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Parametric set-wise injective maps

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ABSTRACT

We introduce the notion of set-wise injective maps and provide results about fiber embeddings. Our results improve some previous results in this area.

1. Introduction

In this paper, unless otherwise noted, all spaces are assumed to be metrizable and all maps continuous. Also, unless stated otherwise, any function space $C(X, M)$ is endowed with the *source limitation topology*. This topology, known also as the *fine topology*, was introduced in [18] and has a base at a given $g \in C(X, M)$ consisting of the sets

$$B_\varrho(g, \varepsilon) = \{h \in C(X, M) : \varrho(h, g) < \varepsilon\},$$

where ϱ is a fixed compatible metric on M and $\varepsilon : X \to (0, 1]$ runs over continuous functions into $(0, 1]$. The symbol $\varrho(h, g) < \varepsilon$ means that $\varrho\big(h(x), g(x)\big) < \varepsilon(x)$ for all $x \in X$. The source limitation topology doesn't

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depend on the metric ϱ [7] and has the Baire property provided M is completely metrizable [9]. Obviously, this topology coincides with the uniform convergence topology when X is compact.

We say that a space M has the *m-DD$^{\{n,k\}}$-property* if any two maps $f : \mathbb{I}^m \times \mathbb{I}^n \to M$, $g : \mathbb{I}^m \times \mathbb{I}^k \to M$ can be approximated by maps $f' : \mathbb{I}^m \times \mathbb{I}^n \to M$ and $g' : \mathbb{I}^m \times \mathbb{I}^k \to M$, respectively, such that $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \varnothing$ for all $z \in \mathbb{I}^m$. Obviously, if M has the m-DD$^{\{n,k\}}$-property, then it also has the m'-DD$^{\{n',k'\}}$-property for all $m' \leq m$, $n' \leq n$ and $k' \leq k$. The 0-DD$^{\{n,k\}}$-property coincides with the well known disjoint (n, k)-cells property. The m-DD$^{\{n,k\}}$-property is very similar to the m-$\overline{\mathrm{DD}}^{\{n,k\}}$-property introduced in [1, Definition 5.1], where it is required for any open cover $\mathcal{U}$ of M the maps f, g to be approximated by maps f', g' such that f', g' are $\mathcal{U}$-homotopic to f and g, respectively and $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \varnothing$ for all $z \in \mathbb{I}^m$.

The notion of continuum-wise injective maps was introduced in [5] for maps between compact spaces. Here we extend this definition for arbitrary spaces and arbitrary closed sets (not necessarily continua as in [5]): A map $g : X \to M$ is *set-wise injective* if for any two closed sets $A, B \subset X$ with $A \neq B$, we have $g(A) \neq g(B)$. We also consider the following specialization of that property: a map $g : X \to M$ is *set-wise injective in dimension k* (see also [5]) if $g(A) \neq g(B)$ for any two closed sets $A, B \subset X$ such that $\dim(A \setminus B) \geq k$. Obviously, every set-wise injective map in dimension 0 is injective. Observe that for any two continua $A, B \subset X$ with $A \setminus B \neq \varnothing$ we have $\dim(A \setminus B) \geq 1$. Hence, every set-wise injective map in dimension 1 is automatically continuum-wise injective.

Recall that a map $f : X \to Y$ is said to be *σ-perfect* if X is a countable union of countably many closed sets X_i such that each restriction $f|X_i : X_i \to f(X_i)$ is a perfect map. Also, a map $f : X \to Y$ is called an *n-dimensional map* if $\dim f^{-1}(y) \leq n$ for each $y \in Y$.

The main results in this paper is the following theorem, which is a parametric version of Theorem 3.11 from [5]:

Theorem 1.1. *Let $f : X \to Y$ be a σ-perfect surjective n-dimensional map between metric spaces such that $\dim Y \leq m$ and let M be a complete separable metric* LC^{2m+n-1}*-space with the m-*DD$^{\{n,k\}}$*-property with $k \leq n$. Then the function space $C(X, M)$ contains a dense G_δ-set of maps g such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are set-wise injective in dimension $n - k$*

For example, it follows from [1, Proposition 5.6 and Theorem 9.1] that every dendrite with a dense set of end-point has both the 0-DD$^{\{0,\infty\}}$-property and the 1-DD$^{\{0,0\}}$-property, while $\mathbb{R}^{m+n+k+1}$ has the m-DD$^{\{n,k\}}$-property. Also, these spaces are LC^n for each $n = 0, 1, 2, \ldots$ (see [8, Theorem 4.2.33]).

Corollary 1.2. *Let X, Y and f be as in Theorem 1.1 and let $P \subset Q \subset X$ be two F_σ-subsets of X such that $\dim(P \cap f^{-1}(y)) \leq p$ and $\dim(Q \cap f^{-1}(y)) \leq q$ for every $y \in Y$, where $0 \leq p \leq q \leq n$. Then for every complete separable metric* LC^{2m+n-1}*-space M with the m-*DD$^{\{q,p\}}$*-property the space $C(X, M)$ contains a dense G_δ-set of maps g satisfying the following condition: $g^{-1}(g(z)) \cap Q \cap f^{-1}(y) = \{z\}$ for all $z \in P \cap f^{-1}(y)$ and all $y \in Y$.*

Note that when $p = q = n$ and M having the m-$\overline{\mathrm{DD}}^{\{n,n\}}$-property, Corollary 1.2 was established in [1, Theorem 3.3]. We already mentioned that the space $\mathbb{R}^l$ has the m-DD$^{\{n,k\}}$-property for all m, n, k with $m + n + k < l$. Hence, Corollary 1.2 is a far reaching generalization of Pasynkov's result [11] stating that for any map $f : X \to Y$ between metrizable compacta the function space $C(X, \mathbb{R}^{\dim Y + 2\dim f + 1})$ contains a dense G_δ-subset of maps that are injective on every fiber of f.

When M is compact and $C(X, M)$ is equipped with the uniform convergence topology, analogues of Theorem 1.1 and Corollary 1.2 also hold. Let us formulate the analogue of Theorem 1.1.

Theorem 1.3. *Let M be a compact metric* LC^{2m+n-1}*-space with the m-*DD$^{\{n,k\}}$*-property with $k \leq n$ and let $f : X \to Y$ be a closed surjective n-dimensional map between normal spaces such that $\dim Y \leq m$ and*

$W(f) \leq \aleph_0$. Then $C(X,M)$ equipped with the uniform convergence topology contains a dense subset of maps g such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are set-wise injective in dimension $n-k$.

Recall that $W(f) \leq \aleph_0$ means that there exists a map $g : X \to \mathbb{I}^{\aleph_0}$ such that $f \triangle g$ embeds X into $Y \times \mathbb{I}^{\aleph_0}$, see [10]. For example, according to [10, Proposition 9.1], $W(f) \leq \aleph_0$ for every closed map f between metrizable spaces provided f has Lindelöf fibers.

We apply Corollary 1.2 to provide a short proof of the following result:

Proposition 1.4. *Suppose n, k are non-negative integers such that $k+1 \leq n$. Then the product $M \times \mathbb{R}^{2l+1}$, where $l = n-k-1$, has the m-$\mathrm{DD}^{\{n,n\}}$-property for every complete separable metric LC^{2m+n-1}-space M with the m-$\mathrm{DD}^{\{n,k\}}$-property.*

The paper is organized as follows: all preliminary results are provided in Section 2, Section 3 contains the proofs of Theorem 1.1, Corollary 1.2 and Theorem 1.3. The proof of Proposition 1.4 is given in Appendix A.

2. Some preliminary results

First, we give some notations. Let $f : X \to Y$ be a surjective map between metric spaces and let M be a complete metric space. For any set $K \subset Y$ and closed disjoint sets $A, B \subset X$ we denote by $C_K(X, M; A, B; f)$ the set of all $g \in C(X,M)$ such that:

- $g(A \cap f^{-1}(y)) \cap g(B \cap f^{-1}(y)) = \varnothing$ for every $y \in K$. If $K = Y$, we write $C(X, M; A, B; f)$ instead of $C_K(X, M; A, B; f)$.

In this section, from here, we suppose that the spaces X, Y, M and the map $f\colon X \to Y$ satisfy the conditions from Theorem 1.1 with the additional assumption that the map f is a perfect surjection. Since M is LC^{2m+n-1}, according to [2, Lemma 4.1], M admits a metric ρ generating its topology and satisfying the following condition:

(1) If Z is an $(2m+n)$-dimensional metric space, $A \subset Z$ its closed set and $h\colon Z \to M$ a map, then for every function $\alpha : Z \to (0,1]$ and every map $g\colon A \to M$ with $\rho(g(z), h(z)) < \alpha(z)/8$ for all $z \in A$ there exists a map $\bar{g}\colon Z \to M$ extending g such that $\rho(\bar{g}(z), h(z)) < \alpha(z)$ for all $z \in Z$.

One can easily show that (1) implies the following condition:

(2) If $F \subset X$ is a closed set, the restriction map $\pi_F : C(X,M) \to C(F,M)$, $\pi_F(g) = g|F$, is an open and surjective map when both $C(X,M)$ and $C(X,M)$ carry the source limitation topology.

The aim of this section is to show that for any closed set $K \subset Y$ and closed disjoint sets $A, B \subset X$, $C_K(X, M; A, B; f)$ is open and dense in $C(X,M)$ with respect to the source limitation topology. Our proofs are based on some ideas from [6] and [17].

Lemma 2.1. *Let $K \subset Y$ be closed and $g_0 \in C_K(X, M; A, B; f)$, where A, B are disjoint closed subsets of X. Then there is a continuous function $\alpha : X \to (0,1]$ and an open set $W \subset Y$ containing K such that $g \in C_W(X, M; A, B; f)$ provided $g \in C(X,M)$ and $\rho(g(x), g_0(x)) < \alpha(x)$ for all $x \in X$.*

Proof. One can show that for every $y \in K$ there exists a neighborhood $V_y \subset Y$ of y and a positive number $\delta_y \leq 1$ such that $y' \in V_y$ and $\rho(g(x), g_0(x)) < \delta_y$ for all $x \in f^{-1}(y')$, where $g \in C(X,M)$, yields

$g \in C_{\{y'\}}(X, M; A, B; f)$. Let $V = \bigcup_{y \in K} V_y$ and $W \subset Y$ be an open set containing K with $\overline{W} \subset V$. The family $\{V_y : y \in K\}$ can be supposed to be locally finite in V. Consider the set-valued lower semi-continuous map $\varphi\colon \overline{W} \to (0,1]$, $\varphi(y) = \cup\{(0, \delta_z] : y \in V_z\}$. By [13, Theorem 6.2, p. 116], φ admits a continuous selection $\beta\colon \overline{W} \to (0,1]$. Let $\overline{\beta} : Y \to (0,1]$ be a continuous extension of β and $\alpha = \overline{\beta} \circ f$. The set W is the required one. □

Corollary 2.2. *Each set $C_K(X, M; A, B; f)$ is open in $C(X, M)$.*

Proof. Let $g_0 \in C_K(X, M; A, B; f)$. By Lemma 2.1, there exists a function $\alpha : X \to (0,1]$ such that $g \in C_K(X, M; A, B; f)$ for any $g \in C(X, M)$ satisfying the inequality $\rho(g(x), g_0(x)) < \alpha(x)$ for all $x \in X$. Then $B_\rho(g_0, \alpha)$ is a neighborhood of g_0 and $B_\rho(g_0, \alpha) \subset C_K(X, M; A, B; f)$. □

Next step is to show that if $K \subset Y$ is closed, A and B are disjoint closed subsets of X with $\dim f|A \leq k$, then $C_K(X, M; A, B; f)$ is dense in $C(X, M)$. To this end we need some preliminary results. The first one is the following characterization of spaces with the m-DD$^{\{n,k\}}$-property, which can be obtain from the proof of [1, Theorem 5.7]:

Proposition 2.3. *Let m, n, k be non-negative integers and $d = m + \max\{n, k\}$. A completely separable metrizable* LC^{d-1}*-space M has the m-*DD$^{\{n,k\}}$*-property if and only if for any separable polyhedron P with $\dim P \leq m$ there are two disjoint σ-compact sets $E_n, E_k \subset P \times M$ such that $E_n \in P$-*MAP^n *and $E_k \in P$-*MAP^k.

The notation $E_n \in P$-MAP^n means that for any n-dimensional map $p : K \to P$ with K being a finite-dimensional metric compactum, a closed subset $F \subset K$, a map $g : K \to M$, and a positive δ there is a map $g' : K \to M$ such that g' is δ-close to g, $g'|F = g|F$ and $(p \triangle g')(K \setminus F) \subset E_n$.

To prove the density of the sets $C_K(X, M; A, B; f)$, where A, B are disjoint closed subsets of X with $\dim f|A \leq k$, we fix a map $g_0 : X \to M$ and a continuous function $\varepsilon : X \to (0,1]$. Define the set-valued map

$$\Phi_\varepsilon : Y \to C(X, M) \text{ by } \Phi_\varepsilon(y) = B_\rho(g_0, \varepsilon) \cap C_{\{y\}}(X, M; A, B; f),$$

where $C(X, M)$ carries the compact-open topology.

Lemma 2.4. *All $\Phi_\varepsilon(y)$ are non-empty sets. Moreover, if $\Phi_\varepsilon(y_0)$ contains a compact set K for some $y_0 \in Y$, then there exists a neighborhood $V(y_0)$ of y_0 such that $K \subset \Phi_\varepsilon(y)$ for every $y \in V(y_0)$.*

Proof. Since M is an LC^{n-1}-space with the disjoint (n, k)-cells property and $\dim f^{-1}(y) \cap A \leq k$, the set of all maps $h \in C(f^{-1}(y), M)$ with $h(A \cap f^{-1}(y)) \cap h(B \cap f^{-1}(y)) = \varnothing$ is dense in $C(f^{-1}(y), M)$ (see the proof of Lemma 3.4 from [5]). So, if $\delta_y = \min\{\varepsilon(x) : x \in f^{-1}(y)\}$, then there exists such a map $h \in C(f^{-1}(y), M)$ with $\rho(h, g_0|f^{-1}(y)) < \delta_y/8$. Then, by the extension property (1), h can be extended to a map $g \in C(X, M)$ such that $\rho(g, g_0) < \varepsilon$. Obviously $g \in C_{\{y\}}(X, M; A, B; f)$, so $\Phi(y) \neq \varnothing$ for all $y \in Y$.

The second part of that lemma can be established following the proof of Lemma 2.5(2) from [6]. □

Lemma 2.5. *Every $\Phi_\varepsilon(y)$ has the following property: If $\hat{v}\colon \mathbb{S}^p \to \Phi_\varepsilon(y)$ is continuous, where $p \leq m - 1$ and $\mathbb{S}^p$ is the p-sphere, then $\hat{v}$ can be extended to a continuous map $\hat{u}\colon \mathbb{I}^{p+1} \to \Phi_{16\varepsilon}(y)$.*

Proof. Let us mention the following property of the function space $C(X, M)$ with the compact open topology: For any metrizable space Z a map $\hat{w}\colon Z \to C(X, M)$ is continuous if and only if the map $w\colon Z \times X \to M$, $w(z, x) = \hat{w}(z)(x)$, is continuous. Hence, every map $\hat{v}\colon \mathbb{S}^p \to \Phi_\varepsilon(y)$ generates a continuous map $v\colon \mathbb{S}^p \times X \to M$ defined by $v(z, x) = \hat{v}(z)(x)$ such that $\rho\big(v(z, x), g_0(x)\big) < \varepsilon(x)$ for all $(z, x) \in \mathbb{S}^p \times X$.

Define the maps $\overline{g}_0 : \mathbb{I}^{p+1} \times X \to M$ and $\overline{\varepsilon} : \mathbb{I}^{p+1} \times X \to (0,1]$ by $\overline{g}_0(t,x) = g_0(x)$ and $\overline{\varepsilon}(t,x) = \varepsilon(x)$ for all $t \in \mathbb{I}^{p+1}$. Since X admits a perfect n-dimensional map onto the m-dimensional space Y, $\dim X \leq n+m$, see [3]. Hence, $\dim(\mathbb{I}^{p+1} \times X) \leq 2m+n$. Then, according to the extension property (1), v can be extended to a map $v_1 : \mathbb{I}^{p+1} \times X \to M$ such that $\rho(v_1, \overline{g}_0) < 8\overline{\varepsilon}$. Let $A_y = A \cap f^{-1}(y)$, $B_y = B \cap f^{-1}(y)$. Denote by $v_{1,A} : \mathbb{I}^{p+1} \times A_y \to M$ and $v_{1,B} : \mathbb{I}^{p+1} \times B_y \to M$, respectively, the restrictions $v_1|(\mathbb{I}^{p+1} \times A_y)$ and $v_1|(\mathbb{I}^{p+1} \times B_y)$. By Proposition 2.3, there exist two disjoint subsets E_k and E_n of $\mathbb{I}^{p+1} \times M$ such that $E_n \in \mathbb{I}^{p+1}\text{-MAP}^n$ and $E_k \in \mathbb{I}^{p+1}\text{-MAP}^k$. Applying the $(\mathbb{I}^{p+1}\text{-MAP}^k)$-property of E_k with respect to the projection $\pi_A : \mathbb{I}^{p+1} \times A_y \to \mathbb{I}^{p+1}$, we find a map $h_A : \mathbb{I}^{p+1} \times A_y \to M$ satisfying the following conditions, where $\delta_y = \min\{8\varepsilon(x) - \rho(v_1(t,x), g_0(x)) : (t,x) \in \mathbb{I}^{p+1} \times f^{-1}(y)\}$:

(3) $h_A|(\mathbb{S}^p \times A_y) = v_1|(\mathbb{S}^p \times A_y)$;
(4) $\rho(h_A, v_{1,A}) < \delta_y$;
(5) $\pi_A \triangle h_A((\mathbb{I}^{p+1} \setminus \mathbb{S}^p) \times A_y) \subset E_k$.

Applying the $(\mathbb{I}^{p+1}\text{-MAP}^n)$-property of E_n with respect to the projection $\pi_B : \mathbb{I}^{p+1} \times B_y \to \mathbb{I}^{p+1}$, we obtain a map $h_B : \mathbb{I}^{p+1} \times B_y \to M$ such that

(6) $h_B|(\mathbb{S}^p \times B_y) = v_1|(\mathbb{S}^p \times B_y)$;
(7) $\rho(h_B, v_{1,B}) < \delta_y$;
(8) $\pi_B \triangle h_B((\mathbb{I}^{p+1} \setminus \mathbb{S}^p) \times B_y) \subset E_n$.

Consider now the map $h : F \to M$, where $F = (\mathbb{S}^p \times X) \cup (\mathbb{I}^{p+1} \times A_y) \cup (\mathbb{I}^{p+1} \times B_y)$, such that $h|(\mathbb{S}^p \times X) = v_1|(\mathbb{S}^p \times X)$, $h|(\mathbb{I}^{p+1} \times A_y) = h_A$ and $h|(\mathbb{I}^{p+1} \times B_y) = h_B$. Observed that $\rho(h(t,x), v_1(t,x)) < \varepsilon(x)$ for all $(t,x) \in F$. So, using again the extension property (1), we extend the map h to a map $\widetilde{h} : \mathbb{I}^{p+1} \times X \to M$ with $\rho(\widetilde{h}, v_1) < 8\overline{\varepsilon}$. Because $\rho(v_1, \overline{g}_0) < 8\overline{\varepsilon}$, we have $\rho(\widetilde{h}, \overline{g}_0) < 16\overline{\varepsilon}$. Then $\widetilde{h}$ provides a map $\hat{u} : \mathbb{I}^{p+1} \to C(X,M)$, defined by $\hat{u}(t)(x) = \widetilde{h}(t,x)$, such that $\hat{u}(t) \in B_\rho(g_0, 16\varepsilon)$ for all $t \in \mathbb{I}^{p+1}$.

It remains to show that $\hat{u}(\mathbb{I}^{p+1}) \subset \Phi_{16\varepsilon}(y)$. To this end, observe that conditions (5) and (8) imply $\widetilde{h}(\{t\} \times A_y) \cap \widetilde{h}(\{t\} \times B_y) = \varnothing$ for all $t \in \mathbb{I}^{p+1} \setminus \mathbb{S}^p$. Because $\widetilde{h}|(\mathbb{S}^p \times f^{-1}(y)) = v|(\mathbb{S}^p \times f^{-1}(y))$ and $\hat{v}(t) \in C_{\{y\}}(X, M; A, B; f)$, $\widetilde{h}(\{t\} \times A_y) \cap \widetilde{h}(\{y\} \times B_y) = \varnothing$ for any $t \in \mathbb{S}^p$. Therefore, $\widetilde{h}(\{t\} \times A_y) \cap \widetilde{h}(\{t\} \times B_y) = \varnothing$ for all $t \in \mathbb{I}^{p+1}$. The last condition yields $\hat{u}(\mathbb{I}^{p+1}) \subset C_{\{y\}}(X, M; A, B; f)$. Hence, $\hat{u}(\mathbb{I}^{p+1}) \subset \Phi_{16\varepsilon}(y)$. $\square$

Proposition 2.6. *$C_K(X, M; A, B; f)$ is a dense subset of $C(X, M)$ with respect to the source limitation topology for every closed $K \subset Y$.*

Proof. Let $p \leq m-1$. Define the set-valued maps $\Phi_i \colon K \to C(X,M)$, $i = 0, 1, ..., m$, $\Phi_i(y) = \Phi_{\varepsilon/16^{m-i+1}}(y)$. Obviously, $\Phi_0(y) \subset \Phi_1(y) \subset ... \subset \Phi_m(y) = \Phi_{\varepsilon/16}(y)$. According to Lemma 2.5, every map from $\mathbb{S}^p$ into $\Phi_i(y)$ can be extended to a map from $\mathbb{I}^{p+1}$ into $\Phi_{i+1}(y)$, where $i = 0, 1, ..., m-1$ and $y \in K$. Moreover, by Lemma 2.4, any $\Phi_i(y)$ has the following property: if $P \subset \Phi_i(y)$ is compact, then there exists a neighborhood V_y of y in Y such that $P \subset \Phi_i(z)$ for all $z \in V_y \cap K$. So, we may apply the proof of [4, Theorem 3.1] to find a continuous selection $\theta \colon K \to C(X,M)$ of Φ_m. Hence, $\theta(y) \in \Phi_{\varepsilon/16}(y)$ for all $y \in K$. Now, consider the map $g \colon f^{-1}(K) \to M$, $g(x) = \theta(f(x))(x)$. Using that $C(X,M)$ carries the compact open topology, one can show that g is continuous. Moreover, $\varrho\big(g(x), g_0(x)\big) < \varepsilon(x)/16$ for all $x \in f^{-1}(K)$. Then, by (1), g can be extended to a continuous map $\bar{g} \colon X \to M$ with $\varrho\big(\bar{g}(x), g_0(x)\big) < \varepsilon(x)$, $x \in X$. It follows from the definition of g that $g|f^{-1}(y) = \theta(y)|f^{-1}(y)$ for every $y \in K$. Since $\theta(y) \in C_{\{y\}}(X, M; A, B; f)$, $\bar{g}(A_y) \cap \bar{g}(B_y) = \varnothing$ for all $y \in K$. Hence, $\bar{g} \in B_\varrho(g_0, \varepsilon) \cap C_K(X, M; A, B; f)$. $\square$

3. Proofs

Proof of Theorem 1.1. Let X be the union of an increasing sequence $\{X_i\}_{i=1}^{\infty}$ of closed sets such that each restriction $f_i = f|X_i : X_i \to f(X_i)$ is a perfect map. So, according to condition (2), the restriction maps $\pi_i : C(X,M) \to C(X_i,M)$ are open surjections when both $C(X,M)$ and $C(X_i,M)$ are equipped with the source limitation topology. Hence, by Corollary 2.2 and Proposition 2.6, the sets $\pi_i^{-1}(C(X_i,M; A\cap X_i, B\cap X_i; f_i))$ are open and dense in $C(X,M)$ for any i, where A and B are closed disjoint subsets of X with $\dim f|A \le k$. Since

$$C(X,M;A,B;f) = \bigcap_{i=1}^{\infty} \pi_i^{-1}(C(X_i,M;A\cap X_i,B\cap X_i;f_i)),$$

any $C(X,M;A,B;f)$ is a dense G_δ-subset of $C(X,M)$.

Suppose first that $k \le n-1$. Since f is σ-perfect and $\dim f \le n$, there exist closed subsets $F_i \subset X$, $i = 1,2,...$, such that $\dim F_i \le k$ for each i and the restriction $f|(X \setminus \bigcup_{i=1}^{\infty} F_i)$ is a map of dimension $\le n-k-1$, see [15, Theorem 1.4]. Because each f_i is a perfect map, by [10, Proposition 9.1], there exist maps $h_i\colon X_i \to \mathbb{I}^{\aleph_0}$ embedding all fibers of f_i, $i = 1,2,...$ We can suppose that each h_i is defined on X. Hence, the diagonal product h of all h_i is a map from X into $\mathbb{I}^{\aleph_0}$ such that $h|f^{-1}(y) : f^{-1}(y) \to \mathbb{I}^{\aleph_0}$ is one-to one for all $y \in Y$. We fix a finitely additive base $\Gamma = \{U_j\}_{j=1}^{\infty}$ for the topology of $\mathbb{I}^{\aleph_0}$ and consider the family $\mathcal{A}$ of all non-empty intersections $h^{-1}(\overline{U}_j) \cap F_i$, $i,j = 1,2,...$, and the family $\mathcal{B} = \{h^{-1}(\overline{U}_j)\}_{j=1}^{\infty}$. Obviously, $\dim A \le k$ for all $A \in \mathcal{A}$. We already observed that the sets $C(X,M;A,B;f)$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are disjoint, are dense and G_δ in $C(X,M)$ with respect to the source limitation topology. Then the intersection $\mathcal{S}$ of all $C(X,M;A,B;f)$ is also a dense G_δ-subset of $C(X,M)$.

Let us show that $\mathcal{S}$ consists of maps g such that each restriction $g|f^{-1}(y)$, $y \in Y$, is set-wise injective in dimension $n-k$. Indeed, suppose $K_1 \ne K_2$ are two non-trivial closed sets, which are contained in some $f^{-1}(y_0)$ and $\dim(K_2 \setminus K_1) \ge n-k$.

Claim 1. *There is $x_0 \in (K_2 \setminus K_1) \cap (\bigcup_{i=1}^{\infty} F_i)$.*

Indeed, otherwise $K_2 \setminus K_1 \subset f^{-1}(y_0) \setminus (\bigcup_{i=1}^{\infty} F_i)$, which implies $\dim K_2 \setminus K_1 \le n-k-1$, a contradiction.

Next claim completes the proof of Theorem 1.1 in the case $k \le n-1$.

Claim 2. *$g(x_0) \notin g(K_1)$ for all $g \in \mathcal{S}$.*

We fix i_0 with $x_0 \in F_{i_0}$. Since $h(x_0) \in h(K_2) \setminus h(K_1 \cap X_i)$ and $h(K_1 \cap X_i)$ is a compact set for every i, there exist $U_{j_i}, U_{l_i} \in \Gamma$ such that $h(x_0) \in U_{j_i}$, $h(K_1 \cap X_i) \subset U_{l_i}$ and $\overline{U}_{j_i} \cap \overline{U}_{l_i} = \varnothing$ (recall that Γ is finitely additive). Then $h^{-1}(\overline{U}_{j_i})$ and $B_i = h^{-1}(\overline{U}_{l_i})$ are also disjoint and $K_1 \cap X_i \subset B_i \cap f^{-1}(y_0)$. Moreover $A_i = h^{-1}(\overline{U}_{j_i}) \cap F_{i_0} \in \mathcal{A}$ and $x_0 \in A_i$. Consequently, $g(x_0) \notin g(K_1 \cap X_i)$ for all $g \in C(X,M;A_i,B_i;f)$ and all i. Finally, since $g(K_1) = \bigcup_{i=1}^{\infty} g(K_1 \cap X_i)$, we have $g(x_0) \in g(K_2) \setminus g(K_1)$.

Suppose now that $k = n$, and let $\Gamma = \{U_j\}_{j=1}^{\infty}$ and $\mathcal{B}$ be as above. Then the intersection of all $C(X,M;A,B;f)$, where $A,B \in \mathcal{B}$ are disjoint, is a dense G_δ-subset of $C(X,M)$ and consists of maps g such that the restrictions $g|f^{-1}(y)$, $y \in Y$, are set-wise injective in dimension 0. □

Proof of Corollary 1.2. Suppose first that $Q \subset X$ is closed, and let $f_Q = f|Q$ and $Y_Q = f(Q)$. Obviously, $f_Q : Q \to Y_Q$ is a σ-perfect surjection with $\dim f_Q \le q$. Then, we apply Theorem 1.1 (with X, Y, f replaced, respectively, by Q, Y_Q, f_Q) to show the existence of a dense G_δ-subset of $C(Q;M)$ of maps g such that all restrictions $g|f_Q^{-1}(y)$, $y \in Y_Q$, are set-wise injective in dimension $q-p$. More precisely, following the notations from the proof of Theorem 1.1, we find countably many disjoint couples (A_i, B_i) of closed subsets of X satisfying the following conditions:

- $A_i, B_i \subset Q$;
- Each $C(Q, M; A_i, B_i; f_Q)$ is a dense G_δ-subset of $C(Q, M)$ and the intersection $\mathcal{S}_Q$ of all $C(Q, M; A_i, B_i; f_Q)$ consists of maps $g \in C(Q, M)$ such that $g|f_Q^{-1}(y)$, $y \in Y_Q$, is set-wise injective in dimension $q - p$;
- If $p \le q - 1$, then for any $y \in Y_Q$ and any two different points $z \in f_Q^{-1}(y) \cap P$ and $x \in f_Q^{-1}(y)$ there exists a couple (A_i, B_i) with $z \in A_i$ and $x \in B_i$;
- If $p = q$, then the couples (A_i, B_i) are separating the points of $f_Q^{-1}(y)$ for all $y \in Y_Q$.

The last two properties yield that $\mathcal{S}_Q$ consists of maps $g \in C(Q, M)$ such that $g^{-1}(g(z)) \cap f_Q^{-1}(y) = \{z\}$ for all $z \in P \cap f_Q^{-1}(y)$ and all $y \in Y_Q$. Let $\pi_Q : C(X, M) \to C(Q, M)$ be the restriction map. According to condition (2), each set $\pi_Q^{-1}(C(Q, M; A_i, B_i; f_Q))$ is dense and G_δ in $C(X, M)$. Then the set $\pi_Q^{-1}(\mathcal{S}_Q)$ is also dense and G_δ in $C(X, M)$, and consists of maps g such that $g^{-1}(g(z)) \cap Q \cap f^{-1}(y) = \{z\}$ for all $z \in P \cap f^{-1}(y)$ and all $y \in Y$.

If $Q = \bigcup_{j=1}^{\infty} Q_j$ is an F_σ-subset of X, we consider the σ-perfect restrictions $f_j = f|Q_j$ and the spaces $Y_j = f_j(Q_j)$. As above, for each j we find countably many couples (A_i^j, B_i^j) of closed disjoint subsets of Q_j such that the intersection $\mathcal{S}_{Q_j}$ of all $C(Q_j, M; A_i^j, B_i^j; f_j)$, $i = 1, 2, ...,$ is dense and G_δ in $C(Q_j, M)$. Consequently, $\mathcal{S} = \bigcap_{j=1}^{\infty} \pi_{Q_j}^{-1}(\mathcal{S}_{Q_j})$ is dense and G_δ in $C(X, M)$. It is easily seen that any $g \in \mathcal{S}$ satisfies the required condition $g^{-1}(g(z)) \cap Q \cap f^{-1}(y) = \{z\}$ for all $z \in P \cap f^{-1}(y)$ and all $y \in Y$. □

Proof of Theorem 1.3. We follow the approach from the proof of [16, Theorem 1.2]. Fix a map $g_0\colon X \to M$ and a number $\varepsilon > 0$. Since $W(f) \le \aleph_0$, there exists a map $\lambda\colon X \to \mathbb{I}^{\aleph_0}$ such that $f \triangle \lambda\colon X \to Y \times \mathbb{I}^{\aleph_0}$ is an embedding. We are going to find a map $g \in C(X, M)$ such that g is ε-close to g_0 and all restrictions $g|f^{-1}(y)$, $y \in Y$, are continuum-wise injective in dimension $n - k$. To this end, let $\overline{\lambda}\colon \beta X \to \mathbb{I}^{\aleph_0}$, $\overline{g}_0\colon \beta X \to M$ and $\overline{f}: \beta X \to \beta Y$ be the Čech–Stone extensions of the maps λ, g_0 and f, respectively. Then $\overline{\lambda} \triangle \overline{g}_0 \in C(\beta X, \mathbb{I}^{\aleph_0} \times M)$. We consider also the constant maps $h\colon \mathbb{I}^{\aleph_0} \times M \to Pt$ and $\eta\colon \beta Y \to Pt$, where Pt is the one-point space. According to Pasynkov's factorization theorem [12, Theorem 13], there exist metrizable compacta K, T and maps $f_*\colon K \to T$, $\xi_1\colon \beta X \to K$, $\xi_2\colon K \to \mathbb{I}^{\aleph_0} \times M$ and $\eta_1\colon \beta Y \to T$ such that:

- $\eta_1 \circ \overline{f} = f_* \circ \xi_1$;
- $\xi_2 \circ \xi_1 = \overline{\lambda} \triangle \overline{g}_0$;
- $\dim T \le \dim \beta Y$ and $\dim f_* \le \dim \overline{f}$.

Since Y is normal, $\dim \beta Y = \dim Y \le m$. Moreover, by [12, Proposition 8], $\dim f \le n$ implies $\dim \overline{f} \le n$. If $p\colon \mathbb{I}^{\aleph_0} \times M \to \mathbb{I}^{\aleph_0}$ and $q\colon \mathbb{I}^{\aleph_0} \times M \to M$ denote the corresponding projections, we have

- $p \circ \xi_2 \circ \xi_1 = \overline{\lambda}$ and $q \circ \xi_2 \circ \xi_1 = \overline{g}_0$.

By Theorem 1.1, there exists a map $\phi\colon K \to M$ such that ϕ is ε-close to $q \circ \xi_2$ and all restrictions $\phi|f_*^{-1}(t)$, $t \in T$, are set-wise injective in dimension $n - k$. Then the map $\overline{g} = \phi \circ \xi_1$ is ε-close to $\overline{g}_0$. Hence, the maps $g = \overline{g}|X$ and g_0 are also ε-close. Because $\lambda = \big(p \circ \xi_2 \circ \xi_1\big)|X$ embeds the fibers of f into $\mathbb{I}^{\aleph_0}$, ξ_1 embeds the fibers of f into K such that $f^{-1}(y) \subset f_*^{-1}(\eta_1(y))$ for all $y \in Y$. Therefore, the restrictions $g|f^{-1}(y)$, $y \in Y$, are set-wise injective in dimension $n - k$.

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Appendix A

Proof of Proposition 1.4. Let $f : \mathbb{I}^m \times \mathbb{I}^n \to M \times \mathbb{R}^{2l+1}$ and $g : \mathbb{I}^m \times \mathbb{I}^n \to M \times \mathbb{R}^{2l+1}$ be two maps. We are going to approximate f and g by maps $f' : \mathbb{I}^m \times \mathbb{I}^n \to M \times \mathbb{R}^{2l+1}$ and $g' : \mathbb{I}^m \times \mathbb{I}^n \to M \times \mathbb{R}^{2l+1}$ such that $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^n) = \varnothing$ for all $z \in \mathbb{I}^m$. To this end, let $\varphi : \mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n) \to M \times \mathbb{R}^{2l+1}$ be the map generated by f and g, where $\oplus$ denotes the discrete sum. Represent φ as the product $\varphi = \varphi_1 \triangle \varphi_2$ of two maps $\varphi_1 : \mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n) \to M$ and $\varphi_2 : \mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n) \to \mathbb{R}^{2l+1}$.

Claim 3. *There exists an F_σ-subset $F \subset \mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n)$ such that $\dim(\mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n) \setminus F) \le n - k - 1$ and $\dim \pi|F \le k$, where $\pi : \mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n) \to \mathbb{I}^m$ is the projection.*

Indeed, denote $X = \mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n)$ and take an F_σ-set $H \subset X$ such that $\dim H \le k$ and $\dim \pi|(X \setminus H) \le n - k - 1$, see [14]. Then H is contained in a G_δ-set $\widetilde{H} \subset X$ with $\dim \tilde{H} \le k$, and the set $F = X \setminus \widetilde{H}$ is the required one.

Since $\dim \pi|F \le k$ and M has the m-$\mathrm{DD}^{\{n,k\}}$-property, by Corollary 1.2, there exists a map $\phi_1 : \mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n) \to M$ sufficiently close to φ_1 with $\phi_1^{-1}(\phi_1(z)) \cap \pi^{-1}(y) = \{z\}$ for all $z \in F \cap \pi^{-1}(y)$ and all $y \in \mathbb{I}^m$. Let $\widetilde{F}$ be the set of all $z \in X$ such that $\phi_1^{-1}(\phi_1(z)) \cap \pi^{-1}(y) = \{z\}$ for all $y \in \mathbb{I}^m$. It is easily seen that $\widetilde{F} = \{z \in X : (\pi \triangle \phi_1)^{-1}((\pi \triangle \phi_1)(z)) = \{z\}\}$, where $\pi \triangle \phi_1 : X \to \mathbb{I}^m \times M$ is the diagonal product of π and ϕ_1. Because $\pi \triangle \phi_1$ is a closed map, $\widetilde{F}$ is a G_δ-subset of X. Moreover, $\widetilde{F}$ contains F and $P = X \setminus \widetilde{F}$ is an σ-compact set of dimension $\dim P \le l$. Thus, there is a map $\phi_2 : X \to \mathbb{R}^{2l+1}$ sufficiently close to φ_2 such that $\phi_2|P$ is one-to-one. Then the map $\phi = \phi_1 \triangle \phi_2 : \mathbb{I}^m \times (\mathbb{I}^n \oplus \mathbb{I}^n) \to M \times \mathbb{R}^{2l+1}$ is close to φ. Consequently, the maps $f' = \phi|(\mathbb{I}^m \times \mathbb{I}^n)$ and $g' = \phi|(\mathbb{I}^m \times \mathbb{I}^n)$ are close, respectively, to f and g. Moreover, $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^n) = \varnothing$ for all $z \in \mathbb{I}^m$. $\square$

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