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# Locally $n$-Connected Compacta and $UV^n$-Maps

**Abstract:** We provide a machinery for transferring some properties of metrizable $ANR$-spaces to metrizable $LC^n$-spaces. As a result, we show that for completely metrizable spaces the properties $ALC^n$, $LC^n$ and $WLC^n$ coincide to each other. We also provide the following spectral characterizations of $ALC^n$ and cell-like compacta: A compactum $X$ is $ALC^n$ if and only if $X$ is the limit space of a $\sigma$-complete inverse system $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ consisting of compact metrizable $LC^n$-spaces $X_\alpha$ such that all bonding projections $p^\beta_\alpha$, as a well all limit projections $p_\alpha$, are $UV^n$-maps.
A compactum $X$ is a cell-like (resp., $UV^n$) space if and only if $X$ is the limit space of a $\sigma$-complete inverse system consisting of cell-like (resp., $UV^n$) metrizable compacta.

**Keywords:** absolute neighborhood retracts; $ALC^n$-spaces; cell-like maps and spaces; $WLC^n$-spaces; $UV^n$-maps and spaces

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## 1 Introduction

Following [7], we say that a space $M$ is weakly locally $n$-connected (briefly, $WLC^n$) in a space $Y$ if $M \subset Y$ is closed and for every point $x \in M$ and its open neighborhood $U$ in $M$ there exists a neighborhood $V$ of $x$ in $M$ such that any map from $\mathbb{S}^k$, $k \leq n$, into $V$ is null-homotopic in $\widetilde{U}$, where $\widetilde{U}$ is any open in $Y$ set with $\widetilde{U} \cap M = U$ (in such a case we say that $\widetilde{U}$ is an open extension of $U$ in $Y$). Dranishnikov [5] also suggested the following notion: a space $M$ is approximately locally $n$-connected (briefly, $ALC^n$) in a space $Y$ if $M \subset Y$ is closed and for every point $x \in M$ and its open neighborhood $U$ in $M$ there exists a neighborhood $V$ of $x$ in $M$ such that for any open in $Y$ extension $\widetilde{U}$ of $U$ there exists and open in $Y$ extension $\widetilde{V}$ of $V$ with any map from $\mathbb{S}^k$, $k \leq n$, into $\widetilde{V}$ being null-homotopic in $\widetilde{U}$. One can show that if $M$ is metrizable (resp., compact), then $M$ is $WLC^n$ in a given space $Y$, where $Y$ is a metrizable (resp., compact) $ANR$, if and only if $M$ is $WLC^n$ in any metrizable (resp., compact) $ANR$ containing $M$ as a closed set. The same is true for the property $ALC^n$. So, the definitions of $WLC^n$ and $ALC^n$ don't depend on the $ANR$-space containing $M$, and we say that $M$ is $WLC^n$ (resp., $ALC^n$). In another words, both $WLC^n$ and $ALC^n$ are in general relative properties but they are absolute properties of a metrizable space with respect to inclusions into "good" spaces.

Dranishnikov [7] proved that both properties $WLC^n$ and $LC^n$ are equivalent in the class of metrizable compacta. Using some results of Gutev [10] and Dugundji-Michael [9], we show in Theorem 2.6 that all properties $WLC^n$, $LC^n$ and $ALC^n$ coincide for completely metrizable spaces. The proof of Theorem 2.6 is based on the technique, developed in Section 2, for transferring properties of metrizable $ANR$'s to $LC^n$-subspaces (in this way well known properties of metrizable $LC^n$-spaces can be obtained from the corresponding properties of

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metrizable $ANR$'s, see for example Proposition 2.2). Section 2 contains also a characterization of metrizable $LC^n$-spaces whose analogue for $ANR$'s was established by Nhu [12].

It is well known that the class of metrizable $LC^n$-spaces are exactly absolute neighborhood extensors for $(n+1)$-dimensional paracompact spaces, and this is not valid for non-metrizable spaces. Outside the class of metrizable spaces we have the following characterization of $ALC^n$ compacta (Theorem 3.1): A compactum $X$ is $ALC^n$ if and only if $X$ is the limit space of a $\sigma$-complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable $LC^n$-spaces $X_\alpha$ such that all bonding projections $p_\alpha^\beta$, as a well all limit projections $p_\alpha$, are $UV^n$-maps. A similar spectral characterization is obtained for cell-like or $UV^n$ compacta, see Theorem 3.3. Both Theorem 3.1 and Theorem 3.3 provide different classes of compacta $\mathcal{C}$ and corresponding classes of maps $\mathcal{M}$ adequate with $\mathcal{C}$ in the following sense (see Shchepin [18]):

- A compactum $X$ belongs to $\mathcal{C}$ if and only if $X$ is the limit space of a $\sigma$-complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable $X_\alpha \in \mathcal{C}$ with all bonding projections $p_\alpha^\beta$ being from $\mathcal{M}$.

For example, according to Theorem 3.1, the class of $ALC^n$ compacta is adequate with the class of $UV^n$-maps. Recall that a closed subset $A \subset X$ is $UV^n$ in $X$ (resp., cell-like in $X$) if every neighborhood $U$ of $A$ in $X$ contains a neighborhood $V$ of $A$ such that, for each $0 \le k \le n$, any map $f\colon \mathbb{S}^k \to V$ is null-homotopic in $U$ (resp., $A$ is contractible in every neighborhood of $A$ in $X$). A space $X$ is said to be $UV^n$ (resp., cell-like) provided it is $UV^n$ (resp., cell-like) in some $ANR$-space containing $X$ as a closed set. It is well known that for metrizable or compact $X$ this definition does not depend on the $ANR$-spaces containing $X$ as a closed set, see for example [17]. A map $f\colon X \to Y$ between compact spaces is called $UV^n$ (resp., cell-like) if all fibres of $f$ are $UV^n$ (resp., cell-like).

Theorem 3.1 yields that any compact $LC^n$-space is $ALC^n$. It is interesting to find an example of an $ALC^n$-compactum which is not $LC^n$ (obviously, such a compactum should be non-metrizable).

## 2 Metrizable $ALC^n$-spaces

We are going to establish some properties of metrizable $LC^n$-spaces using the corresponding properties of metrizable $ANR$-spaces. Recall that a map $f\colon X \to Y$ is *$n$-invertible* if for every Tychonoff space $Z$ with $\dim Z \le n$ and any map $g\colon Z \to Y$ there is a map $h\colon Z \to X$ such that $g = f \circ h$.

The next theorem follows from a stronger result due to Pasynkov [15, Theorem 6], and its proof is based on Dranishnikov results [6, Theorem 1] and [7, Theorem 1.2] (such a result concerning the extension dimension with respect to quasi-finite complexes was established in [14, Proposition 2.7]). We provide here a simple proof based on factorization theorems.

**Theorem 2.1.** *Every Tychonoff space $M$ is the image of a Tychonoff space $X$ with $\dim X \le n$ under a perfect $n$-invertible map. In case $M$ is metrizable, $X$ can be supposed to be also a metrizable space with $w(X) = w(M)$.*

*Proof.* Let $M$ be a Tychonoff space of weight $\tau$. Consider all couples $(Z_\alpha, f_\alpha)$, where $Z_\alpha$ is a Tychonoff space of weight $w(Z_\alpha) \le w(\beta M)$, $\dim Z_\alpha \le n$ and $f_\alpha$ is a map from $Z_\alpha$ into $M$ (here $\beta M$ is the Čech-Stone compactification of $M$). Denote by $Z$ the disjoint sum of all spaces $Z_\alpha$. Obviously, there is a natural map $f\colon Z \to M$ such that $f|Z_\alpha = f_\alpha$ for all $\alpha$. Let $\widetilde{f}\colon \beta Z \to \beta M$ be the extension of $f$. Then, by the Mardešić's factorization theorem [11], there exists a compactum $\widetilde{X}$ of weight $w(\widetilde{X}) \le w(\beta M)$ with $\dim \widetilde{X} \le \dim \beta Z = n$ and maps $h\colon \beta Z \to \widetilde{X}$, $\widetilde{g}\colon \widetilde{X} \to \beta M$ such that $\widetilde{g} \circ h = \widetilde{f}$. Let $X = \widetilde{g}^{-1}(M)$ and $g = \widetilde{g}|X$. According to Corollary 6 and Main Theorem from [15], $\dim X \le n$. To show that $g$ is $n$-invertible, suppose $f_0\colon Z_0 \to M$ is a map with $\dim Z_0 \le n$. Applying again the Mardešić's factorization theorem for the map $\widetilde{f_0}\colon \beta Z_0 \to \beta M$, we obtain a compactum $K$ and maps $h_1\colon \beta Z_0 \to K$ and $f_2\colon K \to \beta M$ such that $\dim K \le n$, $w(K) \le w(\beta M)$ and $f_2 \circ h_1 = \widetilde{f_0}$. Then, as above, $Z' = f_2^{-1}(M)$ is a space of dimension $\le n$ and weight $\le w(\beta M)$. So, there exists $\alpha^*$ and a homeomorphism $j\colon Z' \to Z$ such that $j(Z') = Z_{\alpha^*}$ and $f_{\alpha^*} \circ j = f_2|Z'$. Consequently, $h \circ j$ is a map from $Z'$ to $X$ with $f_2|Z' = g \circ h \circ j$. Finally, $h \circ j \circ h_1$ is a map from $Z_0$ to $X$ such that $g \circ h \circ j \circ h_1 = f_0$.

If $M$ is metrizable, the proof is simpler. Indeed, in this case $P = \widetilde{f}^{-1}(M)$ is a space of dimension $\le n$ and the restriction $\widetilde{f}|P$ is a perfect map. So, by Pasynkov's factorization theorem [16], there exists a metrizable space $X$ and maps $h\colon P \to X$, $g\colon X \to M$ such that $g \circ h = \widetilde{f}$, $w(X) \le w(M)$ and $\dim X \le n$. Then $g$ is a perfect map because so is $\widetilde{f}|P$, and according to the above arguments, $g$ is $n$-invertible. □

The next proposition shows that Theorem 2.1 allows some properties of metrizable $ANR$-spaces to be transferred to metrizable $LC^n$-spaces.

**Proposition 2.2.** *Let $M$ be a metrizable $LC^n$-space and $\alpha$ an open cover of $M$. Then there exists an open cover $\beta$ of $M$ refining $\alpha$ such that for any two $\beta$-near maps $f, g\colon Z \to M$ defined on a metrizable space $Z$ of dimension $\le n$ any $\beta$-homotopy $H\colon A \times [0,1] \to M$ between $f|A$ and $g|A$, where $A$ is closed in $Z$, can be extended to an $\alpha$-homotopy $\widetilde{H}\colon Z \times [0,1] \to M$ connecting $f$ and $g$.*

*Proof.* We embed $M$ as a closed subset of a metrizable $ANR$-space $P$ and let $p\colon Y_P \to P$ be a perfect $(n+1)$-invertible surjection such that $Y_P$ is a metrizable space of dimension $\le n+1$ (see Theorem 2.1). Since $M$ is $LC^n$, there is an open set $G$ in $Y_P$ containing $p^{-1}(M)$ and a map $q\colon G \to M$ extending the restriction $p|p^{-1}(M)$. Then there exists an open set $W \subset P$ containing $M$ with $p^{-1}(W) \subset G$ (recall that $p$ is a perfect map). Obviously $W$ is also an $ANR$-space containing $M$ as a closed set. So, without losing generality, we may assume that $W = P$, $G = Y_P$ and $q$ is a map from $Y_P$ onto $M$. Now, for every open $U \subsetneqq M$, let $\widetilde{U} = P\backslash(p(q^{-1}(M\backslash U)))$. The set $\widetilde{U}$ is non-empty and open in $P$, $\widetilde{U} \cap M = U$ and $p^{-1}(\widetilde{U}) \subset q^{-1}(U)$.

If $\alpha$ is an open cover of $M$ consisting of proper subsets of $M$, the set $L = \bigcup\{\widetilde{U} : U \in \alpha\}$ is open in $P$ and contains $M$. So, $L$ is also an $ANR$ and $\widetilde{\alpha} = \{\widetilde{U} : U \in \alpha\}$ is an open cover of $L$. According to the properties of metrizable $ANR$'s (see for example [13, chapter IV, Theorem 1.2]), there exists an open cover $\widetilde{\beta}$ of $L$ with the following property: for any two $\beta$-near maps $h_1, h_2\colon Z \to M$ defined on a metrizable space $Z$ any $\beta$-homotopy $H\colon A \times [0,1] \to M$ between $h_1|A$ and $h_2|A$, where $A$ is closed in $Z$, can be extended to an $\alpha$-homotopy $F\colon Z \times [0,1]$ connecting $h_1$ and $h_2$. Then $\beta = \{V \cap M : V \in \widetilde{\beta}\}$ is an open cover of $M$ refining $\alpha$ and has the desired property. Indeed, suppose $f, g\colon Z \to M$ are two $\beta$-near maps and $H\colon A \times [0,1] \to M$ is a $\beta$-homotopy between $f|A$ and $g|A$, where $Z$ is a metrizable space of dimension $\le n$ and $A$ is closed in $Z$. According to the choice of $\widetilde{\beta}$, $H$ can be extended to an $\widetilde{\alpha}$-homotopy $F\colon Z \times [0,1] \to L$ connecting $f$ and $g$. Since $\dim Z \times [0,1] \le n+1$ and $p$ is $(n+1)$-invertible, there exists a lifting $F_1\colon Z \times [0,1] \to p^{-1}(L)$ of $F$. Finally, $\widetilde{H} = q \circ F_1$ is an $\alpha$-homotopy between $f$ and $g$. □

We also need the following property of metrizable $LC^n$-spaces.

**Proposition 2.3.** *Suppose both $M$ and $P$ are metrizable $LC^n$-spaces with $M \subset P$ being closed. Then there exists an open set $U \subset P$ containing $M$ with the following property: If $Z$ is a metrizable space with $\dim Z \le n$ and $h\colon Z \to U$ is a map, there exists a map $h_1\colon Z \to M$ such that $h$ and $h_1$ are homotopic in $P$.*

*Proof.* Let $p\colon Y_P \to P$ be a perfect $(n+1)$-invertible surjection such that $Y_P$ is a metrizable space of dimension $\le n+1$, and $q\colon p^{-1}(W) \to M$ extends the restriction $p|p^{-1}(W)$, where $W \subset P$ is an open set containing $M$ (see the proof of Proposition 2.2). Since $W$ (as an open subset of $P$) is $LC^n$, we can apply Proposition 2.2 (with $\alpha = \{W\}$ and $A$ a point), to obtain that $W$ has an open cower $\beta$ such that any two $\beta$-near maps from $Z$ into $W$ are homotopic. For every $V \in \beta$ let $G_V = V\backslash p(q^{-1}(M\backslash(M \cap V)))$ and $U = \bigcup\{G_V : V \in \beta\}$. Because $\beta$ covers $M$, $M \subset U$. If $h\colon Z \to U$ is any map, let $h_1\colon Z \to M$ be the map $h_1 = q \circ \widetilde{h}$, where $\widetilde{h}\colon Z \to p^{-1}(U)$ is a lifting of $h$. It is easily seen that the maps $h$ and $h_1$ are $\beta$-near. So, $h$ and $h_1$ are homotopic in $W$, and hence in $P$. □

**Corollary 2.4.** *If $M$ and $P$ are metrizable $LC^n$-spaces such that $M \subset P$ is closed, then $M$ is $ALC^n$ in $P$.*

*Proof.* Since $M$ is $LC^n$, for every $x \in M$ and its open neighborhood $U$ in $M$ there exists a neighborhood $V$ of $x$ in $M$ such that any map of a $k$-sphere, $k \le n$, into $V$ is contractible in $U$. Let $\widetilde{U} \subset P$ and $G \subset \widetilde{U}$ be open in $P$ extensions of $U$ and $V$, respectively. Then both $V$ and $G$ are $LC^n$ (as open subsets of $LC^n$-spaces), and $V$ is closed in $G$. So, there is an open extension $\widetilde{V} \subset G$ satisfying the conclusion from Proposition 2.3.

Consequently, any map $g: \mathbb{S}^k \to \widetilde{V}$, $0 \le k \le n$, is homotopic in $G$ to a map $g_1: \mathbb{S}^k \to V$. Since $g_1$ is homotopic in $U$ to a constant map, we obtain that $g$ is homotopic in $\widetilde{U}$ to a constant map. □

**Proposition 2.5.** *Let $P$ be metrizable and $M \subset P$ be a closed and $LC^n$-set. Then every closed set $A \subset M$ is $UV^n$ in $M$ provided $A$ is $UV^n$ in $P$.*

*Proof.* Let $p: Y_P \to P$ be a perfect $(n+1)$-invertible surjection such that $Y_P$ is a metrizable space of dimension $\le n+1$, and $q: p^{-1}(W) \to M$ extends the restriction $p|p^{-1}(W)$, where $W \subset P$ is an open set containing $M$. Suppose $U \subset M$ is an open set containing $A$ and let $\widetilde{U} = W \backslash p(q^{-1}(M \backslash U))$. Obviously, $\widetilde{U} \subset W$ is an open extension of $U$. Since $A \in UV^n(P)$, there is an open set $\widetilde{V} \subset \widetilde{U}$ containing $A$ such that any map from $\mathbb{S}^k$ with $0 \le k \le n$ to $\widetilde{V}$ can be extended to a map from $\mathbb{B}^{k+1}$ to $\widetilde{U}$. Let $V = M \cap \widetilde{V}$ and $g: \mathbb{S}^k \to V$ be a map. Then extend $g$ to a map $g_1: \mathbb{B}^{k+1} \to \widetilde{U}$ and take a lifting $g_2: \mathbb{B}^{k+1} \to p^{-1}(\widetilde{U})$ of $g_1$. Finally, $\widetilde{g} = q \circ g_2$ is a map from $\mathbb{B}^{k+1}$ to $U$ extending $g$. □

Now, we are in a position to prove the main theorem in this section.

**Theorem 2.6.** *For a completely metrizable space $M$ the following are equivalent:*

*(i)* *$M$ is $LC^n$;*
*(ii)* *$M$ is $ALC^n$;*
*(iii)* *$M$ is $WLC^n$.*

*Proof.* Implication $(i) \Rightarrow (ii)$ follows from Corollary 2.4, and implication $(ii) \Rightarrow (iii)$ is trivial.
$(iii) \Rightarrow (i)$. We need the following notion introduced in [10]:

(*) A collection $\mathcal{A}$ of subsets a metric space $(Y, d)$ is called uniformly equi-weakly $LC^k$, abbreviated uniformly equi-$WLC^k$, if for every $\epsilon > 0$ there corresponds $\delta(\epsilon) > 0$ such that for every $\mu > 0$ and $A \in \mathcal{A}$, every continuous image of a $k$-sphere in $A$ of diameter $< \delta(\epsilon)$ is contractible over a subset of $B^d_\mu(A) = \{y \in Y : d(y; A) < \mu\}$ of diameter $< \epsilon$.

We embed $M$ as a closed subset of a completely metrizable $ANR$-space $P$ and consider the following relation between the open subsets of $M$: $V\alpha U$ if $V \subset U$ and every map from $\mathbb{S}^k$, $k \le n$, into $V$ is null-homotopic in $\widetilde{U}$, where $\widetilde{U}$ is any open extension of $U$ in $P$. It follows from [9, Theorem 1] the existence of a complete metric $\rho$ on $M$ generating its topology such that

(**) For every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for every $x \in M$ and any open $W \subset P$ with $W \cap M = B^\rho_\epsilon(x)$, every continuous image of a $k$-sphere ($k \le n$) in $B^\rho_{\delta(\epsilon)}(x)$ is contractible in $W$.

The metric $\rho$ can be extended to a complete metric $d$ on $P$, see [1]. Property (**) implies that the single-element collection $\{M\}$ is uniformly equi-$WLC^k$ for every $k \le n$, in the sense of (*). Indeed, to verify (*), take $\delta(\epsilon/2)$, where $\delta(\epsilon) \le \epsilon$ is as in (*). Next, let $\mu > 0$ and $g: \mathbb{S}^k \to M$ be a continuous map such that the diameter of $g(\mathbb{S}^k)$ is less than $\delta(\epsilon/2)$. Fix a point $x \in g(\mathbb{S}^k)$, and an open set $W \subset P$ such that $W \subset B^d_\mu(M) \cap B^d_{\epsilon/2}(x)$ and $W \cap M = B^\rho_{\epsilon/2}(x)$. Then, by (**), $g(\mathbb{S}^k)$ is contractible in $W$. Hence, $g(\mathbb{S}^k)$ is contractible over a subset of $B^d_\mu(M)$ of diameter $< \epsilon$. So, $\{M\}$ is uniformly equi-$WLC^k$, $k \le n$. Finally, by [10, Theorem 3.1], $\{M\}$ is uniformly equi-$LC^n$; equivalently $M$ is $LC^n$. □

A sequence of open covers $\mathcal{U} = (\mathcal{U}_k)_{k\in\mathbb{N}}$ of a metric space $(M, d)$ is called a *zero-sequence* if $\lim_{k\mapsto\infty} \mathrm{mesh}\mathcal{U}_k = 0$. For any such a sequence we define $\mathrm{Tel}(\mathcal{U}) = \bigcup_{k\in\mathbb{N}} \mathrm{N}(\mathcal{U}_k \cup \mathcal{U}_{k+1})$. Here $N(\mathcal{U}_k \cup \mathcal{U}_{k+1})$ is the nerve of $\mathcal{U}_k \cup \mathcal{U}_{k+1}$ with $\mathcal{U}_k$ and $\mathcal{U}_{k+1}$ considered as disjoint sets. For any $\sigma \in \mathrm{Tel}(\mathcal{U})$ let $s(\sigma) = \max\{s : \sigma \in N(\mathcal{U}_s \cup \mathcal{U}_{s+1})\}$.

We complete this section by a characterization of metrizable $LC^n$-spaces similar to the characterization of metrizable $ANR$-spaces provided in [12] (see also [17, Theorem 6.8.1]).

**Proposition 2.7.** *A metric space $(M, d)$ is $LC^n$ if and only if it has a zero-sequence $\mathcal{U}$ of open covers such that any map $f_0: K^{(0)} \to M$ with $f(U) \in U$, $U \in K^{(0)}$, where $K$ is a subcomplex of* $\mathrm{Tel}(\mathcal{U})$, *extends to a map $f: K^{(n+1)} \to M$ satisfying the following condition:*

*(***) For any sequence $\{\sigma_k\}$ of simplexes of $K^{(n+1)}$ with $s(\sigma_k) \to \infty$ we have* $\lim_{k\mapsto\infty} \mathrm{diam}(\mathrm{f}(\sigma_k)) = 0$.

*Proof.* Suppose $M$ is $LC^n$ and embed $(M,d)$ isometrically in a metric $ANR$-space $(P,\rho)$ as a closed subset. According to the proof of Theorem 2.1, there are metrizable space $Y_P$ and two maps $p\colon Y_P \to P$ and $q\colon Y_p \to M$ such that $\dim Y_P \le n+1$, $p$ is $(n+1)$-invertible and $q$ extends the map $p|p^{-1}(M)$. Using the proof of Nhu's theorem [12, Theorem 1.1] for $ANR$'s, we can find a zero-sequence $\mathcal{V} = (\mathcal{V}_k)_{k\in\mathbb{N}}$ of $P$ such that any map $h_0\colon K^{(0)} \to P$ with $h_0(V) \in V$ for each $V \in K^{(0)}$, where $K$ is a subcomplex of $\mathrm{Tel}(\mathcal{U})$, extends to a map $h\colon |K| \to P$ such that $\lim_{k\mapsto\infty} \mathrm{diam}(\mathrm{h}(\sigma_k)) = 0$ for any sequence $\{\sigma_k\}$ of simplex of $K$ with $s(\sigma_k) \to \infty$. Let us show that the sequence $(\mathcal{U}_k)_{k\in\mathbb{N}}$, $\mathcal{U}_k = \{V\cap M : V \in \mathcal{V}_k\}$, is as required. Indeed, for each $U = V\cap M \in \mathcal{U}_k$ define $W(U) = V\backslash(p(q^{-1}(M\backslash U)))$ and consider the open families $\mathcal{W}_k = \{W(U) : U \in \mathcal{U}_k\}$, $k \in \mathbb{N}$. We may assume that each $U$ is a proper subset of $M$, so $W_k \neq \varnothing$ for all $k$. Note that $\mathcal{W}_k$ may not cover $P$, but any $\mathcal{W}_k$ covers $M$. Moreover, $\mathrm{mesh}\mathcal{W}_\mathrm{k} \le \mathrm{mesh}\mathcal{V}_\mathrm{k}$, so $\lim_{k\mapsto\infty} \mathrm{mesh}\mathcal{W}_\mathrm{k} = 0$. If $K$ is a subcomplex of $\mathrm{Tel}(\mathcal{U})$, take any map $f_0\colon K^{(0)} \to M$ with $f_0(U) \in U$. Hence, $f_0$ extends to a map $g\colon |K| \to P$ such that $\lim_{k\mapsto\infty} \mathrm{diam}(\mathrm{g}(\sigma_k)) = 0$ for any sequence $\{\sigma_k\}$ of simplexes of $K$ with $s(\sigma_k) \to \infty$. Finally, let $f\colon |K^{(n+1)}| \to M$ be the map $q \circ \widetilde{g}$, where $\widetilde{g}\colon |K^{(n+1)}| \to Y_P$ is a lifting of $g$. To show that $f$ satisfies condition $(***)$, fix an $\epsilon > 0$, a cover $\mathcal{W}_m$ with $\mathrm{mesh}(\mathcal{W}_\mathrm{m}) < \epsilon$ and a sequence $\{\sigma_k\} \subset K^{(n+1)}$ with $\lim_{k\mapsto\infty} s(\sigma_k) = \infty$. Then $\lim_{k\mapsto\infty} \mathrm{mesh}(\mathrm{g}(\sigma_k)) = 0$, so there exists $k_0$ such that $g(\sigma_k) \subset \bigcup \mathcal{W}_m$ for all $k \ge k_0$ (recall that $g(U) \in M$ for all $U \in K^{(0)}$). Hence, for any two different points $x, y \in \sigma_k$ with $k \ge k_0$ we have $g(x) \in W(U_x)$ and $g(y) \in W(U_y)$ for some $W(U_x), W(U_y) \in \mathcal{W}_m$. So, according to the definition of $W(U)$, $f(x) \in U_x$ and $f(y) \in U_y$. Therefore, $\rho(f(x), g(x)) \le \mathrm{diam}(\mathrm{W}(\mathrm{U_x})) < \epsilon$ and, similarly, $\rho(f(y), g(y)) < \epsilon$. Consequently, $d(f(x), f(y)) < 2\cdot\epsilon + \mathrm{diam}(\mathrm{g}(\sigma_k))$.

To prove the other implication, embed $M$ as a closed subset of a metrizable space $Z$ with $\dim Z\backslash M \le n+1$ and follow the proof of implication $(iii) \Rightarrow (i)$ from [12, Theorem 1.1] to obtain that $M$ is a retract of a neighborhood $W_1$ of $M$ in $Z$ (the only difference is that in Fact 1.2 from [12] we take the cover $\mathcal{V}$ of $W_1\backslash M$ to be of order $\le n+1$, so the nerve $N(\mathcal{V})$ is a complex of dimension $\le n+1$). Then by [13, chapter V, Theorem 3.1], $M$ is $LC^n$. □

# 3 $UV^n$-maps and $ALC^n$-spaces

In this section we provide spectral characterizations of non-metrizable $ALC^n$-compacta and cell-like compacta. Recall that a map $f\colon X \to Y$ between compact spaces is said to be soft [18] if for every compactum $Z$, its closed subset $A \subset Z$ and maps $h\colon A \to X$ and $g\colon Z \to Y$ with $f\circ h = g|A$ there exists a lifting $\overline{g}\colon Z \to X$ of $g$ extending $h$.

**Theorem 3.1.** *A compactum $X$ is $ALC^n$ if and only if $X$ is the limit space of a σ-complete inverse system $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ consisting of compact metrizable $LC^n$-spaces $X_\alpha$ such that all bonding projections $p^\beta_\alpha$, as a well all limit projections $p_\alpha$, are $UV^n$-maps.*

*Proof.* Suppose that $X$ is the limit space of an inverse system $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ such that each $X_\alpha$ is a metrizable $LC^n$-compactum and all $p^\beta_\alpha$ are $UV^n$-maps. We embed $X$ in a Tychonoff cube $\mathbb{I}^B$, where $\mathbb{I} = [0,1]$ and the cardinality of $B$ is equal to $\tau$. According to Shchepin's spectral theorem [18], we can assume that $B$ is the union of countable sets $B_\alpha$, $\alpha \in A$, such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$, $B_\gamma = \bigcup\{B_{\gamma(k)} : k = 1, 2, ..\}$ for any chain $\gamma(1) < \gamma(2) < ..$ with $\gamma = \sup\{\gamma(k) : k \ge 1\}$, and each $p^\beta_\alpha\colon X_\beta \to X_\alpha$ is the restriction of the projection $q^\beta_\alpha\colon \mathbb{I}^{B_\beta} \to \mathbb{I}^{B_\alpha}$. So, each $X_\alpha = q_\alpha(X)$ is a subset of $\mathbb{I}^{B_\alpha}$, where $q_\alpha$ denotes the projection $q_\alpha\colon \mathbb{I}^B \to \mathbb{I}^{B_\alpha}$. Note that $q_\alpha$ is soft. We also denote $q_\alpha|X$ by $p_\alpha$. Choose $x_0 \in X$ and its neighborhood $U \subset X$. There exists $\alpha_0 < \tau$ and an open set $U_0 \subset X_{\alpha_0}$ with $x_0 \in p^{-1}_{\alpha_0}(U_0) \subset \overline{p^{-1}_{\alpha_0}(U_0)} \subset U$. Since $X_{\alpha_0}$ is $LC^n$, there exists a neighborhood $V_0$ of $x_{\alpha_0} = p_{\alpha_0}(x_0)$ such that for all $k \le n$ the inclusion $j_0 : V_0 \hookrightarrow U_0$ generates trivial homomorphisms $j^\star_0 : \pi_k(V_0) \to \pi_k(U_0)$ between the homotopy groups. Let $V = p^{-1}_{\alpha_0}(V_0)$ and $\widetilde{U}$ be an open set in $\mathbb{I}^B$ extending $U$. Choose a finite family $\omega = \{W_1, .., W_m\}$ of open sets from the ordinary base of $\mathbb{I}^B$ such that $W = \bigcup_{i=1}^{i=m} W_i$ covers $\overline{p^{-1}_{\alpha_0}(U_0)}$ and $W \subset \widetilde{U}$. Then we can find $\alpha_1 > \alpha_0$ with $q^{-1}_{\alpha_1}(q_{\alpha_1}(W_i)) = W_i$, $i \le m$. Denote $V_1 = (p^{\alpha_1}_{\alpha_0})^{-1}(V_0)$

and $U_1 = (p^{\alpha_1}_{\alpha_0})^{-1}(U_0)$. Obviously, $q_{\alpha_1}(W)$ is open in $\mathbb{I}^{B_{\alpha_1}}$ containing $U_1$. Take an open in $\mathbb{I}^{B_{\alpha_1}}$ extension $\widetilde{V_1}$ of $V_1$ with $\widetilde{V_1} \subset q_{\alpha_1}(W)$. Since $\widetilde{V_1}$ is an $ANR$ and $V_1$ is $LC^n$ (as an open subset of $X_{\alpha_1}$), by Proposition 2.3, there exists an open in $\mathbb{I}^{B_{\alpha_1}}$ extension $G$ of $V_1$ which is contained in $\widetilde{V_1}$ with the following property: for every map $h\colon Z \to G$, where $Z$ is at most $n$-dimensional metrizable space, there exists a map $h_1\colon Z \to V_1$ such that $h$ and $h_1$ are homotopic in $\widetilde{V_1}$. Finally, let $\widetilde{V} = q^{-1}_{\alpha_1}(G)$. It is easily seen that $\widetilde{V}$ is an open extension of $V$ and $\widetilde{V} \subset W \subset \widetilde{U}$. Consider a map $f\colon \mathbb{S}^k \to \widetilde{V}$, where $k \leq n$. Then there exists a map $g\colon \mathbb{S}^k \to V_1$ such that $q_{\alpha_1} \circ f$ and $g$ are homotopic in $\widetilde{V_1}$. We are going to show that $g$ is homotopic to a constant map in the set $U_1$. This will be done if the inclusion $j_1\colon V_1 \hookrightarrow U_1$ generates a trivial homomorphism $j_1^\star\colon \pi_k(V_1) \to \pi_k(U_1)$. To this end, we consider the following commutative diagram:

$$\begin{array}{ccc} \pi_k(V_1) & \xrightarrow{\ j_1^\star\ } & \pi_k(U_1) \\ \big\downarrow{\scriptstyle (p^{\alpha_1}_{\alpha_0})^\star} & & \big\downarrow{\scriptstyle (p^{\alpha_1}_{\alpha_0})^\star} \\ \pi_k(V_0) & \xrightarrow{\ j_0^\star\ } & \pi_k(U_0) \end{array}$$

Since $X_{\alpha_1}$ is $LC^n$ and the map $p^{\alpha_1}_{\alpha_0}$ is $UV^n$, each fiber of $p^{\alpha_1}_{\alpha_0}$ is an $UV^n$-set in $X_{\alpha_1}$, see Proposition 2.5. Then, according to [8, Theorem 5.3], both vertical homomorphisms from the above diagram are isomorphisms. This implies that $j_1^\star$ is trivial because so is $j_0^\star$. Hence, $q_{\alpha_1} \circ f$ is homotopic to a constant map in the set $\widetilde{V_1} \cup U_1 \subset q_{\alpha_1}(W)$ (recall that $q_{\alpha_1} \circ f$ is homotopic to $g$ in $\widetilde{V_1}$ and $g$ is homotopic to a constant map in $U_1$). Therefore, there is a map $f_1\colon \mathbb{B}^{n+1} \to q_{\alpha_1}(W)$ extending $q_{\alpha_1} \circ f$. Since $q_{\alpha_1}$ is a soft map, and $W = q^{-1}_{\alpha_1}(q_{\alpha_1}(W))$, $f$ can be extended to a map $\tilde{f}\colon \mathbb{B}^{n+1} \to W$. Thus, the inclusion $\widetilde{V} \hookrightarrow \widetilde{U}$ generates a trivial homomorphism between $\pi_k(\widetilde{V})$ and $\pi_k(\widetilde{U})$ for any $k \leq n$. So, $X$ is $ALC^n$.

Now, suppose $X$ is $ALC^n$, and consider $X$ as a subset of some $\mathbb{I}^B$. Since the sets $V$ and $\widetilde{V}$ from the definition of $ALC^n$ depend on the point $x$, the set $U$ and its open extension $\widetilde{U}$, respectively, we use the notations $\lambda(x, U) = V$ and $\lambda(x, U, \widetilde{U}) = \widetilde{V}$. First, we show that the sets $V$ and $\widetilde{V}$ can be chosen to be functionally open in $X$ and in $\mathbb{I}^B$, respectively. Indeed, if $U \subset X$ is a neighborhood of $x \in X$, we take a functionally open in $X$ neighborhood $V^\star$ of $x$ with $V^\star \subset \lambda(x, U)$. Then for a given open in $\mathbb{I}^B$ extension $\widetilde{U}$ of $U$ and every $y \in V^\star$ choose a functionally open in $\mathbb{I}^B$ neighborhood $G(y)$ of $y$ with $G(y) \subset \lambda(x, U, \widetilde{U}) \cap G$, where $G$ is an open in $\mathbb{I}^B$ extension of $V^\star$. Since $V^\star$, as a functionally open subset of $X$, is Lindelöf, there exist countably many sets $G(y_i)$ whose union covers $V^\star$. Obviously the set $\widetilde{G} = \bigcup_{i=1}^\infty W(y_i)$ is a functionally open in $\mathbb{I}^B$ extension of $V^\star$ which is contained in $\lambda(x, U, \widetilde{U})$. So, every map from $\mathbb{S}^k$ to $\widetilde{G}$, $k \leq n$, is homotopic in $\widetilde{U}$ to a constant map. Therefore, for every open set $U \subset X$ and every $x \in U$ there exists a functionally open in $X$ set $V = \lambda(x, U)$ such that for any open in $\mathbb{I}^B$ extension $\widetilde{U}$ of $U$ we can find a functionally open in $\mathbb{I}^B$ extension $\lambda(x, U, \widetilde{U})$ of $V$ contained in $\widetilde{U}$ with all homomorphisms $\pi_k(\lambda(x, U, \widetilde{U}) \to \pi_k(\widetilde{U})$, $k \leq n$, being trivial.

Every functionally open set $U$ in $X$ is Lindelöf, so there are countably many $x_i \in U$ such that $\{\lambda(x_i, U) : i = 1, 2, ..\}$ is a cover of $U$. We fix such a countable cover $\gamma(U)$ for any functionally open set $U \subset X$.

Let $A \subset B$ and $W_0, W_1, .., W_k$ be elements of the standard open base $\mathcal{B}_A$ for $\mathbb{I}^A$ such that

(w) $\varnothing \neq \overline{W_0} \cap X_A \subset \bigcup_{i=1}^{i=k} W_i$.

Here, $X_A = q_A(X)$, where $q_A : \mathbb{I}^B \to \mathbb{I}^A$ is the projection. We also denote by $[W_0, W_1, .., W_k]_A$ the set $q_A^{-1}\big((\bigcup_{i=1}^{i=k} W_i) \backslash (X_A \backslash W_0)\big)$. Observe that $[W_0, W_1, .., W_k]_A \cap X = q_A^{-1}(W_0) \cap X$, so $[W_0, W_1, .., W_k]_A \cap X$ is functionally open in $X$. Moreover, if $\gamma(q_A^{-1}(W_0) \cap X) = \{V(y_i) : i = 1, 2, ..\}$, where $y_i \in q_A^{-1}(W_0) \cap X$ for all $i$, we consider the sets $\widetilde{V(y_i)} = \lambda(y_i, q_A^{-1}(W_0) \cap X, [W_0, W_1, .., W_k]_A)$. So, each $\widetilde{V(y_i)}$ is a functionally open extension of $V(y_i)$ and all homomorphisms $\pi_k(\widetilde{V(y_i)}) \to \pi_k([W_0, W_1, .., W_k]_A)$, $k \leq n$, are trivial. Denote the family $\{\widetilde{V(y_i)} : i = 1, 2, ..\}$ by $\widetilde{\gamma}([W_0, W_1, .., W_k]_A)$. We say that a set $A \subset B$ is admissible if the following holds:

- $q_A^{-1}(q_A(V)) = V$ for all $V \in \widetilde{\gamma}([W_0, W_1, .., W_k]_A)$ and all finitely many elements $W_0, W_1, .., W_k$ of $\mathcal{B}_A$ satisfying condition $(w)$;
- $p_A^{-1}(p_A(V)) = V$ for all $V \in \gamma(q_A^{-1}(W) \cap X)$ and $W \in \mathcal{B}_A$, where $p_A\colon X \to X_A$ is the restriction $q_A|X$.

Recall that for any functionally open set $U$ in $\mathbb{I}^B$ (resp., in $X$) there is a countable set $s(U) \subset B$ such that $q^{-1}_{s(U)}(q_{s(U)}(U)) = U$ (resp., $p^{-1}_{s(U)}(p_{s(U)}(U)) = U$).

*Claim* 1. *For any set $A \subset B$ there exists an admissible set $C \subset B$ of cardinality $|A|.\aleph_0$ containing $A$.*

We construct by induction sets $A = A_0 \subset A_1 \subset \ldots \subset A_k \subset A_{k+1} \subset ..$ of cardinality $|A|.\aleph_0$ such that:

- $s(V) \subset A_{k+1}$ for all $V \in \gamma(q^{-1}_{A_k}(W) \cap X)$ and all $W \in \mathcal{B}_{A_k}$.
- $s(V) \subset A_{k+1}$ for all $V \in \widetilde{\gamma}([W_0, W_1, .., W_m]_{A_k})$ and all finitely many $W_0, W_1, .., W_m \in \mathcal{B}_{A_k}$ satisfying condition $(w)$.

The construction follows from the fact that the cardinality of each base $\mathcal{B}_{A_k}$ is $|A|.\aleph_0$ and the families $\gamma(q^{-1}_{A_k}(W) \cap X)$ and $\widetilde{\gamma}([W_0, W_1, .., W_m]_{A_k})$ are countable provided $W, W_0, .., W_k \in \mathcal{B}_{A_k}$. It is easily seen that the set $C = \bigcup_{k=1}^{\infty} A_k$ is as required.

*Claim* 2. *$X_A$ is an $ALC^n$-space for every admissible set $A \subset B$.*

Let $y \in U$, where $U$ is open in $X_A$. Take $x \in X$ and $W_0 \in \mathcal{B}_A$ containing $y$ such that $\overline{W_0} \cap X_A \subset U$ and $q_A(x) = y$. Then $x$ belongs to some $V_x \in \gamma(q_A^{-1}(W_0) \cap X)$. Since $s(V_x) \subset A$ (recall that $A$ is admissible), $p_A^{-1}(p_A(V_x)) = V_x$. So, $V = p_A(V_x)$ is a functionally open in $X_A$ neighborhood of $y$, which is contained in $U$. Take any open extension $\widetilde{U} \subset \mathbb{I}^A$ of $U$, and finitely many $W_1, .., W_k \in \mathcal{B}_A$ satisfying $\overline{W_0} \cap X_A \subset \bigcup_{i=1}^{i=k} W_i \subset \widetilde{U}$. Since $V_x \in \gamma(q_A^{-1}(W_0) \cap X)$, the set $\widetilde{V_x} = \lambda(x, q_A^{-1}(W_0) \cap X, [W_0, W_1, .., W_k]_A)$ is a functionally open extension of $V_x$ with $\widetilde{V_x} \in \widetilde{\gamma}([W_0, W_1, .., W_k]_A)$. Then $s(\widetilde{V_x}) \subset A$ and $\widetilde{V} = q_A(\widetilde{V_x})$ is a functionally open in $\mathbb{I}^A$ extension of $V$. We are going to show that all homomorphisms $\pi_k(\widetilde{V}) \to \pi_k(\widetilde{U})$, $k \leq n$, are trivial. Indeed, every map $f\colon \mathbb{S}^k \to \widetilde{V}$ can be lifted to a map $g\colon \mathbb{S}^k \to \widetilde{V_x}$ because $q_A^{-1}(\widetilde{V}) = \widetilde{V_x}$ and $q_A$ is a soft map. Recall that $\widetilde{V_x}$ belongs to $\widetilde{\gamma}([W_0, W_1, .., W_k]_A)$, so $g$ can be extended to a map $\widetilde{g} : \mathbb{B}^{k+1} \to [W_0, W_1, .., W_k]_A$. Finally, $q_A \circ \widetilde{g} : \mathbb{B}^{k+1} \to q_A([W_0, W_1, .., W_k]_A) \subset \widetilde{U}$ is an extension of $f$. This completes the proof of Claim 2.

*Claim* 3. *Let $A_2 \subset A_1$ be admissible subsets of $B$. Then each fiber of $p^{A_1}_{A_2} : X_{A_1} \to X_{A_2}$ is $UV^n$.*

Let $x \in X_{A_2}$ and $U \subset \mathbb{I}^{A_1}$ be an open set containing $F = (p^{A_1}_{A_2})^{-1}(x)$. Take $W'_0 \in \mathcal{B}_{A_2}$ with $x \in W'_0$ and $(p^{A_1}_{A_2})^{-1}(\overline{W'_0} \cap X_{A_2}) \subset U$. So, $(q^{A_1}_{A_2})^{-1}(\overline{W'_0}) \cap X_{A_1} \subset U$. Next, choose $W_1, .., W_k \in \mathcal{B}_{A_1}$ such that $(q^{A_1}_{A_2})^{-1}(\overline{W'_0}) \cap X_{A_1} \subset \bigcup_{i=1}^{i=k} W_i \subset U$. Obviously, $W_0 = (q^{A_1}_{A_2})^{-1}(W'_0) \in \mathcal{B}_{A_1}$ and $\overline{W_0} \cap X_{A_1} \subset \bigcup_{i=1}^{i=k} W_i$. Let $y \in X$ with $q_{A_2}(y) = x$. Then $y$ belongs to some $V_y \in \gamma(q^{-1}_{A_1}(W_0) \cap X)$. Because $q^{-1}_{A_1}(W_0) = q^{-1}_{A_2}(W'_0)$ and $A_2$ is admissible, we have $s(V_y) \subset A_2$. Hence, $p^{-1}_{A_2}(p_{A_2}(V_y)) = V_y$ and $p^{-1}_{A_2}(x) = p^{-1}_{A_1}(F) \subset V_y$. Then $\widetilde{V_y} = \lambda(y, q_A^{-1}(W_0) \cap X, [W_0, W_1, .., W_k]_{A_1})$ is a functionally open extension of $V_y$ and $\widetilde{V_y} \in \widetilde{\gamma}([W_0, W_1, .., W_k]_{A_1})$. Because $s(\widetilde{V_y}) \subset A_1$, $q^{-1}_{A_1}(q_{A_1}(\widetilde{V_y})) = \widetilde{V_y}$ and $V = q_{A_1}(\widetilde{V_y})$ is an open subset of $\mathbb{I}^{A_1}$ such that $F \subset V \subset \bigcup_{i=1}^{i=k} W_i \subset U$. Then, as in the proof of Claim 2, we can show that the inclusion $V \hookrightarrow U$ generates trivial homomorphisms $\pi_k(V) \to \pi_k(U)$, $k \leq n$. Hence, $F$ is $UV^n$.

*Claim* 4. *The union of any increasing sequence of admissible subsets of $B$ is also admissible.*

This claim follows directly from the definition of admissible sets.

Now we can complete the proof of Theorem 3.1. According to Claim 1 and Claim 4, the set $B$ is covered by a family $\mathcal{S}$ of countable sets such that $\mathcal{S}$ is stable with respect to taking unions of increasing countable subfamilies. Then, by Claim 2, each $X_A$, $A \in \mathcal{S}$, is a metrizable $ALC^n$-compactum . Hence, Theorem 2.6 yields that all spaces $X_A$, $A \in \mathcal{S}$, are $LC^n$. Moreover, the projections $p^{A_1}_{A_2}$ are $UV^n$-maps for any $A_1, A_2 \in \mathcal{S}$ with $A_2 \subset A_1$. Because the set $B$ is admissible, it follows from Claim 3 that the limit projections $p_A : X \to X_A$, $A \in \mathcal{S}$, are also $UV^n$-maps. Therefore, $X$ is limit space of the $\sigma$-complete inverse system $\{X_A, p^{A_1}_{A_2}, A, A_1, A_2 \in \mathcal{S}\}$. □

**Corollary 3.2.** *Any $LC^n$-compactum $X$ is an $ALC^n$-space.*

*Proof.* We embed $X$ in some $\mathbb{I}^B$ and let $A_0 \subset B$ be a countable set. According to the factorization theorem of Bogatyi-Smirnov [2, Theorem 3], there is a metrizable compactum $Y_1$ and maps $g_1 : X \to Y_1$, $h_1 : Y_1 \to X_{A_0}$ such that $p_{A_0} = h_1 \circ g_1$ and all fibers of $g_1$ are $UV^n$-sets in $X$. Then, by [8, Theorem 5.4], $Y_1$ is $LC^n$. Since $g_1$ depends on countably many coordinates, there is a countable set $A_1 \subset B$ containing $A_0$ and a map $f_1 : X_{A_1} \to Y_1$ such that $f_1 \circ p_{A_1} = g_1$. In this way we construct countable sets $A_k \subset A_{k+1} \subset B$ and $LC^n$ metrizable compacta $Y_k$ together with maps $g_k : X \to Y_k$, $f_k : : X_{A_k} \to Y_k$ and $h_k : Y_k \to X_{A_{k-1}}$ such that $g_k = f_k \circ p_{A_k}$,

$p_{A_{k-1}} = h_k \circ g_k$ and the fibers of each $g_k$ are $UV^n$-sets in $X$. Let $A$ be the union of all $A_k$. Then $X_A$ is the limit space of the inverse sequence $\mathcal{S} = \{Y_k, s_k^{k+1} = f_k \circ h_{k+1}\}$. According to [8, Theorem 5.3], for all open sets $U \subset Y_k$ the group $\pi_m(U)$ is isomorphic to $\pi_m(g_k^{-1}(U))$, $m = 0, 1, .., n$. This property of the maps $g_k$ implies that any $s_k^{k+1}\colon Y_{k+1} \to Y_k$ is an $UV^n$-map. Hence, by Theorem 3.1, $X_A$ is an $ALC^n$-compactum (as the limit of an inverse sequence of metrizable $LC^n$-compacta and bounding $UV^n$-maps). Finally, by Theorem 2.6, $X_A$ is also an $LC^n$-space. Moreover, for any $y \in X_A$ we have $p_A^{-1}(y) = \bigcap_{k\geq 1} g_k^{-1}(y_k)$, where $y_k = s_k(y)$ with $s_k\colon X_A \to Y_k$ being the projections of $\mathcal{S}$. Because all $g_k^{-1}(y_k)$ are $UV^n$-sets in $X$, so is the set $p_A^{-1}(y)$. Therefore, every countable subset $A_0$ of $B$ is contained in an element of the family $\mathcal{A}$ consisting of all countable sets $A \subset B$ such that $X_A$ is $LC^n$ and the fibers of the map $p_A$ are $UV^n$-sets in $X$. It is easily seen that the union of an increasing sequence of elements of $\mathcal{A}$ is again from $\mathcal{A}$, and that $p_A^C\colon X_C \to X_A$ is an $UV^n$-map for all $A, C \in \mathcal{A}$ with $A \subset C$. So, the inverse system $\{X_A, p_A^C, A, C \in \mathcal{A}\}$ is $\sigma$-complete and consists of metrizable $LC^n$-compacta and $UV^n$-bounding maps. Then by Theorem 3.1, $X$ is $ALC^n$ (observe that the proof of the "if" part of Theorem 3.1 does not need the assumption that all limit projections are $UV^n$-maps). □

Theorem 3.1 shows that the class of $ALC^n$-compacta is adequate to the class of $UV^n$-maps. Next theorem provides another classes of compacta adequate to cell-like and $UV^n$-maps.

**Theorem 3.3.** *A compactum $X$ is a cell-like (resp., $UV^n$) space if and only if $X$ is the limit space of a $\sigma$-complete inverse system consisting of cell-like (resp., $UV^n$) metrizable compacta.*

*Proof.* Suppose $X$ is a cell-like compactum. Then $X$ has a shape of a point, so we can apply Corollary 8.4.8 from [3] stating that if $\varphi$ is a shape isomorphism between the limit spaces of two $\sigma$-complete inverse systems $\{X_\alpha, p_\alpha^\beta, \alpha \in A\}$ and $\{Y_\alpha, q_\alpha^\beta, \alpha \in A\}$ of metrizable compacta, then the set of those $\alpha \in A$ for which there exist shape isomorphisms $\varphi_\alpha : X_\alpha \to Y_\alpha$ satisfying $Sh(q_\alpha) \circ \varphi = \varphi_\alpha \circ Sh(p_\alpha)$ is cofinal and closed in $A$. So, according to this corollary, $X$ is the limit space of a $\sigma$-complete inverse system consisting of metrizable cell-like compacta. In case $X$ is an $UV^n$-compactum, it has an $n$-shape of a point (this notion was introduced by Chigogidze in [4]), and the above arguments apply.

Suppose now that $X$ is the limit space of a $\sigma$-complete inverse system $\{X_\alpha, p_\alpha^\beta, \alpha \in A\}$ such that all $X_\alpha$ are metrizable cell-like compacta. As in the proof of Theorem 3.1, we can embed $X$ in a Tychonoff cube $\mathbb{I}^B$, where $B$ is the union of countable sets $B_\alpha$, $\alpha \in A$, such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$, $B_\gamma = \bigcup\{B_{\gamma(k)} : k = 1, 2, ..\}$ for any chain $\gamma(1) < \gamma(2) < ..$ with $\gamma = \sup\{\gamma(k) : k \geq 1\}$, and each $p_\alpha^\beta\colon X_\beta \to X_\alpha$ is the restriction of the projection $q_\alpha^\beta\colon \mathbb{I}^{B_\beta} \to \mathbb{I}^{B_\alpha}$. Then $X_\alpha = q_\alpha(X) \subset \mathbb{I}^{B_\alpha}$ with $q_\alpha$ being the projection from $\mathbb{I}^B$ onto $\mathbb{I}^{B_\alpha}$. If $U$ is a neighborhood of $X$ in $\mathbb{I}^B$, there is $\alpha$ and an open set $U_\alpha$ in $\mathbb{I}^{B_\alpha}$ such that $q_\alpha^{-1}(U_\alpha) \subset U$. Since $X_\alpha$ is a cell-like space, there exists a closed neighborhood $V_\alpha \subset \mathbb{I}^{B_\alpha}$ of $X_\alpha$ contractible in $U_\alpha$. Using that $q_\alpha$ is a soft map, we conclude that $q_\alpha^{-1}(V_\alpha)$ is contractible in $q_\alpha^{-1}(U_\alpha)$. Similarly, we can show that any limit space of a $\sigma$-complete inverse system of metrizable $UV^n$-compacta is also an $UV^n$-compactum. □

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