Topology and its Applications 221 (2017) 38–50

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Homological dimension and homogeneous ANR spaces

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ARTICLE INFO

Article history:
Received 28 March 2016
Accepted 10 November 2016
Available online 14 February 2017

MSC:
primary 55M10, 55M15
secondary 54F45, 54C55

Keywords:
Homology membrane
Homological dimension
Homology groups
Homogeneous metric *ANR*-compacta

ABSTRACT

The homological dimension d_G of metric compacta was introduced by Alexandroff in [1]. In this paper we provide some general properties of d_G, mainly with an eye towards describing the dimensional full-valuedness of compact metric spaces. As a corollary of the established properties of d_G, we prove that any two-dimensional lc^2 metric compactum is dimensionally full-valued. This improves the well known result of Kodama [10] that every two-dimensional *ANR* is dimensionally full-valued. Applications for homogeneous metric *ANR*-compacta are also given.

1. Introduction

Reduced Čech homology $H_n(X;G)$ and cohomology groups $H^n(X;G)$ with coefficient from an abelian group G are considered everywhere below. By a space, unless stated otherwise, we mean a metric compactum.

Alexandroff [1] introduced the homological dimension theory as a further geometrization of the ordinary dimension theory. His definition of the homological dimension d_GX of a space X (with coefficients in an abelian group G) provided a new characterization of the covering dimension $\dim X$: If X is finite-dimensional, then $\dim X$ is the maximum integer n such that $\dim X = d_{\mathbb{Q}_1}X = d_{\mathbb{S}^1}X = n$, where $\mathbb{S}^1$ is the circle group and $\mathbb{Q}_1$ is the group of rational elements of $\mathbb{S}^1$. Here is Alexandroff's definition: the homological dimension d_GX is the maximum integer n such that there exist a closed set $F \subset X$ and a nontrivial element $\gamma \in H_{n-1}(F;G)$ with γ being G-homologous to zero in X. According to [1, p. 207] we have $d_GX \leq \dim X$ for any space X.

The aim of this paper is to provide more properties of the homological dimension. Except some general properties of d_G, we also describe dimensional full-valuedness of compact metric spaces using the equalities

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[1] The author was partially supported by NSERC Grant 261914-13.

http://dx.doi.org/10.1016/j.topol.2017.02.063
0166-8641/

$d_G X = \dim X$ for some specific groups G. The lack of exactness of Čech homology is a real problem. We overcome this problem by using the exact homology, developed by Sklyarenko [15], and its coincidence with Čech homology in some important cases (let us note that the exact homology is equivalent to Steenrod homology in the class of metric compacta).

Because the definition of $d_G X$ does not provide any information for the homology groups $H_{k-1}(F;G)$ when $F \subset X$ is closed and $k < d_G X - 1$, we consider the set $\mathcal{H}_{X,G}$ of all integers $k \geq 1$ such that there exist a closed set $F \subset X$ and a nontrivial element $\gamma \in H_{k-1}(F;G)$, which is homologous to zero in X. Obviously, $d_G X = \max \mathcal{H}_{X,G}$. The set $\mathcal{H}_{X,G}$ was very useful in establishing some local homological properties of homogeneous *ANR*-compacta, see [19]. In this paper we also provide some properties of $\mathcal{H}_{X,G}$.

Suppose (K,A) is a pair of closed subsets of a space X with $A \subset K$. By $i^n_{A,K}$ we denote the homomorphism from $H_n(A;G)$ into $H_n(K;G)$ generated by the inclusion $A \hookrightarrow K$. This inclusion generates also the homomorphism $j^n_{K,A} : H^n(K;G) \to H^n(A;G)$. Following [2], we say that K is an *n-homology membrane spanned on A for some $\gamma \in H_n(A;G)$* provided γ is homologous to zero in K, but not homologous to zero in any proper closed subset of K containing A. It is well known that for space X and a closed set $A \subset X$ the existence of a non-trivial element $\gamma \in H_n(A;G)$ with $i^n_{A,X}(\gamma) = 0$ yields the existence of a closed set $K \subset X$ containing A such that K is an n-homology membrane for γ spanned on A, see [2, Property 5, p. 103]. The above statement remains true for any Hausdorff compactum X and a closed set $A \subset X$, see Lemma 2.6 below.

Recall that the cohomological dimension $\dim_G X$ is the largest integer n such that there exists a closed set $A \subset X$ with $H^n(X,A;G) \neq 0$. It is well known that $\dim_G X \leq n$ iff every map $f\colon A \to K(G,n)$ can be extended to a map $\tilde{f}\colon X \to K(G,n)$, where $K(G,n)$ is the Eilenberg–MacLane space of type (G,n), see [16]. We also say that a space X is dimensionally full-valued if $\dim X \times Y = \dim X + \dim Y$ for all metric compacta Y, or equivalently (see [3] and [11, Theorem 11]), $\dim_G X = \dim_{\mathbb{Z}} X$ for any abelian group G.

The paper is organized as follows. In Section 2 we provide some properties of the dimension d_G and the set $\mathcal{H}_{X,G}$. As stated above, we generally have the inequalities $d_{\mathbb{Z}} X \leq \dim X \geq d_G X$, where the equalities yield peculiar properties of X stated below:

Theorem 2.9. *The following holds for any space X with $\dim X = n$:*

(1) *If $d_G X = n$ for a torsion free abelian group G, then:*
  (1.1) *there exist a point $x \in X$ and a local base $\mathcal{B}_x$ at x consisting of open sets U such that $\mathrm{bd}\,\overline{U}$ is a dimensionally full-valued set of dimension $n-1$, and*
  (1.2) *X is dimensionally full-valued;*

(2) *If X is dimensionally full-valued and X has a base of $\mathcal{B}$ of open sets U with $H^n(\overline{U};\mathbb{Z}) = 0$, then $d_G X = n$ for every field G;*

(3) *If X is dimensionally full-valued and lc^n, then $d_{\mathbb{Z}} X = d_G X = n$ for every field G.*

The assumption in item (1) of the above theorem that G is torsion free is essential. Indeed, we already observed that $\dim X = \dim_{\mathbb{Q}_1}$ for any finite-dimensional metric compactum X, but not any such X is dimensionally full-valued.

It follows from Theorem 2.9 (see Corollary 2.10) that if X is a space such that $\dim X = n$ and X is homologically locally connected up to dimension n with respect to the singular homology with integer coefficients (br., lc^n), then the following conditions are equivalent: X is dimensionally full-valued; there is a point $x \in X$ having a local base $\mathcal{B}_x$ of open sets U such that $\mathrm{bd}\,U$ is dimensionally full-valued for each $U \in \mathcal{B}_x$; $d_G X = n$ for any torsion free field G; $d_{\mathbb{Z}} X = n$.

Let us also mention Corollary 2.11 stating that every two-dimensional lc^2-space is dimensionally full-valued. This improves a result of Kodama [10, Theorem 8] that every two-dimensional *ANR* is dimensionally full-valued.

Using the sets $\mathcal{H}_{X,G}$, we obtain in Section 3 some properties of homogeneous ANR-spaces. In particular, the following proposition is shown:

Theorem 3.3. *Let X be a homogeneous ANR-space.*

(1) *If G is a field and $\dim_G X = n$, then $n \in \mathcal{H}_{X,G}$ and $n+1 \notin \mathcal{H}_{X,G}$;*
(2) *If $\dim X = n$, then $\mathcal{H}_{X,\mathbb{Z}} \cap \{n-1, n\} \neq \varnothing$.*

The final Section 4 contains all preliminary information about the exact and Čech homology groups we are using in this paper.

2. Some properties of d_G

Everywhere below $\widehat{H}_*$ denotes the exact homology (see [13,15]), where G is any module over a commutative ring with unity. This homology is equivalent to Steenrod's homology [17] in the category of compact metric spaces. According to [15], for every module G and a compact metric pair (X, A) there exists a natural transformation $T_{X,A} : \widehat{H}_*(X, A; G) \to H_*(X, A; G)$ between the exact and Čech homologies such that $T^k_{X,A} : \widehat{H}_k(X, A; G) \to H_k(X, A; G)$, is a surjective homomorphism for each k, see [15, Theorem 4] (if A is the empty set, we denote $T^k_{X,\varnothing}$ by T^k_X). This homomorphism is an isomorphism in some situations, see Proposition 4.4.

We already mentioned that $d_G X \leq \dim X$ for any group G. Next proposition shows that in some cases we have a stronger inequality.

Proposition 2.1. *For any X and a field G we have $d_G X \leq \dim_G X$.*

Proof. It suffices to prove the required inequality when $\dim_G X = n < \infty$. The set $\mathcal{H}_{X,G}$ does not contain any $k \geq n+2$ because, according to the Universal Coefficient Theorem for Čech homology with field coefficients, $k-1 > \dim_G X$ implies $H_{k-1}(F; G) = 0$ for every closed set $F \subset X$. So, we need to show that $n+1 \notin \mathcal{H}_{X,G}$. Indeed, if $n+1 \in \mathcal{H}_{X,G}$, then there exist a closed set $A \subset X$ and non-trivial $\gamma \in H_n(A; G)$ with $i^n_{A,X}(\gamma) = 0$. Since G is a field, by Proposition 4.4(i), $H_n(A; G)$ and $H_n(X; G)$ are isomorphic to the groups $\widehat{H}_n(A; G)$ and $\widehat{H}_n(X; G)$, respectively. Let $\widehat{\gamma}$ be the element from $\widehat{H}_n(A; G)$ corresponding to γ and $\widehat{i}^n_{A,X} : \widehat{H}_n(A; G) \to \widehat{H}_n(X; G)$ be the inclusion homomorphism. It follows from the exact sequence

$$\to \widehat{H}_{n+1}(X, A; G) \to \widehat{H}_n(A; G) \to \widehat{H}_n(X; G) \to$$

that $\widehat{H}_{n+1}(X, A; G) \neq 0$. On the other hand, $\dim_G X = n$ implies that $H^{n+2}(X, A; G) = 0$, so $\mathrm{Ext}_G(H^{n+2}(X, A; G), G) = 0$. Hence, by [15, Theorem 3], we have the exact sequence

$$0 \to \widehat{H}_{n+1}(X, A; G) \to \mathrm{Hom}_G(H^{n+1}(X, A; G), G) \to 0.$$

Then the groups $\widehat{H}_{n+1}(X, A; G)$ and $\mathrm{Hom}_G(H^{n+1}(X, A; G), G)$ are isomorphic. Finally, since $\dim_G X = n$, $H^{n+1}(X, A; G) = 0$. Thus, $\widehat{H}_{n+1}(X, A; G)$ is also trivial, a contradiction. □

We consider the set $\mathcal{CH}_{X,G}$ of all integers $k \geq 1$ such that there exist a closed set $F \subset X$ and a nontrivial element $\alpha \in H^{k-1}(F; G)$ with $\alpha \notin j^{k-1}_{X,F}(H^{k-1}(X; G))$ (in such a situation we say that α is not extendable over X). Obviously, the set $\mathcal{CH}_{X,G}$ is a dual notion of the set $\mathcal{H}_{X,G}$.

Next proposition provides a connection between the sets $\mathcal{CH}_{X,G}$ and $\mathcal{H}_{X,G^*}$, where G^* denotes the dual group of G (G is considered as a topological group with the discrete topology).

Proposition 2.2. *Let X be a metric compactum and G a countable abelian group. Then $n \in \mathcal{CH}_{X,G}$ if and only if $n \in \mathcal{H}_{X,G^*}$.*

Proof. The proof follows from the fact that for any space Z and any $k \geq 0$ the group $H_k(Z;G^*)$ is the dual group of $H^k(Z;G)$, where both G and $H^k(Z;G)$ are equipped with discrete topology, see [8, Chap. VIII, §4]. Indeed, if $n \in \mathcal{CH}_{X,G}$, then there exist a closed set $A \subset X$ and a non-trivial element $\gamma \in H^{n-1}(A;G)$ with $\gamma \notin j^{n-1}_{X,A}(H^{n-1}(X;G))$. Choose a homomorphism $\alpha : H^{n-1}(A;G) \to \mathbb{S}^1$ such that $\alpha(\gamma) \neq 0$ and $\alpha\big(j^{n-1}_{X,A}(H^{n-1}(X;G))\big) = 0$. Because of the duality mentioned above, α can be treated as a non-trivial element of $H_{n-1}(A;G^*)$ such that $i^{n-1}_{A,X} = \alpha \circ j^{n-1}_{X,A}$. So, $i^{n-1}_{A,X}(\alpha) = 0$, which yields $n \in \mathcal{H}_{X,G^*}$.

If $n \in \mathcal{H}_{X,G^*}$, we take a closed set $A \subset X$ and a non-trivial $\alpha \in H_{n-1}(A;G^*)$ with $i^{n-1}_{A,X}(\alpha) = 0$. We consider α as a non-trivial element of $H^{n-1}(A;G)^*$. So, $\alpha(\gamma) \neq 0$ for some $\gamma \in H^{n-1}(A;G)$. This implies that $\gamma \notin j^{n-1}_{X,A}(H^{n-1}(X;G))$ because α is homologous to zero in X. □

Proposition 2.3. *For every space X and every abelian group G we have $\max \mathcal{CH}_{X,G} = \dim_G X$.*

Proof. The definition of the set $\mathcal{CH}_{X,G}$ and the exact sequence

$$\to H^{n-1}(X;G) \to H^{n-1}(F;G) \to H^n(X,F;G) \to$$

imply that if $n \in \mathcal{CH}_{X,G}$, then $H^n(X,F;G) \neq 0$. Thus, $\max \mathcal{CH}_{X,G} \leq \dim_G X$.

On the other hand, if $\dim_G X = n$, then there exists a closed set $Y \subset X$ and points $x \in Y$ possessing a basis of open in Y sets W such that all homomorphisms $j^{n-1}_{\overline{W},bd_Y W} : H^{n-1}(\overline{W};G) \to H^{n-1}(bd_Y W;G)$ are not surjective, see [11, Theorem 2]. Hence, the homomorphisms $j^{n-1}_{X,bd_Y W} : H^{n-1}(X;G) \to H^{n-1}(bd_Y W;G)$ are also not surjective, which implies that $n \in \mathcal{CH}_{X,G}$. □

Corollary 2.4. *For every countable abelian group G and a space X we have:*

(1) $d_{G^*} X = \dim_G X$;
(2) *If $\dim_G X$ is finite, then $\mathcal{H}_{X,G^*} = [1, \dim_G X]$.*

Proof. The first item follows from Propositions 2.2–2.3. Obviously, the second item also follows from Propositions 2.2–2.3 provided $\mathcal{CH}_{X,G} = [1, \dim_G X]$. The last equality follows from the inclusion $[1, \dim_G X] \subset \mathcal{CH}_{X,G}$. So, let $n \in [1, \dim_G X]$. Since $\dim_G X > n - 1$, there exists a closed set $A \subset X$ and a non-trivial $\gamma \in H^{n-1}(A;G)$ with $\gamma \notin j^{n-1}_{X,A}(H^{n-1}(X;G))$ (we use the following well known fact: $\dim_G X \leq k$ if and only if the homomorphism $j^k_{X,F} : H^k(X;G) \to H^k(F;G)$ is surjective for every closed set $F \subset X$). Hence, $[1, \dim_G X] \subset \mathcal{CH}_{X,G}$. □

The well known Bockstein theorem (see [4,11]) states that for every group G there is a set of countable groups $\sigma(G)$ such that $\dim_G X = \sup\{\dim_H X : H \in \sigma(G)\}$ for any space X. Corollary 2.4(1) yields a similar equality for $d_{G^*} X$.

Corollary 2.5. *For any space X and any countable abelian group G we have $d_{G^*} X = \sup\{d_{H^*} X : H \in \sigma(G)\}$.*

Next proposition is a non-metrizable homological analogue of Theorem 14 from [1, chapter IV, §6] and Property 5 from [2].

Lemma 2.6. *Let $B \subset A$ be a compact pair and $H_{n-1}(B;G)$ contain a non-trivial element γ, which is homologous to zero in A. Then there exists a closed set $K \subset A$ containing B such that K is a homological membrane for γ.*

Proof. Consider the family $\mathcal{F}$ of all closed sets $F \subset A$ containing B such that $i^{n-1}_{B,F}(\gamma) = 0$. Obviously, $A \in \mathcal{F}$ and any minimal element of $\mathcal{F}$ is a homological membrane for γ. To show that $\mathcal{F}$ has a minimal element, according to Zorn's lemma, we need the following claim.

Claim 1. *Suppose $\{F_\alpha : \alpha \in \Lambda\}$ is an infinite linearly ordered decreasing subfamily of $\mathcal{F}$ and $F = \bigcap F_\alpha$. Then $F \in \mathcal{F}$.*

Denote by $\mathcal{C}_1$ the family of all finite open covers of F_1 and let $\mathcal{C}_\alpha, \mathcal{C}$ be the restrictions of $\mathcal{C}_1$ on the sets F_α and F, respectively. The set $\mathcal{C}_1$ becomes directed with respect to the relation: $\omega' \prec \omega''$ iff ω'' is a refinement of ω'. For every $\omega \in \mathcal{C}_1$ let N_ω be the nerve of ω. There is a natural simplicial map $p_{\omega'',\omega'} : \mathrm{N}_{\omega''} \to \mathrm{N}_{\omega'}$ provided $\omega' \prec \omega''$. The maps $p_{\omega'',\omega'}$ induce corresponding homomorphisms $\pi^k_{\omega'',\omega'} : H_k(\mathrm{N}''_\omega; G) \to H_k(\mathrm{N}'_\omega; G)$ for every $k \geq 0$. Therefore, we have the inverse systems $\mathcal{S}^k_1 = \{H_k(\mathrm{N}_\omega; G), \pi^k_{\omega',\omega} : \omega \prec \omega'\}$, $k \geq 0$, such that $H_k(F_1; G)$ is the limit of $\mathcal{S}^k_1$, see [6]. The sets $\mathcal{C}$ and $\mathcal{C}_\alpha$ also generate the nerves $\mathrm{N}_{\omega|F}$, $\mathrm{N}_{\omega|F_\alpha}$ and the inverse systems

$$\mathcal{S}^k = \{H_k(\mathrm{N}_{\omega|F}; G), \pi^k_{\omega'|F,\omega|F} : \omega \prec \omega'\}$$

and

$$\mathcal{S}^k_\alpha = \{H_k(\mathrm{N}_{\omega|F_\alpha}; G), \pi^k_{\omega'|F_\alpha,\omega|F_\alpha} : \omega \prec \omega'\},$$

where $\omega|F$ and $\omega|F_\alpha$ denote the restrictions of ω on the sets F and F_α, respectively. It is easily seen that for every open cover δ of F_α there exists $\omega \in \mathcal{C}_1$ such that $\omega|F_\alpha$ refines δ. Consequently, $H_k(F_\alpha; G) = \varprojlim \mathcal{S}^k_\alpha$. Similarly, $H_k(F; G) = \varprojlim \mathcal{S}^k$. Note that each $\mathrm{N}_{\omega|F_\alpha}$ is a subcomplex of the finite complex N_ω, $\omega \in \mathcal{C}_1$. Moreover, if $F_\alpha \subset F_\beta$ for some $\alpha, \beta \in \Lambda$, then $\mathrm{N}_{\omega|F_\alpha}$ is a subcomplex of $\mathrm{N}_{\omega|F_\beta}$. So, the families $\{\mathrm{N}_{\omega|F_\alpha} : \alpha \in \Lambda\}$, $\omega \in \mathcal{C}_1$, are decreasing and each one of them consists of finite simplicial complexes. Because Λ is infinite, for every $\omega \in \mathcal{C}_1$ there is $\alpha(\omega) \in \Lambda$ such that $\mathrm{N}_{\omega|F_\beta} = \mathrm{N}_{\omega|F_{\alpha(\omega)}} = \mathrm{N}_{\omega|F}$ for all $\beta \succ \alpha(\omega)$.

Now, we can complete the proof of Claim 1. Let $i^{n-1}_{B,F}(\gamma) = \gamma_F$ and $\omega \in \mathcal{C}_1$. Then, according to the above notations, $\mathrm{N}_{\omega|F} = \mathrm{N}_{\omega|F_{\alpha(\omega)}}$. Consider also the projections $\pi_{F,\omega} : H_{n-1}(F; G) \to H_{n-1}(\mathrm{N}_{\omega|F}; G)$ and $\pi_{F_{\alpha(\omega)},\omega} : H_{n-1}(F_{\alpha(\omega)}; G) \to H_{n-1}(\mathrm{N}_{\omega|F_{\alpha(\omega)}}; G)$. Then we have the commutative diagram

$$\begin{array}{ccc} H_{n-1}(F;G) & \xrightarrow{i^{n-1}_{F,F_{\alpha(\omega)}}} & H_{n-1}(F_{\alpha(\omega)};G) \\ \downarrow \pi_{F,\omega} & & \downarrow \pi_{F_{\alpha(\omega)},\omega} \\ H_{n-1}(\mathrm{N}_{\omega|F};G) & \xrightarrow{i^{n-1}_{\mathrm{N}_{\omega|F},\mathrm{N}_{\omega|F_{\alpha(\omega)}}}} & H_{n-1}(\mathrm{N}_{\omega|F_{\alpha(\omega)}};G) \end{array}$$

Since $i^{n-1}_{F,F_{\alpha(\omega)}}(\gamma_F) = i^{n-1}_{B,F_\alpha(\omega)}(\gamma) = 0$ and $i^{n-1}_{\mathrm{N}_{\omega|F},\mathrm{N}_{\omega|F_{\alpha(\omega)}}}$ is the identity, we obtain $\pi_{F,\omega}(\gamma_F) = 0$. The last equality (being true for each $\omega \in \mathcal{C}_1$) implies $\gamma_F = 0$. Hence, $F \in \mathcal{F}$. $\square$

Corollary 2.7. *Let X be a Hausdorff compactum with $n \in \mathcal{H}_{X,G}$. Then there exists a pair $F \subset K$ of closed subsets of X such that K is a homological membrane for some non-trivial element of $H_{n-1}(F; G)$.*

Proof. Since $n \in \mathcal{H}_{X,G}$, there exists a proper closed subset $F \subset X$ and a non-trivial $\gamma \in H_{n-1}(F; G)$ such that γ is G-homologous to zero in X. Then apply Lemma 2.6 for the pair $F \subset X$. $\square$

Corollary 2.8. *Let X be a space and G any group. Then, $n \in \mathcal{H}_{X,G}$ if and only if there exist $x \in X$ and a local base $\mathcal{B}_x$ at x consisting of open sets U with each $\mathrm{bd}\,\overline{U}$ containing a closed set F_U such that $i^{n-1}_{F_U,\overline{U}}(\gamma_U) = 0$ for some non-trivial $\gamma_U \in H_{n-1}(F_U; G)$.*

Proof. Obviously, the existence of a local base with the stated property yields $n \in \mathcal{H}_{X,G}$. So, we need to show the "only if" part. To this end, let $n \in \mathcal{H}_{X,G}$. Then, there exists a compact pair $F \subset K$ such that K is an $(n-1)$-homology membrane for some $\gamma \in H_{n-1}(F;G)$ (see Lemma 2.6). According to [2, Property 6], every $x \in K \setminus F$ has a local base $\mathcal{B}_x$ of open sets $U \subset X$ such that $\overline{U \cap K}$ is an $(n-1)$-homology membrane for an element $\gamma_U \in H^{n-1}(F_U;G)$ spanned on F_U, where $F_U = \mathrm{bd}_K(U \cap K)$. Because $i^{n-1}_{F_U,\overline{U\cap K}}(\gamma_U) = 0$, $i^{n-1}_{F_U,\overline{U}}(\gamma_U) = 0$. □

A space X is *homologically locally connected in dimension* n (br., $n - lc$) if for every $x \in X$ and a neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the homomorphism $\widetilde{i}^n_{V,U} : \widetilde{H}_n(V;\mathbb{Z}) \to \widetilde{H}_n(U;\mathbb{Z})$ is trivial, where $\widetilde{H}_*(.;\mathbb{Z})$ denotes the singular homology groups with integer coefficients ($\widetilde{H}_*(.;\mathbb{Z})$ should not be confused with the reduced singular homology). If in the above definition the group Z is replaced by a group G, we say that X is $n - lc$ with respect to G. One can show that $X \in n - lc$ with respect to any group G provided $X \in k - lc$ for every $k \in \{n, n-1\}$ (for $n \geq 1$ this follows from Proposition 4.8; for $n = 0$ it is obvious). We say that X is *homologically locally connected up to dimension* n (br., $X \in lc^n$) provided X is $k - lc$ for all $k \leq n$.

According to [12, Theorem 1] and [15, Proposition 9], the homology groups $H_k(X,A;G)$, $\widehat{H}_k(X,A;G)$ and $\widetilde{H}_k(X,A;G)$ are isomorphic for any abelian group G and any $k \leq n$ provided $A \subset X$ is a pair of metric compacta such that both X and A are lc^{n+1}, see Propositions 4.2 and 4.3.

It follows from Proposition 4.8 and Proposition 4.2 that any lc^n space X is homologically locally connected up to dimension n with respect to both singular and Čech homology with arbitrary coefficients, i.e. for every group G, a point $x \in X$ and its neighborhood U there exists a neighborhood V of x such that $V \subset U$ and the homomorphisms $i^k_{V,U} : H_k(V;G) \to H_k(U;G)$ and $\widetilde{i}^k_{V,U} : \widetilde{H}_k(V;G) \to \widetilde{H}_k(U;G)$ are trivial for all $k \leq n$.

Theorem 2.9. *The following holds for any space X with* $\dim X = n$*:*

(1) *If $d_G X = n$ for a torsion free abelian group G, then:*
  (1.1) *there exist a point $x \in X$ and a local base $\mathcal{B}_x$ at x consisting of open sets U such that* $\mathrm{bd}\,\overline{U}$ *is a dimensionally full-valued set of dimension $n-1$, and*
  (1.2) *X is dimensionally full-valued;*
(2) *If X is dimensionally full-valued and X has a base of $\mathcal{B}$ of open sets U with $H^n(\overline{U};\mathbb{Z}) = 0$, then $d_G X = n$ for every field G;*
(3) *If X is dimensionally full-valued and lc^n, then $d_{\mathbb{Z}} X = d_G X = n$ for every field G.*

Proof. (1.1) Suppose $d_G X = n$. Then $n \in \mathcal{H}_{X,G}$, so there exist $x \in X$ and a local base $\mathcal{B}_x$ at x of open sets $U \subset X$ such that each $\mathrm{bd}\,\overline{U}$ contains a closed set F_U with $i^{n-1}_{F_U,X}(\gamma_U) = 0$ for some non-trivial $\gamma_U \in H_{n-1}(F_U;G)$ (see Corollary 2.8). According to Proposition 4.5, $H_{n-1}(F_U;\mathbb{Z}) \neq 0$. Because $\dim X = n$, we can suppose that $\dim \mathrm{bd}\,\overline{U} \leq n-1$ for all $U \in \mathcal{B}_x$. Hence, $\dim F_U \leq n-1$. On the other hand, $H_{n-1}(F_U;G) \neq 0$ implies $\dim F_U \geq n-1$. Thus, $\dim F_U = n-1$ and, by Proposition 4.7, F_U is dimensionally full-valued. This yields that $bd\,\overline{U}$ is also dimensionally full-valued. Indeed, for any finite-dimensional metric compactum Y, we have the inequalities

$$\dim Y + n - 1 = \dim F_U \times Y \leq \dim bd\,\overline{U} \times Y \leq \dim Y + n - 1.$$

Hence, $\dim bd\,\overline{U} \times Y = \dim bd\,\overline{U} + \dim Y$.

(1.2) As above, we can find a closed set $F_U \subset X$ of dimension $n-1$ and a non-trivial $\gamma \in H_{n-1}(F_U;G)$ such that $i^{n-1}_{F_U,X}(\gamma) = 0$. Then, by Proposition 4.6, $H_n(X,F_U;G) \neq 0$. Because G is torsion free, the last relation implies $H_n(X,F_U;\mathbb{Z}) \neq 0$, see Proposition 4.5. Finally, according to Proposition 4.7, X is dimensionally full-valued.

(2) Let G be any field. Because X is dimensionally full-valued, $\dim_G X = \dim X = n$, see [11, Theorem 11]. Since $d_G X \le \dim X = n$, it suffices to show that $n \in \mathcal{H}_{X,G}$. To this end, observe that, since G is a field, $h\dim_G X = \dim_G X = n$, see [7]. Here, $h\dim_{\mathbb{Q}} X$ denotes the maximal integer m such that $\widehat{H}_k(X, P; G) = 0$ for all $k > m$ and all closed sets $P \subset X$. On the other hand, by [14, Corollary 2], if $h\dim_G X$ is finite, it is equal to the maximum integer k such that the module $H^x_k = \varinjlim_{x\in U} \widehat{H}_k(X, X \setminus U; G)$ is non-trivial for some $x \in X$. Therefore, there exists a point x with $\widehat{H}_n(X, X \setminus U; G) \neq 0$ for all sufficiently small neighborhoods U of x. We fix such U with $U \in \mathcal{B}$. Then, by the excision axiom, $\widehat{H}_n(X, X \setminus U; G)$ is isomorphic to $\widehat{H}_n(\overline{U}, bd\,\overline{U}; G)$. Consider the exact sequences (see Theorems 2–3 from [15])

$$\widehat{H}_n(\overline{U}; G) \longrightarrow \widehat{H}_n(\overline{U}, \mathrm{bd}\,\overline{U}; G) \xrightarrow{\partial} \widehat{H}_{n-1}(\mathrm{bd}\,\overline{U}; G) \xrightarrow{\widehat{i}^{\,n-1}_{\mathrm{bd}\,\overline{U},\overline{U}}} \widehat{H}_{n-1}(\overline{U}; G)$$

and

$$0 \to \mathrm{Ext}(H^{n+1}(\overline{U}; \mathbb{Z}), G) \to \widehat{H}_n(\overline{U}; G) \to \mathrm{Hom}(H^n(\overline{U}; \mathbb{Z}), G) \to 0.$$

Since $H^{n+1}(\overline{U}; \mathbb{Z}) = 0$ (recall that $\dim \overline{U} \le n$) and $H^n(\overline{U}; \mathbb{Z}) = 0$, the second sequence yields $\widehat{H}_n(\overline{U}; G) = 0$. Hence, the homomorphism $\partial : \widehat{H}_n(\overline{U}, bd\,\overline{U}; G) \to \widehat{H}_{n-1}(bd\,\overline{U}; G)$ is injective and the image $L = \partial(\widehat{H}_n(\overline{U}, bd\,\overline{U}; G))$ is a non-trivial subgroup of $\widehat{H}_{n-1}(bd\,\overline{U}; G)$ such that $\widehat{i}^{\,n-1}_{bd\,\overline{U},\overline{U}}(L) = 0$, where $\widehat{i}^{\,n-1}_{bd\,\overline{U},\overline{U}} : \widehat{H}_{n-1}(bd\,\overline{U}; G) \to \widehat{H}_{n-1}(\overline{U}; G)$ is the homomorphism generated by the inclusion $bd\,\overline{U} \hookrightarrow \overline{U}$. Because G is a field, the homomorphisms $T^{n-1}_{\overline{U}} : \widehat{H}_{n-1}(\overline{U}; G) \to H_{n-1}(\overline{U}; G)$ and $T^{n-1}_{\mathrm{bd}\,\overline{U}} : \widehat{H}_{n-1}(\mathrm{bd}\,\overline{U}; G) \to H_{n-1}(\mathrm{bd}\,\overline{U}; G)$ are isomorphisms, see Proposition 4.4(i). So, it follows from the commutative diagram

$$\begin{array}{ccc}
\widehat{H}_{n-1}(\mathrm{bd}\,\overline{U}; G) & \xrightarrow{\widehat{i}^{\,n-1}_{\mathrm{bd}\,\overline{U},\overline{U}}} & \widehat{H}_{n-1}(\overline{U}; G) \\
\downarrow T^{n-1}_{\mathrm{bd}\,\overline{U}} & & \downarrow T^{n-1}_{\overline{U}} \\
H_{n-1}(\mathrm{bd}\,\overline{U}; G) & \xrightarrow{i^{n-1}_{\mathrm{bd}\,\overline{U},\overline{U}}} & H_{n-1}(\overline{U}; G)
\end{array}$$

that $T^{n-1}_{bd\,\overline{U}}(L)$ contains a non-trivial element γ with $i^{n-1}_{bd\,\overline{U},\overline{U}}(\gamma) = 0$. The last equality implies $i^{n-1}_{bd\,\overline{U},X}(\gamma) = 0$. Thus, $n \in \mathcal{H}_{X,G}$.

(3) We have $\dim_G X = n$ (recall that X is dimensionally full-valued). Moreover, by [7], $h\dim_G X = \dim_G X = n$. Hence, as in the proof of item (2), we can find a point $x \in X$ such that $\widehat{H}_n(\overline{U}, bd\,\overline{U}; G) \neq 0$ for all sufficiently small neighborhoods U of x. Because X is lc^n, X is also locally homologically connected up to dimension n with respect to Čech homology with arbitrary coefficients. Hence, there exists a neighborhood V of x such that the homomorphism $i^n_{V,X} : H_n(V; G) \to H_n(X; G)$ is trivial.

We claim that $\widehat{H}_n(\overline{U}; G) = 0$ for each neighborhood U of x with $\overline{U} \subset V$. Indeed, suppose there is a neighborhood U of x with $\overline{U} \subset V$ and $\widehat{H}_n(\overline{U}; G) \neq 0$. Then Proposition 4.4(i) implies that $\widehat{H}_n(\overline{U}; G)$ is isomorphic to $H_n(\overline{U}; G)$. So, $H_n(\overline{U}; G)$ is a non-trivial group such that the inclusion homomorphism $i^n_{\overline{U},X} = i^n_{V,X} \circ i^n_{\overline{U},V}$ is trivial. That yields $d_G X \ge n + 1$, which contradicts Proposition 2.1.

Therefore, the homomorphism $\partial : \widehat{H}_n(\overline{U}, bd\,\overline{U}; G) \to \widehat{H}_{n-1}(bd\,\overline{U}; G)$ is injective and the image $L = \partial(\widehat{H}_n(\overline{U}, bd\,\overline{U}; G))$ is a non-trivial subgroup of $\widehat{H}_{n-1}(bd\,\overline{U}; G)$ with $\widehat{i}^{\,n-1}_{bd\,\overline{U},X}(L) = 0$ (see the proof of item (3)). Because the groups $\widehat{H}_{n-1}(bd\,\overline{U}; G)$ and $\widehat{H}_{n-1}(X; G)$ are isomorphic to $H_{n-1}(bd\,\overline{U}; G)$ and $H_{n-1}(X; G)$, respectively, we have $d_G X = n$.

To prove that $d_{\mathbb{Z}} X = n$, consider the field $\mathbb{Q}$ of all rational numbers. According to the previous paragraph, there is a point $x \in X$ with $H_{n-1}(bd\,\overline{U}; \mathbb{Q}) \neq 0$ for all sufficiently small neighborhoods U of x. So, by Proposition 4.5, $H_{n-1}(bd\,\overline{U}; \mathbb{Z}) \neq 0$ for any such U. Finally, Proposition 4.9 (together with the equality $\dim X = n$) implies $d_{\mathbb{Z}} X = n$. □

Corollary 2.10. *Let X be a metric lc^n-compactum, where $\dim X = n$. Then the following conditions are equivalent: X is dimensionally full-valued; there is a point $x \in X$ having a local base $\mathcal{B}_x$ such that $\mathrm{bd}\,\overline{U}$ is dimensionally full-valued for each $U \in \mathcal{B}_x$; $d_G X = n$ for any torsion free field G; $d_{\mathbb{Z}} X = n$.*

Let us also mention the following corollary, which improves a result of Kodama [10, Theorem 8].

Corollary 2.11. *Every two-dimensional lc^2-compactum is dimensionally full-valued.*

Proof. The complete proof is presented in [18, Theorem 3.2], here we provide a sketch only. By [1], $d_{\mathbb{S}^1} X = h\dim_{\mathbb{S}^1} X = 2$. Then, as in the proof of Theorem 2.9(3), there is $x \in X$ such that $\widehat{H}_2(\overline{U}, \mathrm{bd}\,\overline{U}; \mathbb{S}^1)$ and $\widehat{H}_1(\mathrm{bd}\,\overline{U}; \mathbb{S}^1)$ are both non-trivial for sufficiently small neighborhoods U of x, while the homomorphisms $\widehat{i}^1_{\mathrm{bd}\,\overline{U},X} : \widehat{H}_1(\mathrm{bd}\,\overline{U}; \mathbb{S}^1) \to \widehat{H}_1(X; \mathbb{S}^1)$ are trivial. Moreover, $\widehat{i}^2_{\overline{U},X} : \widehat{H}_2(\overline{U}; \mathbb{S}^1) \to \widehat{H}_2(X; \mathbb{S}^1)$ is also trivial for small U because X is lc^2 and the exact homology groups with coefficients $\mathbb{S}^1$ are isomorphic to the corresponding Čech groups (see Proposition 4.4(i)). So, by the Universal Coefficient Theorem for Čech cohomology, $H^2(\overline{U}, \mathrm{bd}\,\overline{U}; \mathbb{Z}) \neq 0$, $H^1(\mathrm{bd}\,\overline{U}; \mathbb{Z}) \neq 0$ and the triviality of the homomorphism $\widehat{i}^1_{\mathrm{bd}\,\overline{U},X}$ yields the triviality of the inclusion homomorphism $j^1_{X\mathrm{bd}\,\overline{U}} : H^1(X; \mathbb{Z}) \to H^1(\mathrm{bd}\,\overline{U}; \mathbb{Z})$. It follows from the exact sequence

$$\cdots \to H^1(X;\mathbb{Z}) \xrightarrow{j^1_{X\mathrm{bd}\,\overline{U}}} H^1(\mathrm{bd}\,\overline{U};\mathbb{Z}) \xrightarrow{\partial_X} H^2(X, \mathrm{bd}\,\overline{U};\mathbb{Z}) \to \cdots$$

that ∂_X is an injective homomorphism. Hence, $H^2(X, \mathrm{bd}\,\overline{U}; \mathbb{Z})$ contains elements of infinite order (recall that the simplicial one-dimensional cohomology groups with integer coefficients are free, so any non-trivial one dimensional Čech cohomology group $H^1(. : \mathbb{Z})$ is torsion free, see for example [4, Theorem 12.5]). This implies that $H^2(X, \mathrm{bd}\,\overline{U}; \mathbb{Q}) \neq 0$. So, $\dim_{\mathbb{Q}} X = 2$, and by [7, p. 364], $h\dim_{\mathbb{Q}} X = 2$. On the other hand, $d_{\mathbb{Q}} X = h\dim_{\mathbb{Q}} X = 2$, see [18, Corollary 2.3]. Finally, Theorem 2.9(1) yields X is dimensionally full-valued. □

3. Homogeneous metric *ANR*-compacta

The following statement was established in [19] (see Theorem 1.1 and Corollary 1.2):

Proposition 3.1. *Let X be a finite dimensional homogeneous ANR-space with $\dim X \geq 2$. Then X has the following properties for any abelian group G and $n \geq 2$ with $n \in \mathcal{H}_{X,G}$ and $n+1 \notin \mathcal{H}_{X,G}$:*

(1) *Every $x \in X$ has a basis of open sets U_k such that $H_{n-1}(\overline{U}_k; G) = 0$ and $H_{n-1}(\mathrm{bd}\,\overline{U}_k; G) \neq 0$;*
(2) *If a closed subset $K \subset X$ is an $(n-1)$-homology membrane spanned on B for some closed set $B \subset X$ and $\gamma \in H_{n-1}(B; G)$, then $(K \setminus B) \cap \overline{X \setminus K} = \varnothing$.*

Here is our first proposition concerning homogeneous *ANR*.

Proposition 3.2. *Let X be as in Proposition 3.1 and $Z \subset X$ a closed set. If $n \geq 2$ is an integer with $n \in \mathcal{H}_{Z,G}$ and $n+1 \notin \mathcal{H}_{X,G}$, then Z has a non-empty interior in X. Moreover $d_G Z = d_G X$ iff $\mathrm{Int}(Z) \neq \varnothing$.*

Proof. Since $n \in \mathcal{H}_{Z,G}$, there exists a closed set $F \subset Z$ and non-trivial $\gamma \in H_{n-1}(F, G)$ homologous to zero in Z. Consequently, by Corollary 2.7, we can find a closed set $K \subset Z$ containing F such that K is a homological membrane for γ spanned on F. Because γ, being homologous to zero in Z, is homologous to zero in X, $n \in \mathcal{H}_{X,G}$. Recall that $n+1 \notin \mathcal{H}_{X,G}$, so we can apply Proposition 3.1(2) to conclude that $(K \setminus F) \cap \overline{X \setminus K} = \varnothing$. This implies that $K \setminus F$ is open in X, i.e., $\mathrm{Int}(Z) \neq \varnothing$.

Suppose now that $d_G Z = d_G X = n$. Then $n \in \mathcal{H}_{Z,G}$ and $n+1 \notin \mathcal{H}_{X,G}$. Hence, $\mathrm{Int}(Z) \neq \varnothing$. Conversely, if $\mathrm{Int}(Z) \neq \varnothing$ and $d_G X = n$, choose $x \in \mathrm{Int}(Z)$. Then, by Proposition 3.1(1), there exists $U \in \mathcal{B}_x$ such that $\overline{U} \subset \mathrm{Int}(Z)$ and $H_{n-1}(\mathrm{bd}\,\overline{U}; G)$ contains a non-trivial element homologous to zero in $\overline{U}$. Hence, $d_G Z \geq n$. Finally, the inequality $d_G Z \leq d_G X$ implies $d_G Z = d_G X$. □

Theorem 3.3. *Let X be a homogeneous ANR-space.*

(1) *If G is a field and $\dim_G X = n$, then $n \in \mathcal{H}_{X,G}$ and $n+1 \notin \mathcal{H}_{X,G}$;*
(2) *If $\dim X = n$, then $\mathcal{H}_{X,\mathbb{Z}} \cap \{n-1, n\} \neq \varnothing$.*

Proof. (1) Since G is a field, by [7], $h\dim_G X = \dim_G X = n$. Then there is a point $x \in X$ having sufficiently small open neighborhoods $U \subset X$ with $\widehat{H}_n(\overline{U}, bd\,\overline{U}; G) \neq 0$ (see the arguments from Theorem 2.9(2)). Consider the exact sequence

$$\to \widehat{H}_n(\overline{U}; G) \to \widehat{H}_n(\overline{U}, bd\,\overline{U}; G) \to \widehat{H}_{n-1}(bd\,\overline{U}; G) \to \widehat{H}_{n-1}(\overline{U}; G).$$

We claim that $\widehat{H}_n(\overline{U}; G) = 0$ for all sufficiently small U. Indeed, since the groups $H_n(\overline{U}; G)$ and $\widehat{H}_n(\overline{U}; G)$ are isomorphic by Proposition 4.4(i) (recall that G is a field), and X is locally contractible, we can choose $\overline{U}$ so small that the inclusion homomorphism $i^n_{\overline{U},X} : H_n(\overline{U}; G) \to H_n(X; G)$ is trivial. So, $\widehat{H}_n(\overline{U}; G) \neq 0$ would imply $d_G X \geq n+1$, which contradicts Proposition 2.1. Hence, $\widehat{H}_n(\overline{U}; G) = 0$. Then, proceeding as in the proof of Theorem 2.9(2), we obtain $n \in \mathcal{H}_{X,G}$. Finally, by Proposition 2.1, $d_G X \leq \dim_G X = n$. Therefore, $n+1 \notin \mathcal{H}_{X,G}$.

(2) By [20, Theorem 1.1], every point $x \in X$ has a basis of neighborhoods U with $\dim bd\,\overline{U} = n-1$ and $H^{n-1}(bd\,\overline{U}; \mathbb{Z}) \neq 0$. For any such U consider the exact sequences, where the coefficient group $\mathbb{Z}$ is suppressed

$$0 \to \mathrm{Ext}(H^n(bd\,\overline{U}), \mathbb{Z}) \to \widehat{H}_{n-1}(bd\,\overline{U}) \to \mathrm{Hom}(H^{n-1}(bd\,\overline{U}), \mathbb{Z}) \to 0$$

and

$$0 \to \mathrm{Ext}(H^{n-1}(bd\,\overline{U}), \mathbb{Z}) \to \widehat{H}_{n-2}(bd\,\overline{U}) \to \mathrm{Hom}(H^{n-2}(bd\,\overline{U}), \mathbb{Z}) \to 0.$$

The equality $\widehat{H}_{n-1}(bd\,\overline{U}; \mathbb{Z}) = \widehat{H}_{n-2}(bd\,\overline{U}; \mathbb{Z}) = 0$ implies the triviality of the groups $\mathrm{Ext}(H^{n-1}(bd\,\overline{U}; \mathbb{Z}), \mathbb{Z})$ and $\mathrm{Hom}(H^{n-1}(bd\,\overline{U}; \mathbb{Z}), \mathbb{Z})$. Thus, $H^{n-1}(bd\,\overline{U}; \mathbb{Z}) = 0$ (see §52, Exercise 9 from [5]), a contradiction. Hence, at least one of the groups $\widehat{H}_{n-1}(bd\,\overline{U}; \mathbb{Z})$ and $\widehat{H}_{n-2}(bd\,\overline{U}; \mathbb{Z})$ is not trivial. According to [20, Theorem 1.1], $H^{n-1}(bd\,\overline{U}; \mathbb{Z})$ is finitely generated. Hence, by [15, Theorem 4] (see also Proposition 4.4(iv) below), $\widehat{H}_{n-2}(bd\,\overline{U}; \mathbb{Z})$ is isomorphic to $H_{n-2}(bd\,\overline{U}; \mathbb{Z})$. Moreover, the equality $\dim bd\,\overline{U} = n-1$ yields that $\widehat{H}_{n-1}(bd\,\overline{U}; \mathbb{Z})$ is isomorphic to $H_{n-1}(bd\,\overline{U}; \mathbb{Z})$. Because X is an ANR, the sets $bd\,\overline{U}$ are contractible in X for sufficiently small U. Therefore, both homomorphisms $i^{n-1}_{bd\,\overline{U},X}$ and $i^{n-2}_{bd\,\overline{U},X}$ are trivial, which implies that $\mathcal{H}_{X,\mathbb{Z}}$ contains at least one of the integers $n-1$ and n. □

4. Appendix

There are two types of Universal Coefficients Theorems for the exact homology, see Theorem 2 and Theorem 3 from [15].

Proposition 4.1. *Let (X, A) be a pair of compact metric spaces and G be a module over a principal ideal domain K. Then we have the exact sequence:*

$$0 \to \mathrm{Ext}_K(H^{n+1}(X, A; K), G) \to \widehat{H}_n(X, A; G) \to \mathrm{Hom}_K(H^n(X, A; K), G) \to 0.$$

If, in addition, G is finitely generated, the sequence

$$0 \to \widehat{H}_n(X,A;K) \otimes_K G \to \widehat{H}_n(X,A;G) \to \widehat{H}_{n-1}(X,A;G) *_K G \to 0$$

is also exact, where $\widehat{H}_{n-1}(X,A;K) *_K G$ *denotes the torsion product* $\mathrm{Tor}_K(\widehat{H}_{n-1}(X,A;K),G)$.

We say that a space X is *homologically locally connected up to dimension* n (br., lc^n) if for every $x \in X$ and a neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the homomorphism $\widetilde{i}^m_{V,U} : \widetilde{H}_m(V;\mathbb{Z}) \to \widetilde{H}_m(U;\mathbb{Z})$ is trivial for all $m \le n$. Here, $\widetilde{H}_*(.;.)$ denotes the singular homology groups. According to [12, Theorem 1], the following is true:

Proposition 4.2. *Let* (X,A) *be a pair of paracompact spaces. If both* X *and* A *are* lc^n*, then there exists a natural transformation* $M_{X,A}$ *between the singular and Čech homologies of* (X,A) *such that* $M^k_{X,A}$: $\widetilde{H}_k(X,A;G) \to H_k(X,A;G)$ *is an isomorphism for each* $k \le n$ *and each group* G.

There is also an analogue of Proposition 4.3 concerning the homology $\widehat{H}_*(X,A;G)$, see [15, Proposition 9]:

Proposition 4.3. *Let* (X,A) *be a pair of compact metric spaces with both* X *and* A *being* lc^n*. Then there is a natural transformation* $S_{X,A}$ *between the singular and the exact homologies of* (X,A) *such that* $S^k_{X,A}$: $\widetilde{H}_k(X,A;G) \to \widehat{H}_k(X,A;G)$ *is an isomorphism for each* $k \le n-1$ *and it is surjective for* $k = n$.

As we already noted, if G is a module over a ring with unity and (X,A) is a compact metric pair, then there exists a natural transformation

$$T_{X,A} : \widehat{H}_*(X,A;G) \to H_*(X,A;G)$$

between the exact and Čech homologies such that the homomorphism $T^k_{X,A} : \widehat{H}_k(X,A;G) \to H_k(X,A;G)$ is surjective for each k. According to [15, Theorem 4], this homomorphism is an isomorphism under certain conditions. We list below some of these conditions.

Proposition 4.4. *The homomorphism* $T^k_{X,A}$ *is an isomorphism in each of the following situations:*

(i) *The group* G *admits a compact topology or* G *is a vector space over a field;*
(ii) *Both* $\widehat{H}_k(X,A;G)$ *and* G *are countable modules;*
(iii) $\dim X = k$*;*
(iv) *both* $H^{k+1}(X,A;\mathbb{Z})$ *and* G *are finitely generated with finite number of relations.*

We don't know if the second Universal Coefficient Theorem from Proposition 4.1 also holds for Čech homologies when $K = \mathbb{Z}$ and G is a torsion free field. Probably $\widehat{H}_n(X,A;G)$ is not isomorphic to $\widehat{H}_n(X,A;K) \otimes \mathbb{G}$, but we have the following conclusion:

Proposition 4.5. *Let* (X,A) *be a compact pair and* n *be a non-negative integer. Then* $H_n(X,A;G) \neq 0$ *implies* $H_n(X,A;\mathbb{Z}) \neq 0$ *for every torsion free abelian group* G.

Proof. We choose a family $\{\omega_\alpha\}$ of finite open covers of X such that $H_n(X,A;G)$ is the limit of the inverse system $\{H_n(N^X_\alpha, N^A_\alpha;G), \pi^*_{\beta,\alpha}\}$, where N^X_α and N^A_α are the nerves of ω_α and the restriction of ω_α over A, and $\pi_{\beta,\alpha} : (N^X_\beta, N^A_\beta) \to (N^X_\alpha, N^A_\alpha)$ are the corresponding simplicial maps with β being a refinement of α. By Proposition 4.2, all $H_n(N^X_\alpha, N^A_\alpha;G)$ (resp., $H_n(N^X_\alpha, N^A_\alpha;\mathbb{Z})$) are isomorphic to the singular homology groups $\widetilde{H}_n(N^X_\alpha, N^A_\alpha;G)$ (resp., $\widetilde{H}_n(N^X_\alpha, N^A_\alpha;\mathbb{Z})$). So, the second Universal Coefficient

Theorem from Proposition 4.1, which holds for the singular homology with arbitrary coefficients, implies that $H_n(N_\alpha^X, N_\alpha^A; G)$ are isomorphic to the tensor products $H_n(N_\alpha^X, N_\alpha^A; \mathbb{Z}) \otimes G$ (recall that, since G is torsion free, the torsion product of G with any abelian group is trivial). Because $H_n(X, A; \mathbb{Z})$ is the limit of the inverse sequence $\{H_n(N_\alpha^X, N_\alpha^A; \mathbb{Z}), \pi_{\beta,\alpha}^*\}$, the triviality of $H_n(X, A; \mathbb{Z})$ would yield that $H_n(N_\alpha^X, N_\alpha^A; \mathbb{Z}) = 0$ for all α. Consequently, all groups $H_n(N_\alpha^X, N_\alpha^A; G)$ would be also trivial, which contradicts the condition $H_n(X, A; G) \neq 0$. □

Proposition 4.6. *Let (X, A) be a compact pair, G be an abelian group and $n \geq 1$ an integer. If there is a non-trivial element $\gamma \in H_{n-1}(A; G)$ such that $i_{A,X}^{n-1}(\gamma) = 0$, then $H_n(X, A; G) \neq 0$.*

Proof. Following the notations from the proof of Proposition 4.5, take a family $\{\omega_\alpha\}$ of finite open covers of X such that the homology groups $H_n(X, A; G)$, $H_{n-1}(X; G)$ and $H_{n-1}(A; G)$ are limit, respectively, of the inverse systems $\{H_n(N_\alpha^X, N_\alpha^A; G), \pi_{\beta,\alpha}^*\}$, $\{H_{n-1}(N_\alpha^X; G), \pi_{\beta,\alpha}^*\}$ and $\{H_{n-1}(N_\alpha^A; G), \pi_{\beta,\alpha}^*\}$. If for each α we denote by $\pi_\alpha : A \to N_\alpha^A$ the natural map, then there exists α_0 such that $\gamma_{\alpha_0} = \pi_{\alpha_0}^*(\gamma)$ is a non-trivial element of $H_{n-1}(N_{\alpha_0}^A; G)$. It follows from the commutative diagram

$$\begin{array}{ccc} H_{n-1}(A;G) & \xrightarrow{i_{A,X}^{n-1}} & H_{n-1}(X;G) \\ \downarrow \pi_{\alpha_0}^* & & \downarrow \pi_{\alpha_0}^* \\ H_{n-1}(N_{\alpha_0}^A;G) & \xrightarrow{i_{N_{\alpha_0}^A, N_{\alpha_0}^X}^{n-1}} & H_{n-1}(N_{\alpha_0}^X;G) \end{array}$$

that $i_{N_{\alpha_0}^A, N_{\alpha_0}^X}^{n-1}(\gamma_{\alpha_0}) = 0$. Because $H_n(N_{\alpha_0}^X, N_{\alpha_0}^A; G)$, $H_{n-1}(N_{\alpha_0}^A; G)$ and $H_{n-1}(N_{\alpha_0}^X; G)$ are naturally isomorphic to the corresponding singular homology groups, we have the exact sequence

$$\to H_n(N_{\alpha_0}^X, N_{\alpha_0}^A; G) \to H_{n-1}(N_{\alpha_0}^A; G) \to H_{n-1}(N_{\alpha_0}^X; G) \to \dots$$

Therefore, $H_n(N_{\alpha_0}^X, N_{\alpha_0}^A; G) \neq 0$, which implies $H_n(X, A; G) \neq 0$. □

Let us also mention the following sufficient condition for a given metric compactum to be dimensionally full-valued, see [9, Corollary 1]. We provide a short proof of this result, different from the original one.

Proposition 4.7. *Let X be a space with $\dim X = n$. If there exists a closed subset $A \subset X$ such that $H_n(X, A; \mathbb{Z}) \neq 0$, then X is dimensionally full-valued.*

Proof. Since $\dim X = n$, $H_n(X, A; \mathbb{Z})$ and $\widehat{H}_n(X, A; \mathbb{Z})$ are isomorphic and $H^{n+1}(X, A; \mathbb{Z}) = 0$. So, the first exact sequence from Proposition 4.1 implies $\mathrm{Hom}(H^n(X, A; \mathbb{Z}), \mathbb{Z}) \neq 0$. This means that $H^n(X, A; \mathbb{Z})$ contains $\mathbb{Z}$ as a direct multiple. Thus, the tensor product $H^n(X, A; \mathbb{Z}) \otimes G$ is not trivial for all abelian groups G. Then the exact sequence

$$0 \to H^n(X, A; \mathbb{Z}) \otimes G \to H^n(X, A; G) \to H^n(X, A; \mathbb{Z}) * G \to 0$$

yields $H^n(X, A; G) \neq 0$. Therefore, $\dim_G X = \dim X$ for all abelian groups G. Finally, by [11, Theorem 11], X is dimensionally full-valued. □

Proposition 4.8. *If X is $k - lc$ and $(k-1) - lc$, where $k \geq 1$, then X is $k - lc$ with respect to every abelian group G.*

Proof. Let U be a neighborhood of a given point x. Since X is $k - lc$ and $(k-1) - lc$, there exist two neighborhoods $W \subset V$ of x such that $V \subset U$ and the inclusion homomorphisms $\widetilde{i}^p_{W,V} : \widetilde{H}_p(W;\mathbb{Z}) \to \widetilde{H}_p(V;\mathbb{Z})$, $\widetilde{i}^p_{V,U} : \widetilde{H}_p(V;\mathbb{Z}) \to \widetilde{H}_p(U;\mathbb{Z})$ are trivial for every $p = k-1, k$. Consider the following diagram, whose rows are exact sequences

$$\begin{array}{ccccc}
\widetilde{H}_k(W;\mathbb{Z}) \otimes G & \longrightarrow & \widetilde{H}_k(W;G) & \longrightarrow & \widetilde{H}_{k-1}(W;\mathbb{Z}) * G \\
\downarrow {\scriptstyle \widetilde{i}^k_{W,V} \otimes id} & & \downarrow {\scriptstyle \widetilde{i}^k_{W,V}} & & \downarrow {\scriptstyle \widetilde{i}^{k-1}_{W,V} * id} \\
\widetilde{H}_k(V;\mathbb{Z}) \otimes G & \longrightarrow & \widetilde{H}_k(V;G) & \longrightarrow & \widetilde{H}_{k-1}(V;\mathbb{Z}) * G \\
\downarrow {\scriptstyle \widetilde{i}^k_{V,U} \otimes id} & & \downarrow {\scriptstyle \widetilde{i}^k_{V,U}} & & \downarrow {\scriptstyle \widetilde{i}^{k-1}_{V,U} * id} \\
\widetilde{H}_k(U;\mathbb{Z}) \otimes G & \longrightarrow & \widetilde{H}_k(U;G) & \longrightarrow & \widetilde{H}_{k-1}(U;\mathbb{Z}) * G
\end{array}$$

Because the homomorphisms $\widetilde{i}^k_{V,U} \otimes id$ and $\widetilde{i}^{k-1}_{W,V} * id$ are trivial, so is the composition $\widetilde{i}^k_{V,U} \circ \widetilde{i}^k_{W,V} = \widetilde{i}^k_{W,U}$. Hence, X is $k - lc$ with respect to G. $\square$

Proposition 4.9. *Let X be a metric lc^{n-1}-space with the following property: there exists a point $x \in X$ such that any sufficiently small neighborhood V of x contains a closed set $F_V \subset X$ with $H_{n-1}(F_V;G) \neq 0$, where G is a given abelian group. Then $n \in \mathcal{H}_{X,G}$.*

Proof. Let G be a fixed group. Then X is $k - lc$ with respect to G for all $k \leq n-1$ (for $k \geq 1$ this follows from Proposition 4.8, and for $k = 0$ it is obvious). On the other hand, by [12, Theorem 1], the singular homology groups $\widetilde{H}_{n-1}(W;G)$ are isomorphic to the groups $H_{n-1}(W;G)$ for every open set $W \subset X$. Thus, the inclusion homomorphisms $i^{n-1}_{U,X} : H_{n-1}(U;G) \to H_{n-1}(X;G)$ are trivial for all small neighborhoods U of x. Then any such U contains F_U and the homomorphism $i^{n-1}_{F_U,X} : H_{n-1}(F_U;G) \to H_{n-1}(X;G)$, being the composition $i^{n-1}_{U,X} \circ i^{n-1}_{F_U,U}$, is trivial. Therefore, $n \in \mathcal{H}_{X,G}$. $\square$

Acknowledgements

The author would like to express his gratitude to K. Kawamura for his helpful comments. The author also thanks the referee for his/her careful reading and suggesting many improvements of the paper.

References

[1] P. Alexandroff, Introduction to Homological Dimension Theory and General Combinatorial Topology, Nauka, Moscow, 1975 (in Russian).
[2] R.H. Bing, K. Borsuk, Some remarks concerning topological homogeneous spaces, Ann. Math. 81 (1) (1965) 100–111.
[3] V. Boltyanskii, On dimensional full valuedness of compacta, Dokl. Akad. Nauk SSSR 67 (1949) 773–777 (in Russian).
[4] A. Dranishnikov, Cohomological dimension theory of compact metric spaces, Topol. Atlas Invit. Contrib. (2004).
[5] L. Fuchs, Infinite Abelian Groups, vol. I, Academic Press, New York, London, 1970.
[6] S. Eilenberg, N. Steenrod, Foundations of Algebraic Topology, Princeton Univ. Press, Princeton, NJ, 1952.
[7] A. Harlap, Local homology and cohomology, homological dimension, and generalized manifolds, Mat. Sb. (N.S.) 96 (138) (1975) 347–373 (in Russian).
[8] W. Hurewicz, H. Wallman, Dimension Theory, Princeton Math. Ser., vol. 4, Princeton University Press, 1941.
[9] Y. Kodama, On a problem of Alexandroff concerning the dimension of product spaces I, J. Math. Soc. Jpn. 10 (4) (1958) 380–404.
[10] Y. Kodama, On homotopically stable points, Fundam. Math. 44 (1957) 171–185.
[11] V. Kuz'minov, Homological dimension theory, Russ. Math. Surv. 23 (1) (1968) 1–45.
[12] S. Mardešic, Comparison of singular and Cech homology in locally connected spaces, Mich. Math. J. 6 (1959) 151–166.
[13] W. Massey, Homology and Cohomology Theory, Dekker, New York, 1978.
[14] E. Sklyarenko, On the theory of generalized manifolds, Izv. Akad. Nauk SSSR, Ser. Mat. 35 (1971) 831–843 (in Russian).
[15] E. Sklyarenko, Homology theory and the exactness axiom, Usp. Mat. Nauk 24 (5(149)) (1969) 87–140 (in Russian).
[16] E. Spanier, Algebraic Topology, McGraw–Hill Book Company, 1966.

[17] N. Steendrod, Regular cycles of compact metric spaces, Ann. Math. 41 (1940) 833–851.
[18] V. Valov, Homological dimension and dimensional full-valuedness, submitted for publication.
[19] V. Valov, Local Homological Properties and Cyclicity of Homogeneous ANR Compacta, Amer. Math. Soc., 2017, in press.
[20] V. Valov, Local cohomological properties of homogeneous ANR compacta, Fundam. Math. 223 (2016) 257–270.