i(p_{\sigma_C}(x)))$ and $p_\omega(i_C(x))$ belong to a simplex of the nerve $|\omega|$ (resp. $|\omega_{F \cap Y}|$). So, the maps $\pi \circ i \circ p_{\sigma_C}$ and $p_\omega \circ i_C$ are homotopic and we have the commutative diagram:

$$\begin{array}{ccc} H^{n-1}(|\omega|, |\omega_{F \cap Y}|) & \xrightarrow{i^* \circ \pi^*} & H^{n-1}(|\sigma_C|, |\sigma_{C \cap F}|) \\ \downarrow p_\omega^* & & \downarrow p_{\sigma_C}^* \\ H^{n-1}(Y, F \cap Y) & \xrightarrow{i_C^*} & H^{n-1}(C, C \cap F) \end{array}$$

Because $\pi^*(e) = 0$, the last diagram implies

$$(3) \quad i_C^*(p_\omega^*(e)) = 0.$$

On the other hand, by (3), the following diagram

$$\begin{array}{ccc} H^{n-1}(|\omega|, |\omega_{F \cap Y}|) & \xrightarrow{i_{\omega_C}^*} & H^{n-1}(|\omega_C|, |\omega_{C \cap F}|) \\ \downarrow p_\omega^* & & \downarrow p_{\omega_C}^* \\ H^{n-1}(Y, F \cap Y) & \xrightarrow{i_C^*} & H^{n-1}(C, C \cap F) \end{array}$$

yields $i_C^*(p_\omega^*(e)) = p_{\omega_C}^*(i_{\omega_C}^*(e)) = 0$, a contradiction.

Hence, W does not contain any partition of X between \bar{P}_1 and \bar{Q}_1 . Choose an open set \tilde{W} such that both sets $\bar{P}_1 \setminus \tilde{W}$ and $\bar{Q}_1 \setminus \tilde{W}$ are non-empty and $\tilde{W} \cap Y = W$. Suppose that $X \setminus \tilde{W}$ is not connected between $\bar{P}_1 \setminus \tilde{W}$ and $\bar{Q}_1 \setminus \tilde{W}$, i.e. $X \setminus \tilde{W} = A \cup B$ with A, B being disjoint closed sets such that $\bar{P}_1 \setminus \tilde{W} \subset A$ and $\bar{Q}_1 \setminus \tilde{W} \subset B$. Then the sets $B' = B \cup \bar{Q}_1$ and $A' = A \cup \bar{P}_1$ are disjoint and closed in X whose union is a proper subset of X . So, there is a partition C in X between A' and B' . Obviously, C is a partition between \bar{P}_1 and \bar{Q}_1 which is contained in W , a contradiction. Hence, $X \setminus \tilde{W}$ is connected between $\bar{P}_1 \setminus \tilde{W}$ and $\bar{Q}_1 \setminus \tilde{W}$. This implies (see [24, §47, Theorem 3, p. 170]) the existence of a continuum $K \subset X \setminus \tilde{W}$ connecting \bar{P}_1 and \bar{Q}_1 . Finally, since $M_1 \subset \tilde{W}$, $K \subset X \setminus M_1$. This contradicts the fact that any continuum connecting \bar{P}_1 and \bar{Q}_1 should meet M_1 . \square

Corollary 3.2. *Every strong K_G^n -manifold is a Mazurkiewicz D_G^{n-2} -manifold.*

The proof of Theorem 3.1 provides a stronger conclusion.

Theorem 3.3. *Let the pair (X, F) be strongly K_G^n -connected between two nonempty open disjoint sets $P, Q \subset X$ and M be a Lindelöff normally placed subset of X*

with $H^{n-1}(M, M \cap F) = 0$. Then in each of the following two cases there exists a continuum $K \subset X \setminus M$ connecting P and Q .

- (1) $M \subset X \setminus (P \cup Q)$;
- (2) $\dim_G M \leq n - 1$ and $F \cap M$ is a G_δ -set in M .

PROOF. It suffices to prove the theorem in case $\overline{P} \cap \overline{Q} = \emptyset$. The proof of item (1) follows from the proof of Theorem 3.1 considering M, P and Q instead of M_1, P_1 and Q_1 , respectively. For the second item, according to the proof of Theorem 3.1, we need to show that $H^{n-1}(M_1, M_1 \cap F) = 0$, where $M_1 = M \setminus (P \cup Q)$. That will be done if we show that the homomorphism $H^{n-1}(M, M \cap F) \rightarrow H^{n-1}(M_1, F \cap M_1)$ is surjective. To this end, consider the quotient map $\pi: M \rightarrow M/(M \cap F)$ (here $M/(M \cap F)$ is the quotient space obtained from M by shrinking $M \cap F$ to a point). Since $F \cap M$ is a G_δ set in M , $M \setminus F$ is an F_σ -subset of M . This implies that $\dim_G M \setminus F \leq n - 1$. On the other hand, $(M/(M \cap F)) \setminus \{\pi(M \cap F)\}$ is homeomorphic to $M \setminus F$, so, $\dim_G M/(M \cap F) \leq n - 1$. Consequently, the homomorphism $H^{n-1}(M/(M \cap F)) \rightarrow H^{n-1}(M_1/(F \cap M_1))$ is surjective. Finally, by a result of Bartik [5], $H^{n-1}(M, M \cap F) = H^{n-1}(M/F)$ and $H^{n-1}(M_1, M_1 \cap F) = H^{n-1}(M_1/(F \cap M_1))$, which completes the proof. \square

We say that a subset $M \subset X$ cuts X between two disjoint sets $A, B \subset X$ if $(A \cup B) \cap M = \emptyset$ and every continuum in X joining A and B meets M . The next corollary follows from Theorem 3.3 and Proposition 2.7 (when M is a partition of X , this corollary was established by the second author in [34]).

Corollary 3.4. *Let X be a homogeneous metric ANR-continuum with $\dim_G X = n$ and $H^n(X) \neq 0$. Then $H^{n-1}(M) \neq 0$ for every set $M \subset X$, which cuts X between two disjoint open subsets of X .*

Corollary 3.4 is interesting because the Bing-Borsuk question [6] whether the group $H^{n-1}(M)$ is non-trivial for any partition M of a homogeneous metric ANR-space of dimension n is still unanswered.

The next corollary can be compared with the classical Mazurkiewicz theorem [26] that any region X in the Euclidean space \mathbb{R}^n has the following property: if $M \subset X$ with $\dim M \leq n - 2$, then every two points from $X \setminus M$ can be joined by a continuum $K \subset X \setminus M$.

Corollary 3.5. *Let M be either a subset of \mathbb{S}^n or a bounded subset of \mathbb{R}^n with $H^{n-1}(M; \mathbb{Z}) = 0$. Then every pair of disjoint open sets $P, Q \subset \mathbb{S}^n$ (resp., $P, Q \subset \mathbb{R}^n$) such that $(P \cup Q) \cap M = \emptyset$ can be joined by a continuum in $\mathbb{S}^n \setminus M$ (resp., in $\mathbb{R}^n \setminus M$). If, in addition $\dim M \leq n - 1$, the requirement $(P \cup Q) \cap M = \emptyset$ can be removed.*

PROOF. Obviously, the case when $M \subset \mathbb{S}^n$ follows from Corollary 3.4. When M is a bounded subset of \mathbb{R}^n , we take an n -dimensional cube B whose interior contains M and meets both P and Q . Then the proof follows from Corollary 2.9 and Theorem 3.3. \square

Let us mention that the requirement in Corollary 3.5 M to be bounded in \mathbb{R}^n is essential. Indeed, any $(n - 1)$ -dimensional hyperplane in \mathbb{R}^n is a counterexample. Also, because every n -dimensional set in \mathbb{R}^n has a non-empty interior, the condition $(P \cup Q) \cap M = \emptyset$ can not be dropped unless $\dim M \leq n - 1$.

4. HOMOLOGY MANIFOLDS AND MAZURKIEWICZ MANIFOLDS

In this section we are going to show that some homological properties of a metric space X imply that X is a *Mazurkiewicz arc n -manifold* in the following sense: If M is an F_σ -subset of X with $\dim M \leq n - 2$, then any two disjoint sets A and B , both having non-empty interiors, can be joined by an arc in $X \setminus M$. Obviously, every Mazurkiewicz arc n -manifold is a Mazurkiewicz manifold with respect to the class of all spaces whose covering dimension \dim is $\leq n - 2$.

Everywhere in this section we consider singular homology groups reduced in dimension zero with coefficients in a given group G (if G is not written then the coefficients are integers). The following notion introduced by Toruńczyk [32] is well known: A closed subset A of a space X is said to be a Z_n -set in X if for any map $f: \mathbb{I}^n \rightarrow X$ and any open cover ω of X there is a map $g: \mathbb{I}^n \rightarrow X \setminus A$ which is ω -close to f (i.e., for all $x \in \mathbb{I}^n$ both points $f(x)$ and $g(x)$ belong to some element of ω). A homological counterpart of this notion was defined by Banach-Cauty-Karashev [4]: A closed subset $A \subset X$ is called a G -homological Z_n -set in X if $H_k(U, U \setminus A; G) = 0$ for all $k \leq n$ and all open $U \subset X$. It follows from the excision axiom that a point $x \in X$ is a G -homological Z_n -set in X provided $H_k(X, X \setminus x; G) = 0$ for all $k \leq n$.

Further, let us remind the definition of the *separating dimension* $t(X)$ of a space X introduced by Steinke [28] and its transfinite extension $trt(X)$ given by Arenas-Chatyrko-Puertas [3]: $trt(X) = -1$ iff $X = \emptyset$; $trt(X) \leq \alpha$ for an ordinal α if any closed set $B \subset X$ containing at least two points can be separated by a closed set $P \subset B$ with $trt(P) < \alpha$. When $trt(X)$ is an integer, then $trt(X) = t(X)$. Moreover, $t(X) \leq \text{ind}(X)$ [28]. Hence, for metrizable X we have $t(X) \leq \text{ind}(X) \leq \dim X$.

Following Mitchell [27], we say that X has the *property $H(n, G)$ at the points of a set $M \subset X$* if $H_k(X, X \setminus x; G) = 0$ for all $k \leq n$ and all $x \in M$. When $M = X$ in the above definition, X is said to have the $H(n, G)$ -property.

Theorem 4.1. *Let X be a complete metric space and M be an F_σ -subset of X such that $\text{trt}(M) \leq n - 2$ and X has the property $H(n - 1, G)$ at the points of M . Suppose $P, Q \subset X$ are open sets which can be joined by an arc in X . Then there is an arc in $X \setminus M$ joining P and Q . In particular, any arcwise connected open subset of X is a Mazurkiewicz arc n -manifold provided X has the property $H(n - 1, G)$.*

PROOF. Let M be a countable union of closed sets $M_i, i = 1, 2, \dots$. Since each $x \in M$ is a G -homological Z_{n-1} -point in X and $\text{trt}(M_i) \leq n - 2$, we can apply Theorem 4.3 from [4] stating that if A is a closed subset of a space X such that $\text{trt}(A) = m$ and all $a \in A$ are G -homological Z_{n+m} -points in X , then A is a G -homological Z_n -set in X (for the covering dimension \dim this was established by Daverman [9, Lemma 2.1]). Therefore, any one of the sets M_i is a G -homological Z_1 -set in X . Then, by [4, Theorem 3.2(6)], M_i are Z_1 -sets in X . Consequently, the spaces $C_i = C(\mathbb{I}, X \setminus M_i)$ are dense (and obviously, open) in the space $C(\mathbb{I}, X)$ of all continuous maps from \mathbb{I} into X equipped with the compact-open topology. Finally, since $C(\mathbb{I}, X)$ is complete, $\bigcap C_i$ is also dense in $C(\mathbb{I}, X)$. Because P and Q can be joined by an arc in X , there exists a map $f: \mathbb{I} \rightarrow X$ with $f(0) \in P$ and $f(1) \in Q$. Since P and Q are open, f can be approximated by maps $g: \mathbb{I} \rightarrow X \setminus M$ with $g(0) \in P$ and $g(1) \in Q$.

The second half follows from the first one and the following observations: if $U \subset X$ is open, then the excision axiom implies $H_k(U, U \setminus x; G) = 0$ for all $k \leq n - 1$ and $x \in U$; moreover, $\text{trt}(M) \leq \dim M$ for any set $M \subset X$. □

Below, by a *homology n -manifold over G* we mean a metric space X such that for every $x \in X$ we have $H_k(X, X \setminus x; G) = 0$ if $k \neq n$ and $H_n(X, X \setminus x; G) = G$.

Corollary 4.2. *Let X be an arcwise connected complete metric space. In each of the following cases any arcwise connected open subset of X is a Mazurkiewicz arc n -manifold:*

- (1) X is a homology n -manifold over a group G ;
- (2) X is a product of at least n metric spaces $X_i, 1 \leq i \leq m$.

PROOF. The first item follows directly from Theorem 4.1. To prove the second one, consider the exact sequence for every i and $x \in X_i$

$$H_0(X_i \setminus x; G) \rightarrow H_0(X_i; G) \rightarrow H_0(X_i, X_i \setminus x; G) \rightarrow 0.$$

Since X_i is arcwise connected (as an image of X), $H_0(X_i; G) = 0$. So, $H_0(X_i, X_i \setminus$

$x; G) = 0$. Then, by [4, Theorem 3.2 and Theorem 6.1(2)], $H_k(X, X \setminus x; G) = 0$ for every $x \in X$ and $k \leq m - 1$. Finally, Theorem 4.1 completes the proof. \square

In some situations the space $X \setminus M$ from Theorem 4.1 is arcwise connected.

Theorem 4.3. *Let $X \in H(n-1, \mathbb{Z})$ be a connected and locally connected complete metric space and M be an F_σ -set in X with $\dim M \leq n-2$. Then $U \setminus M$ is arcwise connected for any open connected set $U \subset X$ with $U \setminus M \neq \emptyset$.*

PROOF. Since X is connected and locally connected, U is arcwise connected and locally arcwise connected. Moreover, by the excision axiom, U has the property $H(n-1, \mathbb{Z})$. Let $M = \bigcup M_i$ with each M_i being closed in X . It follows from the proof of Theorem 4.1 that each $M_i \cap U$ is a Z_1 -set in U . For any two points $a, b \in U \setminus M$ consider the set-valued map $\Phi: \mathbb{I} \rightarrow U$ defined by $\Phi(0) = a$, $\Phi(1) = b$ and $\Phi(t) = U$ for each $t \in (0, 1)$. It is easily seen that Φ is lower semi-continuous and for every $x \in U$ and its neighborhood W there is a neighborhood $V \subset W$ of x with the following property: if $t \in [0, 1]$ and x_1, x_2 are two points from $\Phi(t) \cap V$ there is an arc in $\Phi(t) \cap W$ joining x_1 and x_2 . Then, by [12, Theorem 1.2], Φ admits a continuous selection $g: \mathbb{I} \rightarrow U \setminus M$. Obviously, $g(\mathbb{I})$ is an arc in $U \setminus M$ joining a and b . \square

Next proposition shows that any space with a base consisting of Cantor manifolds is a Mazurkiewicz manifold.

Proposition 4.4. *Suppose X is a connected complete metric space possessing a base \mathcal{B} of open sets such that \bar{U} is a Cantor manifold with respect to a given class \mathcal{C} for every $U \in \mathcal{B}$. Then $X \setminus M$ is arcwise connected for every F_σ -subset $M = \bigcup M_i$ of X with $M_i \in \mathcal{C}$, $i \geq 1$.*

PROOF. According to a result of Hadžiivanov-Hamamdžiev [14, Theorem 1], $X \setminus M$ is connected. Moreover, by the same result, all $U \setminus M$, $U \in \mathcal{B}$, are connected. Hence, $X \setminus M$ is connected and locally connected. Because $X \setminus M$ is complete (as a G_δ -subset of X), it is arcwise connected. \square

A metric space X is said to have the *local separation property in dimension n* (written LS^n) if for every $x \in X$ and every neighborhood U of x there exists another neighborhood $V \subset U$ of x such that any map $f: \mathbb{S}^k \rightarrow V$, $k \leq n$, can be approximated by maps $g: \mathbb{S}^n \rightarrow V$ such that each $g(\mathbb{S}^k)$ does not separate V . It can be shown that if a space has the property LS^n , then the statement in above definition holds with \mathbb{S}^k replaced by \mathbb{I}^k .

The *disjoint (n, m) -cell property* of a metric space X , denoted by $D(n, m)$, is defined as follows: for each $\epsilon > 0$ and any two maps $f: \mathbb{I}^n \rightarrow X$ and $g: \mathbb{I}^m \rightarrow X$

there exist maps $f': \mathbb{I}^n \rightarrow X$ and $g': \mathbb{I}^m \rightarrow X$ such that f' and g' are ϵ -close to f and g , respectively, and $f'(\mathbb{I}^n) \cap g'(\mathbb{I}^m) = \emptyset$. We are interested in the property $D(0, n)$, which implies the property $H(n, G)$ for any G . Using an idea from Krupski [22, Theorem 2.6] we provide some conditions for a homogeneous spaces to have the property $H(n - 1, G)$.

Theorem 4.5. *Any homogeneous locally compact metric ANR-space X with $X \in LS^{n-2}$ has the $D(0, n - 1)$ -property. Thus X has the $H(n - 1, G)$ -property for any group G .*

PROOF. We are going to prove that X has the $D(0, n - 1)$ -property. So, by [22, Theorem 2.5], $X \in H(n - 1, \mathbb{Z})$ which, according to the Universal Coefficients Formula, implies $X \in H(n - 1, G)$ for any group G . By Proposition 1.8 and Theorem 2.5 from [22], it suffices to show that every $a \in X$ is LCC^{n-2} in X , i.e. for every neighborhood U of a there exists another neighborhood $V \subset U$ of a such that any map $f: \mathbb{S}^k \rightarrow V \setminus a$, $k \leq n - 2$, can be extended to a map $\bar{f}: \mathbb{B}^{k+1} \rightarrow U \setminus a$ (here \mathbb{B}^{k+1} is the $(k + 1)$ -dimensional ball). To this end, let $a \in U \subset X$ be an open connected set with a compact closure. Take an open neighborhood $V \subset U$ of a such that V is contractible in U . We may also suppose that V satisfies the requirements from the definition of the LS^{n-2} -property.

Let $f: \mathbb{S}^k \rightarrow V \setminus a$ be a map with $k \leq n - 2$ and $g: \mathbb{S}^k \rightarrow V \setminus a$ be an approximation of f such that $g(\mathbb{S}^k)$ does not separate V . The proof will be done if g can be extended to a map from \mathbb{B}^{k+1} into $U \setminus a$. Indeed, $U \setminus a \in ANR$ implies that if g is close enough to f , then f and g are homotopic in $U \setminus a$ and f is extendable to a map $\bar{f}: \mathbb{B}^{k+1} \rightarrow U \setminus a$ provided g has such an extension. According to the choice of V , there exists an extension $\bar{g}: \mathbb{B}^{k+1} \rightarrow U$ of g , and let $b \in U \setminus \bar{g}(\mathbb{B}^{k+1})$. Since $g(\mathbb{S}^k)$ does not separate V , it does not separate U . Then $U \setminus g(\mathbb{S}^k)$ is connected and locally connected, so there exists an arc C in $U \setminus g(\mathbb{S}^k)$ joining the points a and b . Following an idea from the proof of [22, Theorem 2.6], consider the set

$$A = \{x \in C : \text{there exists a map } g_x: \mathbb{B}^{k+1} \rightarrow U \setminus x \text{ extending } g\}.$$

Obviously, A is open in C and $b \in A$. We are going to show that A is also closed in C . That would imply that $A = C$ and $a \in A$, which will complete the proof.

Suppose $x \in \bar{A}$, and let $\epsilon < d(x, X \setminus U)/2$ be a positive number, where d is a metric on X , such that if a map $g': \mathbb{S}^k \rightarrow U \setminus x$ is ϵ -close to g , then g and g' are homotopic in $U \setminus x$ (such ϵ exists because $U \setminus x$ is an ANR). Proceeding as in the proof of Theorem 2.6 from [22] (with \mathbb{S}^1 replaced by \mathbb{S}^k), we can find $\delta > 0$, a point $y \in A$ with $d(x, y) < \delta$ and a map $h: U \rightarrow U$ such that $h^{-1}(x) = y$ and h is ϵ -close to the identity on U . Since the map $g_y: \mathbb{B}^{k+1} \rightarrow U \setminus y$ (see the definition of

A) extends g , then $h \circ g_y$ maps \mathbb{B}^{k+1} into $U \setminus x$ and $(h \circ g_y)|_{\mathbb{S}^k} = h \circ g$ is homotopic to g in $U \setminus x$. Hence, by the homotopy extension property, g can be extended to a map from \mathbb{B}^{k+1} into $U \setminus x$. So, $x \in A$, which means that A is closed. \square

Recall the property $\Delta(n)$ of Borsuk [7]: $X \in \Delta(n)$ if for every $x \in X$ every neighborhood U of x contains a neighborhood V of x such that each compact nonempty set $B \subset V$ of dimension $\dim B \leq n - 1$ is contractible in a subset of U of dimension $\leq \dim B + 1$. If U in that definition has a compact closure, then $X \in \Delta(n)$ implies that every map $f: K \rightarrow U$, where K is a compactum of dimension $\dim K \leq n$, can be approximated by maps $g: K \rightarrow U$ such that $\dim g(K) \leq \dim K$. On the other hand, if X is a homogeneous locally compact, locally connected metric space of dimension $\dim \geq n$, then every region in X is a Cantor n -manifold, see [21]. Moreover, according to [22, Observation 3.1], $X \times \mathbb{R}$ has the disjoint disk property provided X is homogeneous locally compact ANR of dimension ≥ 4 satisfying $\Delta(2)$. Therefore, Theorem 4.5 implies next corollary which improves [27, Note added in proof].

Corollary 4.6. *Let X be a homogeneous locally compact metric ANR-space such that $\dim X = n \geq 4$ and $X \in \Delta(n - 2)$. Then X has the property $H(n - 1, \mathbb{Z})$ and the product $X \times \mathbb{R}$ has the disjoint disk property.*

It is still unknown whether the dimension of a product of two homogeneous ANR-compacta satisfies the logarithmic law. The next corollary provides a partial answer of this question.

Corollary 4.7. *Let $X \in LS^{n-2}$ be a homogeneous locally compact metric ANR-space such that $\dim X = n$. Then $\dim X \times Y = \dim X + \dim Y$ for every compact metric space Y .*

PROOF. According to the proof of Theorem 4.5, $X \in LCC^{n-2}$. Using the terminology of Kodama's paper [20], this means that every $x \in X$ is an HL^{n-2} -point. On the other hand, X does not have any HL^{n-1} -point. Indeed, otherwise each point of X would be HL^{n-1} by the homogeneity. Then X has $D(0, n)$ by [22, Proposition 1.8] and, by [22, Corollary 2.4], $\dim X > n$, a contradiction. Therefore, we can apply [20, Corollary 2] to conclude that $\dim X \times Y = \dim X + \dim Y$ for any compact metric space Y . \square

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