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V. Valov

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Homological selections and fixed-point theorems

V. Valov

Abstract. A homological selection theorem for C -spaces as well as a finite-dimensional homological selection theorem is established. We apply the finite-dimensional homological selection theorem to obtain fixed-point theorems forusco homologically UV^n set-valued maps.

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1. Introduction

Banach and Cauty [1, Theorem 8] provided a selection theorem for C -spaces, which is a homological version of the Uspenskij's selection theorem [10, Theorem 1.3]. The aim of this paper was to establish a finite-dimensional form of Banach–Cauty theorem, which is the main tool in proving homological analogues of fixed-point theorems forusco maps established in [2], [4] and [7].

All spaces are assumed to be completely regular. Singular homology $H_n(X; G)$, reduced in dimension 0, with a coefficient group G is considered everywhere below. By default, if not explicitly stated otherwise, G is a ring with unit e . Following the notations from [1], for any space X let $S_k(X; G)$ be the group of all singular chains with coefficients from G consisting of singular k -simplexes and $S(X; G)$ denote the singular complex of X , so $S(X; G)$ is the direct sum $\bigoplus_{k=0}^{\infty} S_k(X; G)$. The groups $S_k(X; G)$ are linked via the boundary homomorphisms $\partial : S_k(X; G) \rightarrow S_{k-1}(X; G)$.

If $\sigma : \Delta^k \rightarrow X$ is a singular k -simplex (Δ^k is the standard k -simplex), we denote by $\|\sigma\|$ the carrier $\sigma(\Delta^k)$ of σ . Similarly, we put $\|c\| = \bigcup_i \|\sigma_i\|$ for any chain $c \in S_k(X; G)$, where $c = \sum_i g_i \sigma_i$ is the irreducible representation of c .

For an open cover \mathcal{U} of X let $S(X, \mathcal{U}; G)$ stand for the subgroup of $S(X; G)$ generated by singular simplexes σ with $\|\sigma\| \subset U$ for some $U \in \mathcal{U}$.

A homomorphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$ is said to be a *chain morphism* if $\varphi(S_k(X, \mathcal{U}; G)) \subset S_k(Y; G)$ for all $k \geq 0$ and $\partial \circ \varphi = \varphi \circ \partial$. A point $x \in X$ is called a *fixed point* for a chain morphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(X; G)$ if for any neighborhood V of x in X there exists a chain $c \in S(V; G) \cap S(X, \mathcal{U}; G)$ such that $\|\varphi(c)\| \cap V \neq \emptyset$.

Let $A \subset B$ be two subsets of a space X . We write $A \xrightarrow{H_m} B$ if the embedding $A \hookrightarrow B$ induces a trivial homomorphism $H_m(A; G) \rightarrow H_m(B; G)$.

A set-valued map $\Phi : X \rightarrow 2^Y$ is called *strongly lower semi-continuous* (br., strongly lsc) if for each compact subset $K \subset Y$ the set $\{x \in X : K \subset \Phi(x)\}$ is open in X . For example, every open-graph set-valued map $\Phi : X \rightarrow 2^Y$ is strongly lsc, see [5, Proposition 3.2].

Here is our first homological selection theorem.

Theorem 1.1. *Let X be a paracompact C -space, Y be an arbitrary space and $\Phi_k : X \rightarrow 2^Y$, $m = 0, 1, \dots, n$, be a finite sequence of strongly lsc maps satisfying the following conditions, where G is a fixed ring with unit:*

- (i) $\Phi_m(x) \xrightarrow{H_m} \Phi_{m+1}(x)$ for every $m = 0, \dots, n - 1$ and every $x \in X$;
- (ii) $H_m(\Phi_n(x); G) = 0$ for all $m \geq n$ and all $x \in X$.

Then there exists an open cover \mathcal{U} of X and a chain morphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$ such that $\varphi(S(U; G)) \subset S(\Phi_n(x); G)$ for every $U \in \mathcal{U}$ and every $x \in U$.

Let us mention that the Banach-Cauty result [1, Theorem 8] is a particular case of Theorem 1.1 with $\Phi_m = \Phi_n$ for all m . There is also a finite-dimensional analogue of the above theorem.

Theorem 1.2. *Let X, Y and G be as in Theorem 1.1. The same conclusion holds if $\dim X \leq n$ and the sequence of strongly lsc maps $\Phi_m : X \rightarrow 2^Y$ satisfies only condition (i).*

Theorem 1.2 and [1, Theorem 7] imply the following fixed point theorem for usco (upper semi-continuous and compact-valued) maps:

Theorem 1.3. *Let X be a paracompact space with $\dim X \leq n$, Y a compact metric AR, G a field and $\Phi : X \rightarrow 2^Y$ be a homologically $UV^{n-1}(G)$ usco map. Then for every continuous map $g : Y \rightarrow X$ there exists a point $y_0 \in Y$ with $y_0 \in \Phi(g(y_0))$.*

The particular case of Theorem 1.3 with $X = Y$ and $g = id_X$ is also interesting.

Corollary 1.4. *Let X be a compact metric AR with $\dim X \leq n$, G a field and $\Phi : X \rightarrow 2^X$ be a homologically $UV^{n-1}(G)$ usco map. Then there exists a point $x_0 \in X$ with $x_0 \in \Phi(x_0)$.*

Recall that a closed subset A of a metric space X is called UV^n in X if every neighborhood U of A in X contains another neighborhood V such that the inclusion $V \hookrightarrow U$ generates trivial homomorphisms $\pi_k(V) \rightarrow \pi_k(U)$ between the homotopical groups for all $k = 0, \dots, n$. If considering

the homology groups $H_k(\cdot; G)$ instead of the homotopy groups $\pi_k(V)$ and $\pi_k(U)$ (i.e. requiring $V \xrightarrow{H_m} U$ for all $m = 0, 1, \dots, n$), then A is said to be *homologically $UV^n(G)$ in X* . It follows from the universal formula coefficients that every UV^n -subset of X is homologically $UV^n(G)$ in X for all groups G . Moreover, following the proof of Proposition 7.1.3 from [8], one can show that A is homologically $UV^n(G)$ in a given metric *ANR*-space X if and only if it is homologically $UV^n(G)$ in any metric *ANR*-space that contains A as a closed set. We say that A is *homologically $UV^\omega(G)$ in X* if A is homologically $UV^n(G)$ for all $n \geq 0$. Because of the last two notions, it is convenient to say that A is *homotopically UV^n in X* instead of A being UV^n in X . We also say that a set-valued map $\Phi : X \rightarrow 2^Y$ is *homologically $UV^n(G)$* if all values $\Phi(x)$ are homologically $UV^n(G)$ -subsets of Y .

Theorem 1.5. *Let X be a compact metric *AR*-space, G a field and $\Phi : X \rightarrow 2^X$ be a homologically $UV^\omega(G)$ usco map. Then there exists a point $x_0 \in Y$ with $x_0 \in \Phi(x_0)$.*

Theorem 1.3 was established by Gutev [7] for usco maps with homotopically UV^n values. A homotopical version of Theorem 1.5 is also known, see Corollaries 3.6 and 5.14 from [4], or Theorem 1.3 from [7]. One can also show that if X is a compact metric *AR* and $\Phi : X \rightarrow 2^X$ is a homologically $UV^\omega(G)$ usco map, then each value $\Phi(x)$ has trivial Čech homology groups with coefficients in G . So, in the particular case when G is the group \mathbb{Q} of the rationals, Theorem 1.5 follows from the more general [3, Theorem 7] treating the so-called algebraic *AR*'s. However, in the framework of usual *AR*'s Theorem 1.5 provides a very simple proof.

2. Homological selection theorems

In this section we prove Theorems 1.1 and 1.2. For any simplicial complex K and an integer $m \geq 0$ let $K^{(m)}$ and $C_m(K; G)$ denote, respectively, the m -skeleton of K and the group generated by the oriented m -simplexes of K with coefficients in G .

Lemma 2.1. *Suppose $\{A_k\}_{k=0}^{m+1}$ is a sequence of subsets of Y with $A_k \xrightarrow{H_k} A_{k+1}$, $k = 0, 1, \dots, m$. Let L be a simplicial complex of dimension m and K be the cone of L . Then every chain morphism $\mu_m : C(L; G) \rightarrow S_m(A_m; G)$ such that $\mu_m(C(L^{(k)}; G)) \subset S_k(A_k; G)$ for all $k \leq m$ can be extended to a chain morphism $\mu_{m+1} : C(K; G) \rightarrow S_{m+1}(A_{m+1}; G)$ satisfying the following conditions:*

- $\mu_{m+1}(C(K^{(k)}; G)) \subset S_k(A_k; G)$ for all $k = 0, 1, \dots, m + 1$;
- $\tilde{\mu}_m \circ \partial_{m+1} = \partial_{m+1} \circ \mu_{m+1}$, where $\tilde{\mu}_m = \mu_{m+1}|_{C(K^{(m)}; G)}$.

Proof. We first extend each morphism $\mu_k = \mu_m|_{(C(L^{(k)}; G)}$ to a morphism $\tilde{\mu}_k : C(K^{(k)}; G) \rightarrow S_k(A_k; G)$ such that $\tilde{\mu}_k \circ \partial_{k+1} = \partial_{k+1} \circ \tilde{\mu}_{k+1}$, $k = 0, 1, \dots, m - 1$. To this end, denote by v_0 the vertex of K and consider the augmentation $\epsilon : S_0(A_1; G) \rightarrow G$ defined by $\epsilon(\sigma) = e$ for all singular 0-simplexes $\sigma \in S_0(A_1; G)$. Define $\tilde{\mu}_0(\{v_0\})$ to be a fixed singular simplex $\sigma_0 \in S_0(A_0; G)$ and

$\tilde{\mu}_0(\{v\}) = \mu_0(\{v\})$ for any $v \in L^{(0)}$. Then extend μ_0 to a homomorphism $\tilde{\mu}_0 : C(K^{(0)}; G) \rightarrow S_0(A_0; G)$ by linearity.

If v_0 is a vertex of a simplex $\Delta \in K^{(1)}$, then $\epsilon(\tilde{\mu}_0(\partial_1(\Delta))) = 0$. Since $A_0 \xrightarrow{H_0} A_1$, there is a singular chain $\tau_\Delta \in S_1(A_1; G)$ such that $\tilde{\mu}_0(\partial_1(\Delta)) = \partial_1(\tau_\Delta)$. Letting $\tilde{\mu}_1(\Delta) = \tau_\Delta$ if $\Delta \in K^{(1)} \setminus L^{(1)}$ and $\tilde{\mu}_1(\Delta) = \mu_1(\Delta)$ if $\Delta \in L^{(1)}$, we define the homomorphism $\tilde{\mu}_1$ on every simplex of $K^{(1)}$. Then extend this homomorphism to $\tilde{\mu}_1 : C(K^{(1)}; G) \rightarrow S_1(A_1; G)$ by linearity.

Because $A_{k-1} \xrightarrow{H_{k-1}} A_k$, we can repeat the above construction to obtain the homomorphisms $\tilde{\mu}_k$ for any $k \leq m$. Then $\tilde{\mu}_m : C(K^{(m)}; G) \rightarrow S_m(A_m; G)$. Since $A_m \xrightarrow{H_m} A_{m+1}$, we can use once more the above arguments to obtain the chain morphism $\mu_{m+1} : C(K; G) \rightarrow S_{m+1}(A_{m+1}; G)$ satisfying the required conditions. □

Lemma 2.2. *Let L be a simplicial complex with trivial homology groups and $A \subset B$ be a pair of spaces. Then every chain morphism $\nu : S(A; G) \rightarrow C(L; G)$ can be extended to a chain morphism $\tilde{\nu} : S(B; G) \rightarrow C(L; G)$.*

Proof. We are going to define by induction for each $k \geq 0$ a homomorphism $\tilde{\nu}_k : S_k(B; G) \rightarrow C_k(L; G)$ extending $\nu_k : S_k(A; G) \rightarrow C_k(L; G)$ such that $\tilde{\nu}_k(\partial_{k+1}(c)) = \partial_{k+1}(\tilde{\nu}_{k+1}(c))$ for every singular chain $c \in S_{k+1}(B; G)$ and $k \geq 0$. For every singular 0-simplex $\sigma \in S_0(B; G)$ we define $\tilde{\nu}_0(\sigma) = \nu_0(\sigma)$ if $\sigma \in S_0(A; G)$ and $\tilde{\nu}_0(\sigma) = v_0$ if $\sigma \notin S_0(A; G)$, where v_0 is a fixed vertex of L . Then extend this homomorphism over $S_0(B; G)$ by linearity.

To define $\tilde{\nu}_1$ we consider the augmentation $\epsilon : C_0(L; G) \rightarrow G$ defined by $\epsilon(v) = e$ for all vertexes of L , see [9]. Thus, $\epsilon(\tilde{\nu}_0(\partial_1(\sigma))) = 0$ for every singular 1-simplex $\sigma \in S_1(B; G)$. Because $H_0(L; G) = 0$, $\partial_1(C_1(L; G)) = \epsilon^{-1}(0)$. Therefore, for every singular simplex $\sigma \in S_1(B; G) \setminus S_1(A; G)$ there exists a chain $c_\sigma \in C_1(L; G)$ such that $\partial_1(c_\sigma) = \tilde{\nu}_0(\partial_1(\sigma))$. We define $\tilde{\nu}_1(\sigma) = \nu_1(\sigma)$ if $\sigma \in S_1(A; G)$ and $\tilde{\nu}_1(\sigma) = c_\sigma$ if $\sigma \in S_1(B; G) \setminus S_1(A; G)$, and extend $\tilde{\nu}_1$ over $S_1(B; G)$ by linearity.

Suppose the homomorphism $\tilde{\nu}_k : S_k(B; G) \rightarrow C_k(L; G)$ was already constructed. Then, using that the kernel of the boundary homomorphism $\partial_k : C_k(L; G) \rightarrow C_{k-1}(L; G)$ coincides with the image $\partial_{k+1}(C_{k+1}(L; G))$, we can define $\tilde{\nu}_{k+1}$ extending $\tilde{\nu}_k$ and satisfying the equality $\tilde{\nu}_k \circ \partial_{k+1} = \partial_{k+1} \circ \tilde{\nu}_{k+1}$. □

Proof of Theorem 1.1. We modify the proof of [1, Theorem 8]. By induction we are going to construct two sequences of locally finite open covers of X , $\mathcal{V}_m = \{V_\alpha : \alpha \in \Gamma_m\}$ and $\mathcal{W}_m = \{W_\alpha : \alpha \in \Gamma_m\}$, $m \geq 0$, an increasing sequence $K_0 \subset K_1 \subset \dots$ of simplicial complexes and chain morphisms $\mu_m : C(K_m; G) \rightarrow S_m(Y; G)$ such that

- (1) $\overline{W}_\alpha \subset V_\alpha$ for all $\alpha \in \Gamma_m$, $m \geq 0$;
- (2) $\dim K_m = m$;
- (3) $\mu_{m+1}|_{C(K_m; G)} = \mu_m$ and $\partial_{m+1} \circ \mu_{m+1} = \tilde{\mu}_m \circ \partial_{m+1}$, where $\tilde{\mu}_m = \mu_{m+1}|_{C(K_m^{(m)}; G)}$.

Moreover, for every m and $\alpha \in \Gamma_m$ we shall assign a finite sub-complex L_α of K_m and a set $\Omega_\alpha = \bigcup_{\sigma \in L_\alpha} \|\mu_m(\sigma)\|$ satisfying the following conditions:

- (4) $\dim L_\alpha = m$ and L_α is a cone with a vertex α whose base is a sub-complex $M_\alpha \subset K_{m-1}$;
- (5) if $m \leq n$ and $\alpha \in \Gamma_m$, then $\Omega_\alpha \subset \Phi_m(x)$ and $\Omega_\alpha^{(k)} \subset \Phi_k(x)$ for all $k \leq m - 1$ and all $x \in V_\alpha$, where $\Omega_\alpha^{(k)} = \bigcup_{\sigma \in L_\alpha} \|\mu_m(\sigma^{(k)})\|$;
- (6) If $m > n$ and $\alpha \in \Gamma_m$, then $\Omega_\alpha \subset \Phi_n(x)$ for all $x \in V_\alpha$.

To start our construction, for every $x \in X$ we fix a point $y_x \in \Phi_0(x)$ and consider the set $O_x = \{x' \in X : y_x \in \Phi_0(x')\}$. Since Φ_0 is strongly lsc, O_x is open in X . Let $\mathcal{V}_0 = \{V_\alpha : \alpha \in \Gamma_0\}$ be a locally finite open cover of X refining the cover $\{O_x : x \in X\}$, and choose $\mathcal{W}_0 = \{W_\alpha : \alpha \in \Gamma_0\}$ to be a locally finite open cover of X with $\overline{W}_\alpha \subset V_\alpha$ for all $\alpha \in \Gamma_0$. Let the complex K_0 be the zero-dimensional complex whose set of vertices is Γ_0 . For every $\alpha \in \Gamma_0$ we set $L_\alpha = \{\alpha\}$ and choose $x_\alpha \in X$ such that $V_\alpha \subset O_{x_\alpha}$. Define $\mu_0 : C(K_0; G) \rightarrow S_0(Y; G)$ to be the homomorphism assigning to each generator corresponding to α the singular 0-simplex y_{x_α} , and let $\Omega_\alpha = \{y_{x_\alpha}\}$.

Suppose for some $m < n - 1$ and all $k \leq m$ we already performed the construction satisfying conditions (1) – (5). Then for every $x \in X$ choose an open neighborhood G_x of x meeting only finitely many elements of the cover $\bigcup_{k \leq m} \mathcal{V}_k$ such that $G_x \subset V_\alpha$ for all $\alpha \in \bigcup_{k=0}^m \Gamma_k$ with $G_x \cap \overline{W}_\alpha \neq \emptyset$. Let $J(x) = \{\alpha \in \bigcup_{k=0}^m \Gamma_k : G_x \subset V_\alpha\}$ and $D_x^{(k)} = \bigcup \{\Omega_\alpha^{(k)} : \alpha \in J(x)\}$, $k \leq m$. Since $J(x)$ is finite, all $D_x^{(k)}$ are compact subsets of Y with $D_x^{(k)} \subset D_x^{(m)} = D_x$. Moreover, condition (5) implies $D_x \subset \Phi_m(x)$. Consider the finite sub-complex $M_x = \bigcup \{L_\alpha : \alpha \in J(x)\}$ of K_m and the cone L_x with a vertex $v_x \notin K_m$ and a base M_x . Then, according to the definition of Ω_α and condition (5), we have $\mu_m(C(M_x^{(k)}; G)) \subset S_k(D_x^{(k)}; G) \subset S_k(\Phi_k(x); G)$, $k \leq m$. Therefore, we can apply Lemma 2.1 to find a chain morphism $\mu_x : C(L_x; G) \rightarrow S_{m+1}(\Phi_{m+1}(x); G)$ extending $\mu_m|C(M_x; G)$ such that $\mu_x(C(L_x^{(k)}; G)) \rightarrow S_k(\Phi_k(x); G)$ and $\partial_{m+1} \circ \mu_x = (\mu_x|C(L_x^{(m)}; G)) \circ \partial_x$, where $\partial_x : C(L_x; G) \rightarrow C(L_x^{(m)}; G)$ is the boundary homomorphism. Then $\Omega_x = \bigcup_{\sigma \in L_x} \|\mu_x(\sigma)\|$ is a compact subset of $\Phi_{m+1}(x)$ containing D_x . The strong lower semi-continuity of Φ_{m+1} yields that $O_x^m = \{x' \in G_x : \Omega_x \subset \Phi_{m+1}(x')\}$ is an open neighborhood of x . So, there exists a locally finite open cover $\mathcal{V}_{m+1} = \{V_\alpha : \alpha \in \Gamma_{m+1}\}$ of X refining the cover $\{O_x^m : x \in X\}$, and take a locally finite open cover $\mathcal{W}_{m+1} = \{W_\alpha : \alpha \in \Gamma_{m+1}\}$ satisfying condition (1). Now, for every $\alpha \in \Gamma_{m+1}$ choose $x_\alpha \in X$ with $V_\alpha \subset O_{x_\alpha}^m$ and let L_α be the cone with base M_{x_α} and vertex α . Define K_{m+1} to be the union $K_m \cup \bigcup_{\alpha \in \Gamma_{m+1}} L_\alpha$. Identifying the cones L_α and L_{x_α} , we define the morphism $\mu_{m+1} : C(K_{m+1}; G) \rightarrow S_{m+1}(Y; G)$ by $\mu_{m+1}|C(K_m; G) = \mu_m$ and $\mu_{m+1}|C(L_\alpha; G) = \mu_{x_\alpha}$. Finally, let $\Omega_\alpha = \Omega_{x_\alpha}$. It is easily seen that conditions (1) – (5) are satisfied. Moreover, the definition of G_x and the inclusion $O_{x_\alpha}^m \subset G_{x_\alpha}$ yield that

- (7) For every $\beta \in \bigcup_{k=0}^m \Gamma_k$ and $\alpha \in \Gamma_{m+1}$ with $W_\alpha \cap W_\beta \neq \emptyset$ we have $W_\alpha \subset V_\beta$, and thus $L_\beta \subset L_\alpha$.

In this way we can perform our construction for all $m \leq n$. If we substitute $\Phi_m = \Phi_n$ for all $m \geq n$, we have also $\Phi_m(x) \xrightarrow{H^m} \Phi_{m+1}(x)$ because

$H_m(\Phi_n(x); G) = 0$ for all $x \in X$. Therefore, following the above arguments, we can perform for all m the steps from m to $m + 1$ satisfying conditions (1)–(7).

Let $K = \bigcup_{m=0}^\infty K_m$. Then the morphisms μ_m define a chin morphism $\mu : C(K; G) \rightarrow S(Y; G)$. Because X is a C -space, there exists a sequence of disjoint open families $\mathcal{U}_m = \{U_\lambda : \lambda \in \Lambda_m\}$, $m \geq 0$, such that each \mathcal{U}_m refines \mathcal{W}_m and the family $\mathcal{U} = \bigcup_{m=0}^\infty \mathcal{U}_m$ covers X .

The final step of the proof is to construct of a chain morphism from $S(X, \mathcal{U}; G)$ into $C(K; G)$. To this end, let $\Lambda = \bigcup_{k=0}^\infty \Lambda_k$ and $\Lambda(m) = \bigcup_{k=0}^m \Lambda_k$. Consider also the sub-complexes $S_m = \sum_{\lambda \in \Lambda(m)} S(U_\lambda; G)$ of $S(X; G)$, $m \geq 0$, whose union is $S(X, \mathcal{U}; G)$. For every $\lambda \in \Lambda_m$ select an $\alpha_\lambda \in \Gamma_m$ with $U_\lambda \subset W_{\alpha_\lambda}$. We are going to construct a chain morphism $\nu : S(X, \mathcal{U}; G) \rightarrow C(K; G)$ such that

$$(8) \nu(S(U_\lambda; G)) \subset C(L_{\alpha_\lambda}; G) \text{ for all } \lambda \in \Lambda.$$

For any $\lambda \in \Lambda_0$ the complex L_{α_λ} is a single point. So, we can find a chain morphism $\nu_\lambda : S(U_\lambda; G) \rightarrow C(L_{\alpha_\lambda}; G)$. Since the family \mathcal{U}_0 is disjoint, S_0 is the direct sum of all $S(U_\lambda; G)$, $\lambda \in \Lambda_0$. Hence, the chain morphism $\nu_0 : S_0 \rightarrow C(K; G)$ with $\nu_0|S(U_\lambda; G) = \nu_\lambda$ for all $\lambda \in \Lambda_0$ is well defined and $\nu_0(S(U_\lambda; G)) \subset C(L_{\alpha_\lambda}; G)$.

Suppose that for some m we have constructed chain morphisms $\nu_k : S_k \rightarrow C(K; G)$, $k \leq m$, such that ν_k extends ν_{k-1} and $\nu_k(S(U_\lambda; G)) \subset C(L_{\alpha_\lambda}; G)$ for all $\lambda \in \Lambda(k)$. Because \mathcal{U}_{m+1} is a disjoint family, so is the family $\{S(U_\lambda; G) : \lambda \in \Lambda_{m+1}\}$. Therefore, to extend ν_m over S_{m+1} , it suffices for every $\lambda \in \Lambda_{m+1}$ to extend $\nu_m|S(U_\lambda; G) \cap S_m$ over $S(U_\lambda; G)$. To this end, observe that if $\lambda \in \Lambda_{m+1}$ and $\lambda' \in \Lambda(m)$ with $U_\lambda \cap U_{\lambda'} \neq \emptyset$, then $W_{\alpha_\lambda} \cap W_{\alpha_{\lambda'}} \neq \emptyset$. Thus, according to condition (7), $L_{\alpha_{\lambda'}} \subset L_{\alpha_\lambda}$. Consequently, by (8), $\nu_m(S(U_\lambda; G) \cap S_m) \subset C(L_{\alpha_\lambda}; G)$ for any $\lambda \in \Lambda_{m+1}$. Since L_{α_λ} is contractible, we can apply Lemma 2.2 (with $A = U_\lambda \cap \bigcup_{\lambda' \in \Lambda(m)} U_{\lambda'}$ and $B = U_\lambda$) to find a chain morphism $\nu_\lambda : S(U_\lambda; G) \rightarrow C(L_{\alpha_\lambda}; G)$ extending $\nu_m|S(U_\lambda; G)$. This completes the induction, so the construction of the required chain morphism $\nu : S(X, \mathcal{U}; G) \rightarrow C(K; G)$ is done.

Finally, let $\varphi : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$ be the composition $\varphi = \mu \circ \nu$. Then, according to (7) and the definitions of Ω_α , for every $\lambda \in \Lambda$ we have

$$\varphi(S(U_\lambda; G)) \subset \mu(C(L_{\alpha_\lambda}; G)) \subset S(\Omega_{\alpha_\lambda}; G).$$

Since $U_\lambda \subset W_{\alpha_\lambda}$, conditions (4) and (5) yield that $\Omega_{\alpha_\lambda} \subset \Phi_n(x)$ for all $x \in U_\lambda$. Therefore, $\varphi(S(U_\lambda; G))$ is contained in $S(\Phi_n(x); G)$ whenever $x \in U_\lambda$. □

Proof of Theorem 1.2. Since the sequence $\{\Phi_m\}_{m=0}^n$ satisfies condition (i) from Theorem 1.1, we can perform the construction from the proof of Theorem 1.1 for every $m = 0, 1, \dots, n$. So, we construct the locally finite covers $\mathcal{V}_m = \{V_\alpha : \alpha \in \Gamma_m\}$ and $\mathcal{W}_m = \{W_\alpha : \alpha \in \Gamma_m\}$ of X , the complexes $K_0 \subset K_1 \subset \dots \subset K_n$, the sets Ω_α for any $\alpha \in \bigcup_{k=0}^n \Gamma_k$ and the chain morphisms $\mu_m : C(K_m; G) \rightarrow S_m(Y; G)$ satisfying conditions (1)–(5) and the particular case of condition (7) with $m \leq n - 1$. Since the complex $K = \bigcup_{m=0}^n K_m$ is n -dimensional, $K^{(m)} = \emptyset$ for all $m > n$. So, we can

suppose that $\mu_m = \mu_n$ for $m \geq n$. In this way we obtain a chain morphism $\mu : C(K; G) \rightarrow S(Y; G)$. Because $\dim X \leq n$, according to Corollary 5.3 from [6], for every $m = 0, 1, \dots, n$ there exists a disjoint family $\mathcal{U}_m = \{U_\lambda : \lambda \in \Lambda_m\}$ such that each \mathcal{U}_m refines \mathcal{W}_m and the family $\mathcal{U} = \bigcup_{m=0}^n \mathcal{U}_m$ covers X . Then, repeating the arguments from the final part of the proof of Theorem 1.1, we construct the required chain morphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$. \square

3. Fixed-point theorems for homologically $UV^n(G)$ usco maps

In this section we prove Theorems 1.3 and 1.5. For a set-valued map $\Phi : X \rightarrow 2^Y$ we denote by $\mathcal{O}(\Phi)$ the family of the open-graph maps $\Theta : X \rightarrow 2^Y$ such that $\Phi(x) \subset \Theta(x)$ for all $x \in X$. Next proposition is a homological version (and its proof is a small modification) of [5, Proposition 4.2].

Proposition 3.1. *Let X be a paracompact space, Y be a space and let $\Phi : X \rightarrow 2^Y$ be an usco map such that for every $x \in X$ and a neighborhood U of $\Phi(x)$ there exists a neighborhood V of $\Phi(x)$ with $V \xrightarrow{H_m} U$. Then for every $\varphi \in \mathcal{O}(\Phi)$ there exists $\Theta \in \mathcal{O}(\Phi)$ such that $\Theta(x)$ is open in Y and $\overline{\Theta(x)} \xrightarrow{H_m} \varphi(x)$ for all $x \in X$.*

Proof. Let $\varphi \in \mathcal{O}(\Phi)$. Then the graph $G(\varphi)$ is open in $X \times Y$ and contains the compact set $\{x\} \times \Phi(x)$ for every $x \in X$. So, there exist neighborhoods $W_1(x)$ and $U(x)$ of x and $\Phi(x)$, respectively, with $W_1(x) \times U(x) \subset G(\varphi)$. Thus, $\Phi(x) \subset U(x) \subset \varphi(x')$ for all $x' \in W_1(x)$. Then $\overline{V(x)} \xrightarrow{H_m} U(x)$ for some open neighborhood $V(x)$ of $\Phi(x)$. Since Φ is upper semi-continuous, we can find a neighborhood $W(x) \subset W_1(x)$ such that $x' \in W(x)$ implies $\Phi(x') \subset V(x)$. Hence, for all $x' \in W(x)$ we have

$$(9) \quad \Phi(x') \subset \overline{V(x)} \xrightarrow{H_m} U(x) \subset \varphi(x').$$

Next, let $\gamma = \{P_\alpha : \alpha \in A\}$ be a locally finite closed cover of X refining the cover $\{W(x) : x \in X\}$ (recall that X is paracompact), and for every α fix $x_\alpha \in X$ such that $P_\alpha \subset W(x_\alpha)$. For every $x \in X$ the set $A(x) = \{\alpha \in A : x \in P_\alpha\}$ is finite, and define $\Theta(x) = \bigcap \{V(x_\alpha) : \alpha \in A(x)\}$. One can show that Θ is open-graph (see the proof of [5, Proposition 4.2]). Moreover, since $x \in \bigcap \{W(x_\alpha) : \alpha \in A(x)\}$, it follows from (9) that

$$\Phi(x) \subset \overline{V(x_\alpha)} \xrightarrow{H_m} U(x_\alpha) \subset \varphi(x)$$

for all $\alpha \in A(x)$. This yields $\Phi(x) \subset \Theta(x) \subset \overline{\Theta(x)} \xrightarrow{H_m} \varphi(x)$. \square

Proof of Theorem 1.3. Let $g : Y \rightarrow X$ be a continuous (single-valued) map. Without loss of generality, we may assume that $g(Y) = X$. We need to show that the set-valued map $\Phi_g = \Phi \circ g : Y \rightarrow 2^Y$ has a fixed-point. Suppose this is not true. So $y \notin \Phi_g(y)$ for all $y \in Y$, or equivalently $\Phi(x) \subset Y \setminus g^{-1}(x)$ for all $x \in X$. Consider the set-valued map $\varphi : X \rightarrow 2^Y$, $\varphi(x) = Y \setminus g^{-1}(x)$. Then Φ is a selection for φ and it is easily seen that φ has an open graph.

Because Φ is homologically $UV^{n-1}(G)$, we can apply Proposition 3.1 to find for each $m = 0, 1, \dots, n$ a set valued map $\Theta_m : X \rightarrow 2^Y$ such that

- $\Phi(x) \subset \Theta_0(x)$, $x \in X$;
- $\Theta_m(x) \xrightarrow{H_m} \Theta_{m+1}(x)$ for all $x \in X$ and $m = 0, \dots, n - 1$;
- Each $\Theta_n(x)$ is open in Y and $\overline{\Theta_n(x)} \subset \varphi(x)$, $x \in X$.

Then, according to Theorem 1.2, there exists an open cover \mathcal{U} of X and a chain morphism $\phi : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$ such that $\phi(S(U; G)) \subset S(\Theta_n(x); G)$ for every $U \in \mathcal{U}$ and every $x \in U$. Consider the open cover $\mathcal{U}_g = g^{-1}(\mathcal{U})$ of Y and the chain morphism $g_{\#} : S(Y, \mathcal{U}_g; G) \rightarrow S(X, \mathcal{U}; G)$ generated by g . Then $\phi_g = \phi \circ g_{\#} : S(Y, \mathcal{U}_g; G) \rightarrow S(Y; G)$ is a chain morphism with

$$(10) \quad \phi_g(S(g^{-1}(U); G)) \subset S(\Theta_n(g(y)); G) \text{ for all } U \in \mathcal{U} \text{ and } y \in g^{-1}(U).$$

So, we can apply the homological fixed-point theorem [1, Theorem 7] to conclude that the chain morphism ϕ_g has a fixed point $y_0 \in Y$. This means that for any neighborhood $W \subset Y$ of y_0 there is a chain $c_W \in S(W; G) \cap S(Y, \mathcal{U}_g; G)$ such that $\|\phi_g(c_W)\| \cap W \neq \emptyset$. Choose $U_0 \in \mathcal{U}$ with $y_0 \in g^{-1}(U_0)$ and let $V \subset g^{-1}(U_0)$ be a neighborhood of y_0 . Then $c_V \in S(V; G) \subset S(g^{-1}(U_0); G)$. Thus, we have

$$(11) \quad \|\phi_g(c_V)\| \cap V \neq \emptyset \text{ and, by (10), } \|\phi_g(c_V)\| \subset \Theta_n(g(y_0)).$$

On the other hand, since $\overline{\Theta_n(x_0)} \subset \varphi(x_0)$, where $x_0 = g(y_0)$, we can choose V to be so small that $V \cap \overline{\Theta_n(x_0)} = \emptyset$. The last relation contradicts condition (11). □

Proof of Theorem 1.5. The arguments from the proof of [5, Theorem 1.3] work in our situation. For completeness, we provide a sketch. Since X can be embedded in the Hilbert cube Q as a retract, we may suppose that $\Phi : Q \rightarrow 2^Q$ is a homologically $UV^\omega(G)$ usco map. Identifying Q with the product $\prod\{\mathbb{I}_k : k \in \mathbb{N}\}$, where $\mathbb{I} = [0, 1]$, let $\pi_n : Q \rightarrow \prod\{\mathbb{I}_k : k \leq n\}$ be the projection onto the cube \mathbb{I}^n and $h_n : \mathbb{I}^n \rightarrow Q$ be the embedding assigning to every point $x = (x_1, \dots, x_n) \in \mathbb{I}^n$ the point $h(x)$ having the same first n -coordinates and all other coordinates 0. For every n consider the homologically $UV^\omega(G)$ usco map $\Phi_n : \mathbb{I}^n \rightarrow 2^Q$ defined by $\Phi_n(x) = \Phi(h_n(x))$. Then, according to Theorem 1.3 (with $X = \mathbb{I}^n$, $Y = Q$, $g = \pi_n$ and $\Phi = \Phi_n$), there is a point $x^n \in Q$ with $x^n \in \Phi_n(\pi_n(x^n))$. If $x^0 \in Q$ is the limit point of a convergent subsequence of $\{x^n\}_{n \geq 1}$, one can see that $x^0 \in \Phi(x^0)$. □

4. Fixed point-theorems for homological $UV^n(G)$ and $UV^\omega(G)$ decompositions

In this section we provide some fixed point-theorems for homological UV^n or homological $UV^\omega(G)$ decompositions of compact metric AR 's, where G is a field. Our results are homological analogues of Gutev's theorems [5, Theorems 7.1–7.3] for homotopical UV^n and UV^ω decompositions. We also follow Gutev's scheme of proofs applying our Theorem 1.3, Corollary 1.4 and Theorem 1.5 instead of their homotopical versions.

By a homological $UV^n(G)$ (resp., homological $UV^\omega(G)$) decomposition of a compactum X we mean an upper semi-continuous decomposition of X into compact homologically $UV^n(G)$ (resp., homologically $UV^\omega(G)$) sets. The decomposition space is denoted by X/\sim and $\pi : X \rightarrow X/\sim$ is the quotient map.

Theorem 4.1. *Let X be a compact metric AR with $\dim X \leq n$ and X/\sim be a homological $UV^{n-1}(G)$ decomposition of X . Then X/\sim has the fixed-point property.*

Proof. For any map $f : X/\sim \rightarrow X/\sim$ the set-valued map $\Phi := \pi^{-1} \circ f \circ \pi : X \rightarrow 2^X$ is usc and homologically $UV^{n-1}(G)$. Then, by Corollary 1.4, $x_0 \in \Phi(x_0)$ for some $x_0 \in X$. Hence, $f(\pi(x_0)) = \pi(x_0)$. \square

Theorem 4.2. *Let X be a compact metric AR and X/\sim be a homological $UV^{n-1}(G)$ decomposition of X with $\dim X/\sim \leq n$. Then X/\sim has the fixed-point property.*

Proof. For any map $f : X/\sim \rightarrow X/\sim$ consider the set-valued map $\Phi := \pi^{-1} \circ f : X/\sim \rightarrow 2^X$ and apply Theorem 1.3 to find a point $x_0 \in X$ with $x_0 \in \Phi(\pi(x_0))$. The last equality implies $f(\pi(x_0)) = \pi(x_0)$. \square

Theorem 4.3. *Let X be a compact metric AR and X/\sim be a homological $UV^\omega(G)$ decomposition of X . Then X/\sim has the fixed-point property.*

Proof. We repeat the proof of Theorem 4.1 using now Theorem 1.5 instead of Corollary 1.4. \square

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V. Valov
Department of Computer Science and Mathematics
Nipissing University
100 College Drive
P.O. Box 5002 North Bay ON P1B 8L7
Canada
e-mail: veskov@nipissingu.ca