

SOME PROPERTIES OF SELECTORS

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1. Introduction. The following definitions are motivated by the well-known Michael theorem from [6, 7].

Definition [10]. Let \mathcal{K} be a class of topological spaces and let the symbols \mathcal{L} and \mathcal{M} assign the families $\mathcal{L}(X)$ and $\mathcal{M}(X)$ of subsets of X whenever X is a topological space. A space X is called a $\mathcal{K}-\mathcal{L}(X)-\mathcal{M}(X)$ -selector provided every l. s. c. set-valued mapping $\Phi: Y \rightarrow \mathcal{L}(X)$ with $Y \in \mathcal{K}$ has a u. s. c. selection $\psi: Y \rightarrow \mathcal{M}(X)$. A space X is called a convex- $\mathcal{K}-\mathcal{L}(X)-\mathcal{M}(X)$ -selector provided X is a closed convex subset of a locally convex vector space and every l. s. c. mapping $\Phi: Y \rightarrow \overline{\text{conv}} \mathcal{L}(X)$ has a u. s. c. selection $\psi: Y \rightarrow \mathcal{M}(X)$ (here $\overline{\text{conv}} \mathcal{L}(X)$ stands for the family of all closed convex hulls of the $\mathcal{L}(X)$ members).

Let us recall that a set-valued mapping $\theta: Y \rightarrow 2^X$ is said to be lower semicontinuous, or l. s. c. (upper semi-continuous, or u. s. c.) iff $\theta^{-1}(\Gamma) = \{y \in Y: \theta(y) \cap \Gamma \neq \emptyset\}$ is open (closed) in Y for every open (closed) Γ in X .

Next, denote by $(\text{convex-}) \mathcal{S}(\mathcal{K}; \mathcal{L}, \mathcal{M})$ the class of all (convex-) $\mathcal{K}-\mathcal{L}(X)-\mathcal{M}(X)$ -selectors. Our main problem concerning so defined notions of selectors could be formulated as follows: determine the class $(\text{convex-}) \mathcal{S}(\mathcal{K}; \mathcal{L}, \mathcal{M})$ for any fixed triple $(\mathcal{K}; \mathcal{L}, \mathcal{M})$ or, at least, the class $(\text{convex-}) \mathcal{S}(\mathcal{K}; \mathcal{L}, \mathcal{M}) \cap \mathcal{K}$.

Let us recall, for example, the following results:

1. [9]. $\mathcal{S}(\mathcal{C}; \mathcal{F}, \mathcal{F}) \cap \mathcal{C}$ is precisely the class of all compact metric spaces and $\text{convex-}\mathcal{S}(\mathcal{C}; \mathcal{F}, \mathcal{F}) \cap \mathcal{C}$ is the class of all convex compact subsets of Frechet spaces. Here (and below) \mathcal{C} stands for the class of all compact Hausdorff spaces and \mathcal{F} assigns to X the family $\mathcal{F}(X)$ of its closed nonvoid subsets. 'Frechet space' means locally convex complete metrizable space.

2. [10]. $\mathcal{S}(\mathcal{N}; \mathcal{C}', \mathcal{F}) \cap \mathcal{N} = \mathcal{S}(\mathcal{P}_{\aleph_0}; \mathcal{F}, \mathcal{F}) \cap \mathcal{P}_{\aleph_0} = \{\text{the class of all complete separable metric spaces}\}$, and $\text{convex-}\mathcal{S}(\mathcal{N}; \mathcal{C}', \mathcal{F}) \cap \mathcal{N} = \text{convex-}\mathcal{S}(\mathcal{P}_{\aleph_0}; \mathcal{F}, \mathcal{F}) \cap \mathcal{P}_{\aleph_0} = \{\text{the class of all closed convex separable subspaces of Frechet spaces}\}$ and $\mathcal{S}(\mathcal{N}; \mathcal{F}, \mathcal{F}) \cap \mathcal{N} = \{\text{the class of all compact metric spaces}\}$, $\text{convex-}\mathcal{S}(\mathcal{N}; \mathcal{F}, \mathcal{F}) \cap \mathcal{N} = \{\text{the class of all closed convex separable, weakly compact subspaces of Frechet spaces}\}$. \mathcal{N} stands for the class of all normal Hausdorff spaces; the class of all τ -paracompact members of \mathcal{N} is denoted by \mathcal{P}_τ and $\mathcal{C}'(X) = \mathcal{C}(X) \cup \{X\}$, where $\mathcal{C}(X) = \{F \in \mathcal{F}(X): F \text{ is compact}\}$.

2. Results. Proposition 1. Let \mathcal{K} be one of the following classes: \mathcal{C} , $p\mathcal{L}$, $p\mathcal{P}$, \mathcal{P} , \mathcal{P}_τ , $cw\mathcal{N}$ or $cw\mathcal{N}_\tau$. If $X \in (\text{convex-}) \mathcal{S}(\mathcal{K}; \mathcal{C}', \mathcal{F}) \cap \mathcal{K}$, then X^{\aleph_0} is perfectly normal.

Here $p\mathcal{P}$ ($p\mathcal{L}$) stands for the class of all p -paracompact (p -Lindeloff) in the sense of Arkhangel'skii [1] members of \mathcal{N} and \mathcal{P} is the class of all paracompact Hausdorff spaces; $cw\mathcal{N}_\tau$ denotes the class of all τ -collectionwise normal spaces and $cw\mathcal{N}$ is the class of all collectionwise normal spaces.

Proposition 2. Let \mathcal{K} denote either the class $cw\mathcal{N}$ or the class $cw\mathcal{N}_\tau$. If $X \in (\text{convex-}) \mathcal{S}(\mathcal{K}; \mathcal{L}, \mathcal{F}) \cap \mathcal{K}$ with $\mathcal{C}'(X) \subset \mathcal{L}(X) \subset \mathcal{F}(X)$, then each decreasing sequence of members of $\mathcal{L}(X)$ (resp. $\overline{\text{conv}} \mathcal{L}(X)$) has no empty intersection.

Corollary 1. Let \mathcal{K} denote either the class $p\mathcal{P}$ or the class $p\mathcal{L}$. If $X \in (\text{convex-}) \mathcal{S}(\mathcal{K}; \mathcal{C}', \mathcal{F}) \cap \mathcal{K}$, then X is metrizable.

Note that Corollary 1 improves the corresponding results from [9].

Corollary 2. Let \mathcal{K} denote either the class $cw\mathcal{N}$ or the class $cw\mathcal{N}_\tau$. Then the class $\mathcal{S}(\mathcal{K}; \mathcal{F}, \mathcal{F}) \cap \mathcal{K}$ coincides with the class of all compact metric spaces.

Corollary 3. Every Lindeloff member of the class $\mathcal{S}(\mathcal{P}; \mathcal{C}', \mathcal{F})$ is completely metrizable.

3. Proofs. Corollary 1 follows immediately from Proposition 1 and from a result of Arkhangel'skii [1]. The one half of Corollary 2 follows from the result of Chamber [2], the latter being applicable in view of Propositions 1 and 2, and the other half of Corollary 2 is covered by known results on selections [6], [7]. What concerns Corollary 3, the Lindeloffness of X permits us to apply the very same arguments as in [10] (the proof of Theorem 1(a)). All we have to prove here are, therefore, Propositions 1 and 2.

We begin with the following definition. For an arbitrary topological space Z , denote its topology by $T(Z)$. A space X will be said to be e -embeddable in a space Y if X can be embedded in Y in such a manner that there is a function $e: T(X) \rightarrow T(Y)$ with the properties:

- (i) $e(U) \cap X = U$ for every $U \in T(X)$;
- (ii) if $\bigcap \{U_\alpha: \alpha \in A\} = \emptyset$ with $U_\alpha \in T(X)$, then $\bigcap \{e(U_\alpha): \alpha \in A\} = \emptyset$.

Proposition 3. (V. Valov). Let X be e -embeddable in a product of metrizable spaces and X does not contain one-point compactifications of uncountable discrete spaces. Then X^{\aleph_0} is perfectly normal.

Proof. Let X be e -embedded in $Z = \prod \{Y_\alpha: \alpha \in A\}$ with metrizable factors Y_α . Denote $cZ = \prod \{\beta Y_\alpha: \alpha \in A\}$ and $cX = \text{cl}_{cZ}(X)$, where βY_α stands for the Čech-Stone compactification of Y_α . Next, define a set-valued mapping $r: Z \rightarrow \mathcal{F}(cX)$ by setting $r(z) = \bigcap \{\text{cl}_{cX}(U): z \in e(U)\}$ if $z \in \bigcup \{e(U): U \in T(X)\}$ and $r(z) = cX$ otherwise. It can easily be seen that r is u. s. c., that $r(x) = \{x\}$ for every $x \in X$ and $r(z) \cap X \neq \emptyset$ for every $z \in Z$. As in [10] (the proof of Lemma 2) it can be seen that for every $x \in X$ there is a G_δ -subset $H(x)$ in Z with $x \in H(x) \subset r^{-1}(x)$. Let P be a closed subset of X and let $H(P) = \text{cl}_Z(\bigcup \{H(x): x \in P\})$. Since $P \subset H(P) \subset r^{-1}(\text{cl}_{cX}(P))$, we have $H(P) \cap X = P$. The set $H(P)$, being the closure of a union of G_δ -subsets of Z , is a zero-set in Z [5]. Thus P is a zero-set in X .

Next, we have to show that X is normal. Let P and Q be disjoint closed subsets of X . The set $H(P) \setminus H(Q)$ is both G_δ and F_σ in Z , so that $H(P) \setminus H(Q)$

$= \bigcup_{i=1}^{\infty} P_i$, where $P_1 \subset P_2 \subset \dots$ is a sequence of closed subsets of Z . Analogously,

$H(Q) \setminus H(P) = \bigcup_{i=1}^{\infty} Q_i$ with the sequence $Q_1 \subset Q_2 \subset \dots$ of closed subsets of Z .

For $L \subset Z$ denote by L^* the set $\bigcup \{K \subset L: K \text{ is a } G_\delta\text{-set in } Z\}$. From a result of R. Pol a. E. Pol [11] (Lemma 1) we have $H(P) \setminus H(Q) = \bigcup_{i=1}^{\infty} \text{cl}_Z(P_i^*)$

as well as $H(Q) \setminus H(P) = \bigcup_{i=1}^{\infty} \text{cl}_Z(Q_i^*)$. By [11] (Proposition 2) and [4], every closed

G_δ -subset of Z is C -embedded in Z . Thus, for each n there are open subsets U'_n and V'_n of Z with $\text{cl}_Z(P_n^*) \subset U'_n$ and $\text{cl}_Z(U'_n) \cap H(Q) = \emptyset$, $\text{cl}_Z(Q_n^*) \subset V'_n$ and $\text{cl}_Z(V'_n) \cup H(P) = \emptyset$.

Let $U_i = U'_i \setminus \bigcup_{j \leq i} \text{cl}_Z(V'_j)$, $V_i = V'_i \setminus \bigcup_{j \leq i} \text{cl}_Z(U'_j)$. Then $U = \bigcup_{i=1}^{\infty} U_i$ and $V = \bigcup_{i=1}^{\infty} V_i$ are disjoint neighbourhoods of P and Q respectively. Thus X is proved to be perfectly normal. But the class of all spaces, which are both e -embeddable in a product of metric spaces and do not contain one-point compactifications of uncountable discrete spaces, is closed with respect to countable products, so that X^{\aleph_0} is perfectly normal.

Proof of Proposition 1. In view of Proposition 3, it suffices to show that X is both e -embeddable in a product of metric spaces and does not contain one-point compactification of any uncountable discrete space. Let us therefore, consider several cases.

Case 1. X is a $\mathcal{K} - \mathcal{C}'(X) - \mathcal{F}(X)$ -selector. For $\mathcal{K} = p\mathcal{P}$, $\mathcal{K} = pL$ or $\mathcal{K} = \mathcal{C}$ the e -embeddability of X in a product of metric spaces follows immediately from the known fact that each member of such a \mathcal{K} lies as a closed subset of a product $M \times I^c \in \mathcal{K}$ ($I = [0, 1]$) where M is metrizable. Thus, there exists a u. s. c. retraction $r: M \times I^c \rightarrow \mathcal{F}(X)$ and one can put $e(U) = r^\#(U) = \{y \in M \times I^c: r(y) \in U\}$ for every open set U in X .

Let now \mathcal{K} be one of the classes \mathcal{P} , \mathcal{P}_τ , $cw\mathcal{N}$ or $cw\mathcal{N}_\tau$. For $X \in \mathcal{K}$ let $\lambda = \sup\{|X'|: X' \text{ be a discrete closed subset of } X\}$ and denote by $J(\lambda)$ the hedgehog with τ spikes (see [3]). Then X can be considered as a subspace of $(J(\lambda))^A = Z$ where A stands for the set of all continuous mappings from X to $J(\lambda)$. Let Y be the space obtained from Z by making the points of $Z \setminus X$ isolated. Then X is closed in Y and $Y \in \mathcal{K}$ (see [13], p. 66). Thus, there is a u. s. c. retraction $r: Y \rightarrow \mathcal{F}(X)$ and one can put $e(U) = \text{Int}_Z(r^\#(U))$ for each open set U in X . The e -embeddability of X in a product of metric spaces has thus been proved in Case 1. Since, at the same case, each compact subset of X is metrizable [9], X does not contain one-point compactifications of uncountable discrete spaces.

Case 2. X is a convex- $\mathcal{K} - \mathcal{C}'(X) - \mathcal{F}(X)$ -selector. The e -embeddability of X in a product of metrizable spaces can be established as in the previous case. Now, suppose $\alpha T \subset X$ is the one-point compactification of the uncountable discrete set $T \subset X$. The set $P = \overline{\text{conv}}(\alpha T)$ is a convex- $\mathcal{K} - \mathcal{C}'(P) - \mathcal{F}(P)$ -selector and thus P is e -embeddable in a product of metrizable spaces. Taking into account that the set $\text{conv}(\alpha T)$ is both Lindeloff and dense in P , we note that in fact P is e -embeddable in a product of separable metric spaces, so that P does not contain an uncountable disjoint family of open subsets. On the other hand, as it is actually proved in [9] (Proposition 2), P must contain such a family. The latter contradiction completes the proof of Proposition 1.

Proof of Proposition 2 (see [10]). Suppose $F_1 \supset F_2 \supset \dots$ is a strictly decreasing sequence of members of $\mathcal{L}(X)$ with an empty intersection. Let us take a collectionwise-normal space Y having an open covering $\omega = \{U_1 \subset U_2 \subset \dots\}$ with no closed index-refinement (a Dowker space; for the existence of such a space see the paper of Rudin [13]). Next, as in [10], define a mapping $\Phi: Y \rightarrow \mathcal{L}(X)$ by setting $\Phi(y) = F_{n_y}$, where $n_y = \min\{n: y \in U_n\}$. Obviously Φ is l. s. c., so let $\psi: Y \rightarrow \mathcal{F}(X)$ be a u. s. c. selection for Φ . Next, fix a point $x_n \in F_n \setminus F_{n+1}$ for each n . Since $\bigcap \{F_n: n = 1, 2, \dots\} = \emptyset$, the set $H = \{x_n: n = 1, 2, \dots\}$ is closed and discrete in F_1 (even in X). For $x \in F_1$ define $\phi(x) = \{x_{n_x}, x_{n_x+1}, \dots\}$ where $n_x = \max\{n: x \in F_n\}$. The mapping $\phi: F_1 \rightarrow \mathcal{F}(H)$ is

l. s. c. and the space F_1 being perfectly normal (by Proposition 1), is countably paracompact. So, there exists (see [6], [8]) a u. s. c. selection $r: F_1 \rightarrow \mathcal{C}(H)$ for φ . Denote $P_n = (r \cdot \psi)^{-1}(\{x_1, \dots, x_n\})$. Since $r \cdot \psi$ is u. s. c., P_n is closed in Y and one can easily check that $P_n \subset U_n$ for each n and $Y = \bigcup_{n=1}^{\infty} P_n$. The latter contradiction completes the proof of Proposition 2.

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