http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 51 (2018)
Pages 277-292

E-Published on January 18, 2018

I-FAVORABLE SPACES: REVISITED

VESKO VALOV

Abstract. The aim of this paper is to extend the external characterization of I-favorable spaces obtained in [13]. This allows us to obtain a characterization of compact I-favorable spaces in terms of quasi κ-metrics. We also provide proofs of some author's results announced in [14].

1. Introduction

The aim of this paper is to extend the external characterization of I-favorable spaces obtained in [13]. We also provide proofs of some author's results announced in [14]. All topological spaces are Tychonoff and the single-valued maps are continuous.

P. Daniels, K. Kunen and H. Zhou [2] introduced the so called *open-open game*: Two players take countably many turns, a round consists of player I choosing a non-empty open set $U \subset X$ and II choosing a non-empty open set $V \subset U$. Player I wins if the union of II's open sets is dense in X, otherwise II wins. A space X is called I-*favorable* if player I has a winning strategy. This means, see [6], there exists a function $\sigma : \bigcup_{n\geq 0} \mathcal{T}_X^n \to \mathcal{T}_X$ such that the union $\bigcup_{n\geq 0} U_n$ is dense in X for each game

$$\big(\sigma(\varnothing), U_0, \sigma(U_0), U_1, \sigma(U_0, U_1), U_2, ..., U_n, \sigma(U_0, U_1, .., U_n), U_{n+1}, , , \big),$$

where all U_k and $\sigma(\varnothing)$ are non-empty open sets in X, $U_0 \subset \sigma(\varnothing)$ and $U_{k+1} \subset \sigma(U_0, U_1, .., U_k)$ for every $k \geq 0$ (here $\mathcal{T}_X$ is the topology of X).

2010 *Mathematics Subject Classification.* Primary 54C10; Secondary 54F65.
Key words and phrases. Compact spaces, continuous inverse systems, I-favorable spaces, skeletal maps.
Research supported in part by NSERC Grant 261914-13.

Recently A. Kucharski and S. Plewik (see [6], [7]) investigated the connection of I-favorable spaces and skeletal maps. In particular, they proved in [7] that the class of compact I-favorable spaces and the skeletal maps are adequate in the sense of E. Shchepin [9]. Recall that a map $f : X \to Y$ is skeletal if $\mathrm{Int}\overline{f(U)} \neq \varnothing$) for every open $U \subset X$. On the other hand, the author announced [14, Theorem 3.1] a characterization of the spaces X such that there is an inverse system $S = \{X_\alpha, p^\beta_\alpha, A\}$ of separable metric spaces X_α and skeletal surjective bounding maps p^β_α satisfying the following conditions: (1) the index set A is σ-complete (every countable chain in A has a supremum in A); (2) for every countable chain $\{\alpha_n\}_{n\geq 1} \subset A$ with $\beta = \sup\{\alpha_n\}_{n\geq 1}$ the space X_β is a (dense) subset of $\varprojlim\{X_{\alpha_n}, p^{\alpha_{n+1}}_{\alpha_n}\}$; (3) X is embedded in $\varprojlim S$ and $p_\alpha(X) = X_\alpha$ for each α, where $p_\alpha\colon \varprojlim S \to X_\alpha$ is the α-th limit projection. An inverse system satisfying (1) and (2) is called *almost σ-continuous.* If condition (3) is satisfied, we say that X is the *almost limit of* S, notation $X = \mathrm{a}-\varprojlim S$. Spaces X such that $X = \mathrm{a}-\varprojlim S$, where S is almost σ-continuous inverse system with skeletal bounding maps and second countable spaces, are called *skeletally generated* [13].

The following theorem is our first main result:

Theorem 1.1. *For a space X the following conditions are equivalent:*

(1) *X is* I-*favorable;*
(2) *Every embedding of X in another space Y is π-regular;*
(3) *X is skeletally generated.*

Here, we say that a subspace $X \subset Y$ is *π-regularly embedded* in Y [14] if there exists a function $\mathrm{e}\colon \mathcal{T}_X \to \mathcal{T}_Y$ such that for every $U, V \in \mathcal{T}_X$ we have: (i) $\mathrm{e}(U) \cap \mathrm{e}(V) = \varnothing$ provided $U \cap V = \varnothing$; (ii) $\mathrm{e}(U) \cap X$ is a dense subset of U. If, $\mathrm{e}(U) \cap X = U$, we say that X is *regularly embedded in Y*. An external characterization of κ-metrizable compacta, similar to condition (2), was established in [11].

Corollary 1.2. *Every* I-*favorable subset of an extremally disconnected space is also extremally disconnected.*

Corollary 1.3. *Every open subset of an* I-*favorable space is* I-*favorable.*

A version of Theorem 1.1 was established in [13], but we used slightly different notions. First, we considered I-favorable spaces with respect to the family of co-zero sets. Also, in the definition of skeletally generated spaces we required the system S to be factorizable (i.e. for each continuous function f on X there exists $\alpha \in A$ and a continuous function h on X_α

with $f = h \circ p_\alpha$). Moreover, in item (2) X was supposed to be C^*-embedded in Y. Corollary 1.2 was also established in [13] under the assumption of C^*-embedability.

Recall that a *κ-metric* [9] on a space X is a non-negative function $\rho(x, C)$ of two variables, a point $x \in X$ and a canonically closed set $C \subset X$, satisfying the following axioms:

K1) $\rho(x, C) = 0$ iff $x \in C$;
K2) If $C \subset C'$, then $\rho(x, C') \leq \rho(x, C)$ for every $x \in X$;
K3) $\rho(x, C)$ is continuous function of x for every C;
K4) $\rho(x, \overline{\bigcup C_\alpha}) = \inf_\alpha \rho(x, C_\alpha)$ for every increasing transfinite family $\{C_\alpha\}$ of canonically closed sets in X.

We say that a function $\rho(x, C)$ is an *quasi κ-metric* on X if it satisfies the axioms $K2) - K4)$ and the following one:

K1*) For any C there is a dense open subset V of $X \setminus C$ such that $\rho(x, C) = 0$ iff $x \in X \setminus V$.

Our second result provides a characterization of compact I-favorable spaces, which is similar to Shchepin's characterization ([9], [10]) of openly generated compacta as compact spaces admitting a κ-metric.

Theorem 1.4. *A compact space X is* I*-favorable iff X is quasi κ-metrizable.*

Corollary 1.5. *Every* I*-favorable space is quasi κ-metrizable.*

The paper is organized as follows: Section 2 contains the proof of Theorem 1.1 and Corollaries 1.2-1.3. The proofs of Theorem 1.4 and Corollary 1.5 are contained in section 3. In section 4 we provide the proof of some results concerning almost continuous inverse systems with nearly open bounding maps, which were announced in [14].

2. Proof of Theorem 1.1

If follows from the definition of I-favorability that a given space is I-favorable if and only if there are a π-base $\mathcal{B}$ and a function $\sigma : \bigcup_{n\geq 0} \mathcal{B}^n \to \mathcal{B}$ such that the union $\bigcup_{n\geq 0} U_n$ is dense in X for any sequence

$$\big(\sigma(\varnothing), U_0, \sigma(U_0), U_1, \sigma(U_0, U_1), U_2, ..., U_n, \sigma(U_0, U_1, .., U_n), U_{n+1}, , , \big),$$

where U_k and $\sigma(\varnothing)$ belong to $\mathcal{B}$, $U_0 \subset \sigma(\varnothing)$ and $U_{k+1} \subset \sigma(U_0, U_1, .., U_k)$ for every $k \geq 0$. Such a function will be also called a winning strategy. Recall that $\mathcal{B}$ is a π-base for X if every open set in X contains an element from $\mathcal{B}$.

Proposition 2.1. [3] *Let $\mathcal{B}$ and $\mathcal{P}$ be two π-bases for X. Then there is a winning strategy $\sigma : \bigcup_{n\geq 0} \mathcal{B}^n \to \mathcal{B}$ if and only if there is a winning strategy $\mu : \bigcup_{n\geq 0} \mathcal{P}^n \to \mathcal{P}$.*

Proof. Suppose $\sigma : \bigcup_{n\geq 0}\mathcal{B}^n \to \mathcal{B}$ is a winning strategy. We define a winning strategy $\mu : \bigcup_{n\geq 0}\mathcal{P}^n \to \mathcal{P}$ by induction. We choose any open non-empty set $\mu(\varnothing) \in \mathcal{P}$ such that $\mu(\varnothing) \subset \sigma(\varnothing)$. If $V_0 \in \mathcal{P}$ is the answer of player II in the game played on $\mathcal{P}$ (i.e., $V_0 \subset \mu(\varnothing)$), then we choose $U_0 \in \mathcal{B}$ with $U_0 \subset V_0$ (U_0 can be considered as the answer of player II in the game played on $\mathcal{B}$). Assume we already defined $V_0, .., V_n \in \mathcal{P}$ and $U_0, .., U_n \in \mathcal{B}$ such that $U_{k+1} \subset V_{k+1} \subset \mu(V_0, .., V_k) \subset \sigma(U_0, .., U_k)$ for all $k \leq n-1$. Then, we choose $\mu(V_0, .., V_n) \in \mathcal{P}$ such that $\mu(V_0, .., V_n) \subset \sigma(U_0, .., U_n)$. If $V_{n+1} \in \mathcal{P}$ is the choice of player II in the game played on $\mathcal{P}$ such that $V_{n+1} \subset \mu(V_0, .., V_n)$, we choose $U_{n+1} \in \mathcal{B}$ with $U_{n+1} \subset V_{n+1}$. This complete the induction. Since σ is a winning strategy and $U_k \subset V_k$ for each k, the union $\bigcup_{n\geq 0} V_n$ is dense in X. So, μ is also a winning strategy. □

In [13] we considered I-favorable spaces X with respect to the co-zero sets meaning that there is a winning strategy $\sigma : \bigcup_{n\geq 0}\Sigma^n \to \Sigma$, where Σ is the family of all co-zero subsets of X. Proposition 2.1 shows that this is equivalent to X being I-favorable. So, all results from [13] are valid for I-favorable spaces.

According to [2, Corollary 1.4], if Y is a dense subset of X, then X is I-favorable if and only Y is I-favorable. So, every compactification of a space X is I-favorable provided X is I-favorable. And conversely, if a compactification of X is I-favorable, then so is X. Because of that, very often when dealing with I-favorable spaces, we can suppose that they are compact.

Let us introduce a few more notations. Suppose $X \subset \mathbb{I}^A$ is a compact space and $B \subset A$, where $\mathbb{I} = [0, 1]$. Let $\pi_B \colon \mathbb{I}^A \to \mathbb{I}^B$ be the natural projection and p_B be a restriction map $\pi_B|X$. Let also $X_B = p_B(X)$. If $U \subset X$ we write $B \in k(U)$ to denote that $p_B^{-1}\big(p_B(U)\big) = U$. A base $\mathcal{A}$ for the topology of $X \subset \mathbb{I}^A$ consisting of open sets is called *special* if for every finite $B \subset A$ the family $\{p_B(U) : U \in \mathcal{A}, B \in k(U)\}$ is a base for $p_B(X)$ and for each $U \in \mathcal{A}$ there is a finite set $B \subset A$ with $B \in k(U)$.

Proposition 2.2. *Let X be a compact* I*-favorable space and $w(X) = \tau$ is uncountable. Then there exists a continuous inverse system $S = \{X_\delta, p^\delta_\gamma, \gamma < \delta < \lambda\}$, where $\lambda = \mathrm{cf}(\tau)$, of compact* I*-favorable spaces X_δ and skeletal bonding maps p^δ_γ such that $w(X_\delta) < \tau$ for each $\delta < \lambda$ and $X = \varprojlim S$.*

Proof. We embed X in a Tychonoff cube $\mathbb{I}^A$ with $|A| = \tau$ and fix a special open base $\mathcal{A} = \{U_\alpha : \alpha \in A\}$ for X of cardinality τ which consists of open sets such that for each α there exists a finite set $H_\alpha \subset A$ with $H_\alpha \in k(U_\alpha)$.

Let $\sigma : \bigcup_{n\geq 0} \mathcal{A}^n \to \mathcal{A}$ be a winning strategy. We represent A as the union of an increasing transfinite family $\{A_\delta : \delta < \lambda\}$ with $|A_\delta| < \tau$, and let $\mathcal{A}_\delta = \{U_\alpha : \alpha \in A_\delta\}$ for each $\delta < \lambda$.

For any finite set $C \subset A$ let γ_C be a fixed countable base for X_C. Observe that for every $U \in \mathcal{A}$ there exists a finite set $B(U) \subset A$ such that $B(U) \in k(U)$ and $p_{B(U)}(U)$ is open in $X_{B(U)}$. We are going to construct by transfinite induction increasing families $\{B_\delta : \delta < \lambda\}$ and $\{\mathcal{B}_\delta : \delta < \lambda\} \subset \mathcal{A}$ satisfying the following conditions for every $\delta < \lambda$:

(1) $A_\delta \subset B_\delta \subset A$, $\mathcal{A}_\delta \subset \mathcal{B}_\delta$, $|B_\delta| = |\mathcal{B}_\delta| < \tau$;
(2) $B_\delta \in k(U)$ for all $U \in \mathcal{B}_\delta$;
(3) $p_C^{-1}(\gamma_C) \subset \mathcal{B}_\delta$ for each finite $C \subset B_\delta$;
(4) $\sigma(U_1, .., U_n) \in \mathcal{B}_\delta$ for every finite family $\{U_1, .., U_n\} \subset \mathcal{B}_\delta$;
(5) $B_\delta = \bigcup_{\gamma<\delta} B_\gamma$ and $\mathcal{B}_\delta = \bigcup_{\gamma<\delta} \mathcal{B}_\gamma$ for all limit cardinals δ.

Suppose all B_γ and $\mathcal{B}_\gamma$, $\gamma < \delta$, have already been constructed for some $\delta < \lambda$. If δ is a limit cardinal, we put $B_\delta = \bigcup_{\gamma<\delta} B_\gamma$ and $\mathcal{B}_\delta = \bigcup_{\gamma<\delta} \mathcal{B}_\gamma$. If $\delta = \gamma + 1$, we construct by induction a sequence $\{C(m)\}_{m\geq 0}$ of subsets of A, and a sequence $\{\mathcal{V}_m\}_{m\geq 0}$ of subfamilies of $\mathcal{A}$ such that:

- $C_0 = B_\gamma$ and $\mathcal{V}_0 = \mathcal{B}_\gamma$;
- $C(m+1) = C(m) \bigcup \{B(U) : U \in \mathcal{V}_m\}$;
- $\mathcal{V}_{2m+1} = \mathcal{V}_{2m} \bigcup \{\sigma(U_1, .., U_s) : U_1, .., U_s \in \mathcal{V}_{2m}, s \geq 1\}$;
- $\mathcal{V}_{2m+2} = \mathcal{V}_{2m+1} \bigcup \{p_C^{-1}(\gamma_C) : C \subset C(2m+1) \text{ is finite}\}$.

Now, we define $B_\delta = \bigcup_{m\geq 0} C(m)$ and $\mathcal{B}_\delta = \bigcup_{m\geq 0} \mathcal{V}_m$. It is easily seen that B_δ and $\mathcal{B}_\delta$ satisfy conditions (1)-(5).

For every $\delta < \lambda$ let $X_\delta = X_{B_\delta}$ and $p_\delta = p_{B_\delta}$. Moreover, if $\gamma < \delta$, we have $B_\gamma \subset B_\delta$, and let $p_\gamma^\delta = p_{B_\gamma}^{B_\delta}$. Since $A = \bigcup_{\delta<\lambda} B_\delta$, we obtain a continuous inverse system $S = \{X_\delta, p_\gamma^\delta, \gamma < \delta < \lambda\}$ whose limit is X. Observe also that each X_δ is of weight $< \tau$ because $p_\delta(\mathcal{B}_\delta)$ is a base for X_δ (see condition (3)).

Claim 1. *All bonding maps p_γ^δ are skeletal.*

It suffices to show that all p_δ are skeletal. And this is really true because each family $\mathcal{B}_\delta$ is stable with respect to σ, see (4). Hence, by [6, Lemma 9], for every open set $V \subset X$ there exists $W \in \mathcal{B}_\delta$ such that whenever $U \subset W$ and $U \in \mathcal{B}_\delta$ we have $V \cap U \neq \varnothing$. The last statement yields that p_δ is skeletal. Indeed, let $V \subset X$ be open, and $W \in \mathcal{B}_\delta$ be as above. Then $p_\delta(W)$ is open in X_δ because of condition (2). We claim that $p_\delta(W) \subset \overline{p_\delta(V)}$. Indeed, otherwise $p_\delta(W) \backslash \overline{p_\delta(V)}$ would be a non-empty open subset of X_δ. So, $p_\delta(U) \subset p_\delta(W) \backslash \overline{p_\delta(V)}$ for some $U \in \mathcal{B}_\delta$ (recall that $p_\delta(\mathcal{B}_\delta)$ is a base for X_δ). Since, by (2), $p_\delta^{-1}(p_\delta(U)) = U$ and $p_\delta^{-1}(p_\delta(W)) = W$, we obtain $U \subset W$ and $U \cap V = \varnothing$ which is a contradiction.

Finally, since the class of I-favorable spaces is closed with respect to skeletal images [5, Lemma 1], all X_δ are I-favorable. □

An inverse system $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$, where τ is a given cardinal, is said to be *almost continuous* provided for every limit cardinal γ the space X_γ is the almost limit of the inverse system $S_\gamma = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \gamma\}$. If $X = \mathrm{a} - \varprojlim S$ of an almost continuous inverse system S and $H \subset X$, the set

$$q(H) = \{\alpha : \mathrm{Int}\big(\big((p^{\alpha+1}_\alpha)^{-1}(\overline{p_\alpha(H)})\big)\backslash\overline{p_{\alpha+1}(H)}\big) \neq \varnothing\}$$

is called a *rank of H*.

Lemma 2.3. [13, Lemma 3.1] *Let $X = \mathrm{a} - \varprojlim S$ and $U \subset X$ be open, where $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ is almost continuous inverse system with skeletal bonding maps. Then we have:*

(1) *$\alpha \notin q(U)$ if and only if $(p^{\alpha+1}_\alpha)^{-1}\big(\mathrm{Int}\overline{p_\alpha(U)}\big) \subset \overline{p_{\alpha+1}(U)}$;*
(2) *$q(U) \cap [\alpha, \tau) = \varnothing$ provided $U = p^{-1}_\alpha(V)$ for some open $V \subset X_\alpha$.*

Lemma 2.4. *Let $S = \{X_\alpha, p^\beta_\alpha, 1 \leq \alpha < \beta < \tau\}$ be an almost continuous inverse system with skeletal bonding maps and $X = \mathrm{a} - \varprojlim S$. The the following hold for any open $U \subset X$:*

(1) *If $(p^\alpha_1)^{-1}\big(\mathrm{Int}\overline{p_1(U)}\big) \subset \mathrm{Int}\overline{p_\alpha(U)}$ for all $\alpha < \tau$, then $p^{-1}_1\big(\mathrm{Int}\overline{p_1(U)}\big) \subset \overline{U}$;*
(2) *If $\lambda < \tau$ and $q(U) \cap [\lambda, \tau) = \varnothing$, then $p^{-1}_\lambda\big(\mathrm{Int}\overline{p_\lambda(U)}\big) \subset \mathrm{Int}\overline{U}$.*

Proof. The first item was proved in [13, Lemma 3.2] under the assumption that $X = \varprojlim S$, but the same arguments work in our situation. Item (2) is equivalent to the inclusion $(p_\lambda)^{-1}\big(\mathrm{Int}\overline{p_\lambda(U)}\big) \subset \overline{U}$. Let A be the set of all $\alpha \in (\lambda, \tau)$ with $(p^\alpha_\lambda)^{-1}\big(\mathrm{Int}\overline{p_\lambda(U)}\big) \setminus \overline{p_\alpha(U)} \neq \varnothing$. Suppose A is non-empty and let $\gamma = \min A$. Observe that γ is a limit cardinal. Indeed, otherwise $\gamma = \beta + 1$ with $\beta \geq \lambda$, so $(p^\beta_\lambda)^{-1}\big(\mathrm{Int}\overline{p_\lambda(U)}\big) \subset \mathrm{Int}\overline{p_\beta(U)}$. Since $\beta \notin q(U)$, according to Lemma 2.3(1), we have $(p^\gamma_\beta)^{-1}\big(\mathrm{Int}\overline{p_\beta(U)}\big) \subset \overline{p_\gamma(U)}$. Hence, $(p^\gamma_\lambda)^{-1}\big(\mathrm{Int}\overline{p_\lambda(U)}\big) \subset \overline{p_\gamma(U)}$, a contradiction.

Since S is almost continuous and γ is a limit cardinal, we have $X_\gamma = \mathrm{a} - \varprojlim S_\gamma$, where S_γ is the inverse system $\{X_\alpha, p^\beta_\alpha, \lambda \leq \alpha < \beta < \gamma\}$. Because p_γ is skeletal, $U_\gamma = \mathrm{Int}\overline{p_\gamma(U)} \neq \varnothing$. So, we can apply item (1) to X_γ, the inverse system S_γ and the open set $U_\gamma \subset X_\gamma$, to conclude that $(p^\gamma_\lambda)^{-1}\big(\mathrm{Int}\overline{p_\lambda(U)}\big) \subset \overline{p_\gamma(U)}$. So, we obtain again a contradiction, which shows that $(p^\alpha_\lambda)^{-1}\big(\mathrm{Int}\overline{p_\lambda(U)}\big) \subset \overline{p_\alpha(U)}$ for all $\alpha \in [\lambda, \tau)$. Finally, because the system $\widetilde{S}_\lambda = \{X_\alpha, p^\beta_\alpha, \lambda \leq \alpha < \beta < \tau\}$ is almost continuous and $X = \mathrm{a} - \varprojlim \widetilde{S}_\lambda$, by item (1) we have $p^{-1}_\lambda\big(\mathrm{Int}\overline{p_\lambda(U)}\big) \subset \mathrm{Int}\overline{U}$. □

The next lemma was established in [13] for continuous inverse systems. We present here a simplified proof concerning almost continuous systems.

Lemma 2.5. [13, Lemma 3.3] *Let $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ be an almost continuous inverse system with skeletal bonding maps and $X = \mathrm{a}-\varprojlim S$. Assume $U, V \subset X$ are open with $q(U)$ and $q(V)$ finite and $\overline{U} \cap \overline{V} = \varnothing$. If $q(U) \cap q(V) \cap [\gamma, \tau) = \varnothing$ for some $\gamma < \tau$, then $\mathrm{Int}\overline{p_\gamma(U)}$ and $\mathrm{Int}\overline{p_\gamma(V)}$ are disjoint.*

Proof. Suppose $\mathrm{Int}\overline{p_\gamma(U)} \cap \mathrm{Int}\overline{p_\gamma(V)} \neq \varnothing$. We are going to show by transfinite induction that $\mathrm{Int}\overline{p_\beta(U)} \cap \mathrm{Int}\overline{p_\beta(V)} \neq \varnothing$ for all $\beta \geq \gamma$. Assume this is done for all $\beta \in (\gamma, \alpha)$ with $\alpha < \tau$. If α is not a limit cardinal, then $\alpha - 1$ belongs to at most one of the sets $q(U)$ and $q(V)$. Suppose $\alpha - 1 \notin q(V)$. Hence, $(p^\alpha_{\alpha-1})^{-1}\big(\mathrm{Int}\overline{p_{\alpha-1}(V)}\big) \subset \mathrm{Int}\overline{p_\alpha(V)}$ (see Lemma 2.3(1)). Due to our assumption, $\mathrm{Int}\overline{p_{\alpha-1}(U)} \cap \mathrm{Int}\overline{p_{\alpha-1}(V)} \neq \varnothing$. Moreover, $p^\alpha_{\alpha-1}\big(\overline{p_\alpha(U)}\big)$ is dense in $\overline{p_{\alpha-1}(U)}$. Hence, $\mathrm{Int}\overline{p_{\alpha-1}(V)}$ meets $p^\alpha_{\alpha-1}\big(\overline{p_\alpha(U)}\big)$. This yields $\mathrm{Int}\overline{p_\alpha(V)} \cap \overline{p_\alpha(U)} \neq \varnothing$. Finally, since $\overline{p_\alpha(U)}$ is the closure of its interior, $\mathrm{Int}\overline{p_\alpha(V)} \cap \mathrm{Int}\overline{p_\alpha(U)} \neq \varnothing$.

Suppose $\alpha > \gamma$ is a limit cardinal. Since $q(U) \cup q(V)$ is a finite set, there exists $\lambda \in (\gamma, \alpha)$ such that $\beta \notin q(U) \cup q(V)$ for all $\beta \in [\lambda, \alpha)$. Now, we consider the almost continuous inverse system $S_\alpha = \{X_\delta, p^\beta_\delta, \lambda \leq \delta < \beta < \alpha\}$ with $X_\alpha = \mathrm{a}-\varprojlim S_\alpha$. Let $U_\alpha = \mathrm{Int}\overline{p_\alpha(U)}$ and $V_\alpha = \mathrm{Int}\overline{p_\alpha(V)}$ and denote by $q_\alpha(U_\alpha)$ and $q_\alpha(V_\alpha)$ the ranks of U_α and V_α with respect to the system S_α. Then, according to Lemma 2.3(1), $\beta \in [\lambda, \alpha)$ does not belong to $q_\alpha(U_\alpha)$ if and only if $(p^{\beta+1}_\beta)^{-1}\big(\mathrm{Int}\overline{p^\alpha_\beta(U_\alpha)} \subset \overline{p^\alpha_{\beta+1}(U_\alpha)}$. Since $\overline{p^\alpha_\beta(U_\alpha)} = \overline{p_\beta(U)}$ and $\overline{p^\alpha_{\beta+1}(U_\alpha)} = \overline{p_{\beta+1}(U)}$, we obtain that $\beta \notin q_\alpha(U_\alpha)$ is equivalent to $\beta \notin q(U)$. Similarly, $\beta \notin q_\alpha(V_\alpha)$ iff $\beta \notin q(V)$. Consequently, $\beta \notin q_\alpha(U_\alpha) \cup q_\alpha(V_\alpha)$ for all $\beta \in [\lambda, \alpha)$. Then, according to Lemma 2.4(2), $(p^\alpha_\lambda)^{-1}\big(\mathrm{Int}\overline{p\lambda(U)} \subset \mathrm{Int}\overline{p_\alpha(U)}$ and $(p^\alpha_\lambda)^{-1}\big(\mathrm{Int}\overline{p\lambda(V)} \subset \mathrm{Int}\overline{p_\alpha(V)}$. Because $\mathrm{Int}\overline{p_\lambda(U)} \cap \mathrm{Int}\overline{p_\lambda(V)} \neq \varnothing$, we finally have $\mathrm{Int}\overline{p_\alpha(U)} \cap \mathrm{Int}\overline{p_\alpha(V)} \neq \varnothing$. This completes the transfinite induction.

Therefore, $\mathrm{Int}\overline{p_\beta(U)} \cap \mathrm{Int}\overline{p_\beta(V)} \neq \varnothing$ for all $\beta \in [\gamma, \tau)$. To finish the proof of this lemma, take $\lambda(0) \in (\gamma, \tau)$ such that $\big(q(U) \cup q(V)\big) \cap [\lambda(0), \tau) = \varnothing$. Then, according to Lemma 2.4(2) we have the following inclusions:

- $p^{-1}_{\lambda(0)}\big(\mathrm{Int}\overline{p_{\lambda(0)}(U)}\big) \subset \mathrm{Int}\overline{U}$;
- $p^{-1}_{\lambda(0)}\big(\mathrm{Int}\overline{p_{\lambda(0)}(V)}\big) \subset \mathrm{Int}\overline{V}$.

Since $\mathrm{Int}\overline{p_{\lambda(0)}(U)} \cap \mathrm{Int}\overline{p_{\lambda(0)}(V)} \neq \varnothing$, the above inclusions imply $\overline{U} \cap \overline{V} \neq \varnothing$, a contradiction. Hence, $\mathrm{Int}\overline{p_\gamma(U)} \cap \mathrm{Int}\overline{p_\gamma(V)} = \varnothing$. □

The next proposition was announced in [14, Proposition 3.2] and a proof was presented in [13, Proposition 3.4] (see Proposition 3.2 below for a similar statement concerning inverse systems with nearly open projections).

Proposition 2.6. [14] *Let $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ be an almost continuous inverse system with skeletal bonding maps such that $X = \mathrm{a} - \varprojlim S$. Then the family of all open subsets of X having a finite rank is a π-base for X.*

Proposition 2.7. *Let X be a compact* I*-favorable space. Then every embedding of X in another space is π-regular.*

Proof. We are going to prove this proposition by transfinite induction with respect to the weight $w(X)$. This is true if X is metrizable, see for example [8, §21, XI, Theorem 2]. Assume the proposition is true for any compact I-favorable space Y of weight $< \tau$, where τ is an uncountable cardinal. Suppose X is compact I-favorable with $w(X) = \tau$. Then, by Proposition 2.2, X is the limit space of a continuous inverse system $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \lambda\}$, where $\lambda = \mathrm{cf}(\tau)$, such that all X_α are compact I-favorable spaces of weight $< \tau$ and all bonding maps are surjective and skeletal. If suffices to show that there exists a π-regular embedding of X in a Tychonoff cube $\mathbb{I}^A$ for some set A.

By Proposition 2.6, X has a π-base $\mathcal{B}$ consisting of open sets $U \subset X$ with finite rank. For every $U \in \mathcal{B}$ let $\Omega(U) = \{\alpha_0, \alpha, \alpha+1 : \alpha \in q(U)\}$, where $\alpha_0 < \lambda$ is fixed. Obviously, X is a subset of $\prod\{X_\alpha : \alpha < \lambda\}$. For every $U \in \mathcal{B}$ we consider the open set $\Gamma(U) \subset \prod\{X_\alpha : \alpha < \lambda\}$ defined by $\Gamma(U) = \prod\{\mathrm{Int}\overline{p_\alpha(U)} : \alpha \in \Omega(U)\} \times \prod\{X_\alpha : \alpha \not\in \Omega(U)\}$.

Claim 2. *$\Gamma(U_1) \cap \Gamma(U_2) = \varnothing$ whenever $\overline{U_1} \cap \overline{U_2} = \varnothing$. Moreover, there exists $\beta \in \Omega(U_1) \cap \Omega(U_2)$ with $\overline{p_\beta(U_1)} \cap \overline{p_\beta(U_2)} = \varnothing$.*

Let $\beta = \max\{\Omega(U_1) \cap \Omega(U_2)\}$. Then β is either α_0 or $\max\{q(U_1) \cap q(U_2)\} + 1$. In both cases $q(U_1) \cap q(U_2) \cap [\beta, \lambda) = \varnothing$. According to Lemma 2.5, $\mathrm{Int}\overline{p_\beta(U_1)} \cap \mathrm{Int}\overline{p_\beta(U_2)} = \varnothing$. Since $\beta \in \Omega(U_1) \cap \Omega(U_2)$, $\Gamma(U_1) \cap \Gamma(U_2) = \varnothing$.

For every $U \in \mathcal{B}$ and α let $U_\alpha = \mathrm{Int}\overline{p_\alpha(U)}$.

Claim 3. *$\bigcap_{\alpha \in \Delta} p^{-1}_\alpha(V_\alpha) \cap U \neq \varnothing$ for every finite set $\Delta \subset \{\alpha : \alpha < \lambda\}$, where each V_α is an open and dense subset of U_α.*

Obviously, this is true if $|\Delta| = 1$. Suppose it is true for all Δ with $|\Delta| \leq n$ for some n, and let $\{\alpha_1, .., \alpha_n, \alpha_{n+1}\}$ be a finite set of $n+1$ cardinals $< \tau$. Then $V = \bigcap_{i \leq n} p^{-1}_{\alpha_i}(V_{\alpha_i}) \cap U \neq \varnothing$. Since $p_{\alpha_{n+1}}$ is a closed and skeletal map, $W = \mathrm{Int}\overline{p_{\alpha_{n+1}}(V)}$ is a non-empty subset of $X_{\alpha_{n+1}}$ and $W \subset U_{\alpha_{n+1}}$. Consequently $V_{\alpha_{n+1}} \cap W \neq \varnothing$. So, $V_{\alpha_{n+1}} \cap p_{\alpha_{n+1}}(V) \neq \varnothing$ and $\bigcap_{i \leq n+1} p_{\alpha_i}^{-1}(V_{\alpha_i}) \cap U \neq \varnothing$.

Claim 4. *$\Gamma(U) \cap X$ is a non-empty subset of $\overline{U}$ for all $U \in \mathcal{B}$.*

We are going to show first that $\Gamma(U) \cap X \neq \varnothing$ for all $U \in \mathcal{B}$. Indeed, we fix such U and let $\Omega(U) = \{\alpha_i : i \leq k\}$ with $\alpha_i \leq \alpha_j$ for $i \leq j$. By Claim 3, there exists $x \in \bigcap_{i \leq k} p_{\alpha_i}^{-1}(U_{\alpha_i}) \cap U$. So, $p_{\alpha_i}(x) \in U_{\alpha_i}$ for all $i \leq k$. This implies $\Gamma(U) \cap X \neq \varnothing$. To show that $\Gamma(U) \cap X \subset \overline{U}$, let $y \in \Gamma(U) \cap X$ and $\beta(U) = \max q(U) + 1$. Then $p_{\beta(U)}(y) \in \mathrm{Int}\overline{p_{\beta(U)}(U)}$. Since $\alpha \notin q(U)$ for all $\alpha \geq \beta(U)$, according to Lemma 2.4(2), we have $y \in p_{\beta(U)}^{-1}\big(\mathrm{Int}\overline{p_{\beta(U)}(U)}\big) \subset \overline{U}$. This completes the proof of Claim 4.

According to our assumption, each X_α is π-regularly embedded in $\mathbb{I}^{A(\alpha)}$ for some $A(\alpha)$. So, there exists a π-regular operator $\mathrm{e}_\alpha : \mathcal{T}_{X_\alpha} \to \mathcal{T}_{\mathbb{I}^{A(\alpha)}}$. For every $U \in \mathcal{B}$ define the open set $\theta_1(U) \subset \prod_{\alpha<\lambda} \mathbb{I}^{A(\alpha)}$,

$$\theta_1(U) = \prod_{\alpha \in \Omega(U)} \mathrm{e}_\alpha\big(\mathrm{Int}\overline{p_\alpha(U)}\big) \times \prod_{\alpha \notin \Omega(U)} \mathbb{I}^{A(\alpha)}.$$

Now, we define a function θ from $\mathcal{T}_X$ to the topology of $\prod_{\alpha<\lambda} \mathbb{I}^{A(\alpha)}$ by

$$\theta(G) = \bigcup\{\theta_1(U) : U \in \mathcal{B} \text{ and } \overline{U} \subset G\}.$$

Let show that θ is π-regular. It follows from Claim 2 that $\theta(G_1) \cap \theta(G_2) = \varnothing$ provided $G_1 \cap G_2 = \varnothing$. On the other hand, for every open $G \subset X$ we have $\theta(G) \cap X \subset \bigcup\{\Gamma(U) \cap X : U \in \mathcal{B} \text{ and } \overline{U} \subset G\}$. Hence, by Claim 4, $\theta(G) \cap X \subset \bigcup\{\overline{U} : U \in \mathcal{B} \text{ and } \overline{U} \subset G\} \subset G$. To prove that $\theta(G) \cap X$ a dense subset of G it suffices to show that $\theta_1(U) \cap X \neq \varnothing$ for all $U \in \mathcal{B}$ with $\overline{U} \subset G$. To this end, we fix such U and let $V_\alpha = \mathrm{e}_\alpha(U_\alpha) \cap X_\alpha$ for every $\alpha \in \Omega(U)$. Then V_α is a dense open subset of U_α, and by Claim 3, $V = \bigcap_{\alpha \in \Omega(U)} p_\alpha^{-1}(V_\alpha) \cap U$ is a non-empty subset of $\theta_1(U) \cap X$. Therefore, X is π-regularly embedded in $\mathbb{I}^A = \prod_{\alpha<\lambda} \mathbb{I}^{A(\alpha)}$. □

The next proposition was established in [13] (Proposition 3.7) assuming that X is a π-regularly C^*-embedded subset of the limit space of a σ-complete inverse system with open bounding maps and second countable spaces. The arguments there work if X is just a π-regularly embedded subset of a product of second countable spaces.

Proposition 2.8. *Let X be a π-regularly embedded subspace of a product of second countable spaces. Then X is skeletally generated.*

Proof of Theorem 1.1. To prove implication (1) $\Rightarrow$ (2), suppose X is I-favorable subspace of a space Y. Then $\widetilde{X} = \overline{X}^{\beta Y}$ is a compactification of X. Since $\widetilde{X}$ is also I-favorable, according to Proposition 2.7, $\widetilde{X}$ is π-regularly embedded in βY. This yields that X is π-regularly embedded in Y.

(2) $\Rightarrow$ (3) Let X be a subset of a Tychonoff cube $\mathbb{I}^A$. Then X is π-regularly embedded in $\mathbb{I}^A$, and by Proposition 2.8, X is skeletally generated.

The implication (3) $\Rightarrow$ (1) follows as follows. If X is skeletally generated, then $X = \mathrm{a} - \varprojlim S$, where S is an almost σ-continuous inverse system of second countable spaces X_α, $\alpha \in A$, and skeletal bounding maps p^α_β. Because each X_α is I-favorable, it follows from [4, Theorem 3.3] (see also [6, Theorem 13]) that X is I-favorable too. □

Proof of Corollary 1.2. Suppose X is an I-favorable subspace of an extremally disconnected space Y. Then there exists a π-regular operator $\mathrm{e}\colon \mathcal{T}_X \to \mathcal{T}_Y$. We need to show that the closure (in X) of every open subset of X is also open. Since Y is extremally disconnected, $\overline{\mathrm{e}(U)}^Y$ is open in Y. So, the proof will be done (finished) if we prove that $\overline{\mathrm{e}(U)}^Y \cap X = \overline{U}^X$ for all $U \in \mathcal{T}_X$. Because $\mathrm{e}(U) \cap X$ is a dense subset of U, we have $\overline{U}^X \subset \overline{\mathrm{e}(U)}^Y \cap X$. Assume $\overline{\mathrm{e}(U)}^Y \cap X \backslash \overline{U}^X \neq \varnothing$ and choose $V \in \mathcal{T}_X$ with $V \subset \overline{\mathrm{e}(U)}^Y \backslash \overline{U}^X$. Then $\mathrm{e}(V) \cap \overline{\mathrm{e}(U)}^Y \neq \varnothing$, so $\mathrm{e}(V) \cap \mathrm{e}(U) \neq \varnothing$. The last one contradicts $U \cap V = \varnothing$. □

Proof of Corollary 1.3. Suppose X is I-favorable and $W \subset X$ is open. Then there is a π-regular embedding of X into a product Π of lines. Obviously, W is also π-regularly embedded in Π, and by Proposition 2.8, W is I-favorable. □

3. Quasi κ-metrizable spaces

Proof of Theorem 1.4. Suppose X is a compact I-favorable. We embed X in $\mathbb{R}^\tau$ for some cardinal τ, and let $\rho(z, C)$ be a κ-metric on $\mathbb{R}^\tau$, see [9]. According to Theorem 1.1, there exists a π-regular function $\mathrm{e} : \mathcal{T}_X \to \mathcal{T}_{\mathbb{R}^\tau}$. We define a new function $\mathrm{e}_1 : \mathcal{T}_X \to \mathcal{T}_{\mathbb{R}^\tau}$,

$$\mathrm{e}_1(U) = \bigcup\{\mathrm{e}(V) : V \in \mathcal{T}_X \text{ and } \overline{V} \subset U\}.$$

Obviously e_1 is π-regular and it is also monotone, i.e. $U \subset V$ implies $\mathrm{e}_1(U) \subset \mathrm{e}_1(V)$. Moreover, for every increasing transfinite family $\gamma = \{U_\alpha\}$ of open sets in Y we have $\mathrm{e}_1(\bigcup_\alpha U_\alpha) = \bigcup_\alpha \mathrm{e}_1(U_\alpha)$. Indeed, if $z \in \mathrm{e}_1(\bigcup_\alpha U_\alpha)$, then there is an open set $V \in \mathcal{T}_X$ with $\overline{V} \subset \bigcup_\alpha U_\alpha$ and $z \in \mathrm{e}(V)$. Since $\overline{V}$ is compact and the family is increasing, $\overline{V}$ is contained in some U_{α_0}. Hence, $z \in \mathrm{e}(V) \subset \mathrm{e}_1(U_{\alpha_0})$. Consequently, $\mathrm{e}_1(\bigcup_\alpha U_\alpha) \subset \bigcup_\alpha \mathrm{e}_1(U_\alpha)$. The other inclusion follows from monotonicity of e_1.

Now, for every open $U \subset X$ and $x \in X$ we can define the function $d(x,\overline{U}) = \rho(x,\overline{\mathrm{e}_1(U)})$, where $\overline{\mathrm{e}_1(U)}$ is the closure of $\mathrm{e}_1(U)$ in $\mathbb{R}^\tau$. It is easily seen that $d(x,\overline{U})$ satisfies axioms $K2) - K3)$. Let show that it also satisfies $K4)$ and $K1^*)$. Indeed, assume $\{C_\alpha\}$ is an increasing transfinite family of regularly closet sets in X. We put $U_\alpha = \mathrm{Int}C_\alpha$ for every α and $U = \bigcup_\alpha U_\alpha$. Thus, $\mathrm{e}_1(U) = \bigcup_\alpha \mathrm{e}_1(U_\alpha)$. Since $\{\overline{\mathrm{e}_1(U_\alpha)}\}$ is an increasing transfinite family of regularly closed sets in $\mathbb{R}^\tau$,

$$d(x,\overline{\bigcup_\alpha C_\alpha}) = \rho(x,\overline{\bigcup_\alpha \mathrm{e}_1(U_\alpha)}) = \inf_\alpha \rho(x,\overline{\mathrm{e}_1(U_\alpha)}) = \inf_\alpha d(x,C_\alpha).$$

To show that $K1^*)$ also holds, observe that $d(x,\overline{U}) = 0$ if and only if $x \in X \cap \overline{\mathrm{e}_1(U)}$. Thus, we need to show that there is an open dense subset V of $X \setminus \overline{U}$ such that $X \cap \overline{\mathrm{e}_1(U)} = X \setminus V$. Because $\mathrm{e}_1(U) \cap X$ is dense in U, $\overline{U} \subset \overline{\mathrm{e}_1(U)}$. Hence, $V = X \setminus \overline{\mathrm{e}_1(U)}$ is contained in $X \setminus \overline{U}$. To prove V is dense in $X \setminus \overline{U}$, let $x \in X \setminus \overline{U}$ and $W_x \subset X \setminus \overline{U}$ be an open neighborhood of x. Then $W \cap U$ is empty, so $\mathrm{e}_1(W) \cap \mathrm{e}_1(U) = \varnothing$. This yields $\mathrm{e}_1(W) \cap X \subset V$. On the other hand, $\mathrm{e}_1(W) \cap X$ is a non-empty subset of W, hence $W \cap V \neq \varnothing$. Therefore, d is a quasi κ-metric on X.

Suppose X is a compact space and let $d(x,\overline{U})$ be a quasi κ-metric on X. We are going to show that X is skeletally generated. To this end we embed X in $\mathbb{I}^A$ for some A. Following the notations from the proof of Proposition 2.2, for any countable set $B \subset A$ let $\mathcal{A}_B$ be the countable base for $X_B = p_B(X)$ consisting of all open sets in X_B of the form $X_B \cap \prod_{\alpha\in B} V_\alpha$, where each $V\alpha$ is an open subinterval of $\mathbb{I} = [0,1]$ with rational end-points and $V_\alpha \neq \mathbb{I}$ for finitely many α. For any open $U \subset X$ denote by f_U the function $d(x,\overline{U})$. We also write $p_B \prec g$, where g is a map defined on X, if there is a map $h : p_B(X) \to g(X)$ such that $g = h \circ p_B$. Since X is compact this is equivalent to the following: if $p_B(x_1) = p_B(x_2)$ for some $x_1, x_2 \in X$, then $g(x_1) = g(x_2)$. We say that a countable set $B \subset A$ is *d-admissible* if $p_B \prec f_{p_B^{-1}(V)}$ for every $V \in \mathcal{A}_B$. Denote by $\mathcal{D}$ the family of all d-admissible subsets of A. We are going to show that all maps $p_B : X \to X_B$, $B \in \mathcal{D}$, are skeletal and the inverse system $S = \{X_B : p^B_C : C \subset B, C, B \in \mathcal{D}\}$ is σ-continuous with $X = \varprojlim S$.

Claim 5. *For every countable set $C \subset A$ there is $B \in \mathcal{D}$ with $C \subset B$.*

We are going to construct a sequence of countable sets $B_n \subset A$ such that for every $n \geq 1$ we have:

- $C \subset B_n \subset B_{n+1}$;
- $p_{B_{n+1}} \prec f_{p^{-1}_{B_n}}(V)$ for all $V \in \mathcal{A}_{B_n}$.

We show the construction of B_1; the other sets B_n can be obtained in a similar way. Every function $f_{p_C{}^{-1}(V)}$, $V \in \mathcal{A}_C$, has a continuous extension $\widetilde{f}_{p_C{}^{-1}(V)}$ on $\mathbb{I}^A$. Moreover, every continuous function g on $\mathbb{I}^A$ depends on countably many coordinates (i.e., there exists a countable set $B_g \subset A$ with $\pi_{B_g} \prec g$). This fact allows us to find a countable set $B_1 \subset A$ containing C such that $p_{B_1} \prec f_{p_C{}^{-1}(V)}$ for all $V \in \mathcal{A}_C$. Next, let $B = \bigcup_{n=1} B_n$. Since $\mathcal{A}_B$ is the union of all families $\{(p^B_{B_n})^{-1}(V) : V \in \mathcal{A}_{B_n}\}$, $n \geq 1$, for every $W \in \mathcal{A}_B$ there is m and $V \in \mathcal{A}_{B_m}$ with $p_B^{-1}(W) = p_{B_m}^{-1}(V)$. Then, according to the construction of the sets B_n, we have $p_{B_{m+1}} \prec f_{p_B^{-1}(W)}$. Hence $p_B \prec f_{p_B^{-1}(W)}$ for all $W \in \mathcal{A}_B$, which means that B is d-admissible.

Claim 6. *For every $B \in \mathcal{D}$ the map p_B is skeletal.*

Suppose there is an open set $U \subset X$ such that the interior in X_B of the closure $\overline{p_B(U)}$ is empty. Then $W = X_B \setminus \overline{p_B(U)}$ is dense in X_B. Let $\{W_m\}_{m\geq 1}$ be a countable cover of W with $W_m \in \mathcal{A}_B$ for all m. Since $\mathcal{A}_B$ is finitely additive, we may assume that $W_m \subset W_{m+1}$, $m \geq 1$. Because B is d-admissible, $p_B \prec f_{p_B^{-1}(W_m)}$ for all m. Hence, there are continuous functions $h_m : X_B \to \mathbb{R}$ with $f_{p_B^{-1}(W_m)} = h_m \circ p_B$, $m \geq 1$. Recall that $f_{p_B^{-1}(W_m)}(x) = d(x, \overline{p_B^{-1}(W_m)})$ and $p_B^{-1}(W) = \bigcup_{m\geq 1} p_B^{-1}(W_m)$. Therefore, $f_{p_B^{-1}(W)}(x) = d(x, \overline{p_B^{-1}(W)}) = \inf_m f_{p_B^{-1}(W_m)}(x)$ for all $x \in X$. Moreover, $f_{p_B^{-1}(W_{m+1})}(x) \leq f_{p_B^{-1}(W_m)}(x)$ because $W_m \subset W_{m+1}$. The last inequalities together with $p_B \prec f_{p_B^{-1}(W_m)}$ yields that $p_B \prec f_{p_B^{-1}(W)}$. So, there exists a continuous function h on X_B with $d(x, \overline{p_B^{-1}(W)}) = h(p_B(x))$ for all $x \in X$. Since $p_B(\overline{p_B^{-1}(W)}) = \overline{W} = X_B$, we have that h is the constant function zero. Then $d(x, \overline{p_B^{-1}(W)}) = 0$ for all $x \in X$. But $\overline{p_B^{-1}(W)} \cap U = \varnothing$. So, according to $K1^*$), there is a dense open subset U' of U with $d(x, \overline{p_B^{-1}(W)}) > 0$ for each $x \in U'$, a contradiction.

It is easily seen that the union of any increasing sequence of d-admissible sets is also d-admissible. This fact and Claim 5 yield that the inverse system $S = \{X_B : p^B_C : C \subset B, C, B \in \mathcal{D}\}$ is σ-continuous and $X = \varprojlim S$. Finally, by Claim 6, all maps p_B, $B \in \mathcal{D}$, are skeletal. So are the bounding maps p^B_C in S. Therefore, X is skeletally generated, and hence I-favorable by Theorem 1.1.

Proof of Corollary 1.5. Since $Y = \beta X$ is I-favorable, by Theorem 1.4 there is a quasi κ-metric d on Y. We are going to show that $d_X(x, \overline{U}^X) = d(x, \overline{U})$, $U \in \mathcal{T}_X$, defines a quasi κ-metric on X, where $\overline{U}^X$ and $\overline{U}$ is the closure of U in X and Y respectively. Since $\overline{U}$ is regularly closed in Y, this definition is correct. It follows directly from the definition that

d_X satisfies axioms $K2$) and $K3$). Because for any increasing transfinite family $\{C_\alpha\}$ of regularly closed sets in X the family $\{\overline{C_\alpha}\}$ is also increasing and consists of regularly closed sets in Y,

$$d_X(x, \overline{\bigcup_\alpha C_\alpha}^X) = d(x, \overline{\bigcup_\alpha C_\alpha}) = \inf_\alpha d(x, \overline{C_\alpha}) = \inf_\alpha d_X(x, C_\alpha),$$

d_X satisfies $K4$). Finally, d_X satisfies also $K1^*$). Indeed, for any $U \in \mathcal{T}_X$ there exists $V \in \mathcal{T}_Y$ such that V is dense in $Y \setminus \overline{U}$ and $d(x, \overline{U}) > 0$ if and only if $x \in V$. This implies that the set $V \cap X$ is dense in $X \setminus \overline{U}^X$ and $d_X(x, \overline{U}^X) > 0$ iff $x \in V \cap X$. So, d_X is a quasi κ-metric on X.

4. Inverse systems with nearly open bounding maps

In this section we consider almost continuous inverse systems with nearly open bounding maps. Recall that a map $f : X \to Y$ is nearly open [1] if $f(U) \subset \mathrm{Int}\overline{f(U)}$ for every open $U \subset X$. Nearly open maps were considered by Tkachenko [12] under the name d-open maps. The following properties of ranks were established in Lemmas 2.3-2.5 when considering almost continuous inverse systems with skeletal bounding maps. The same proofs remain valid and for inverse systems with nearly open bounding maps.

Lemma 4.1. *Let $X = \mathrm{a} - \varprojlim S$, where $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ is almost continuous with nearly open bonding maps. Then for every open sets $U, V \subset X$ we have:*

1. *$\alpha \notin q(U)$ if and only if $(p^{\alpha+1}_\alpha)^{-1}\big(\mathrm{Int}\overline{p_\alpha(U)}\big) \subset \overline{p_{\alpha+1}(U)}$;*
2. *$q(U) \cap [\alpha, \tau) = \varnothing$ provided $U = p^{-1}_\alpha(W)$ for some open $W \subset X_\alpha$;*
3. *Suppose $q(U)$ and $q(V)$ are finite and $\overline{U} \cap \overline{V} = \varnothing$. If $q(U) \cap q(V) \cap [\gamma, \tau) = \varnothing$ for some $\gamma < \tau$, then $\mathrm{Int}\overline{p_\gamma(U)}$ and $\mathrm{Int}\overline{p_\gamma(V)}$ are disjoint.*

The next proposition was announced in [14, Proposition 2.2] without a proof. Note that a similar statement was established in [9] for inverse systems with open bounding maps.

Proposition 4.2. [14] *Let $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ be an almost continuous inverse system with nearly open bonding maps such that $X = \mathrm{a} - \varprojlim S$. Then the family of all open subsets of X having a finite rank is a base for X.*

Proof. We are going to show by transfinite induction that for every $\alpha < \tau$ the open subsets $U \subset X$ with $q(U) \cap [1, \alpha]$ being finite form a base for X. Obviously, this is true for finite α, and it holds for $\alpha + 1$ provided it is true for α. So, it remains to prove this statement for a limit cardinal α if it is true for any $\beta < \alpha$. Suppose $G \subset X$ is open and $x \in G$.

Since p_α is nearly open, $G_\alpha = \mathrm{Int}\overline{p_\alpha(G)}$ contains $p_\alpha(G)$ (here both interior and closure are taken in X_α). Let $S_\alpha = \{X_\gamma, p^\beta_\gamma, \gamma < \beta < \alpha\}$, $Y_\alpha = \varprojlim S_\alpha$ and $\widetilde{p}^\alpha_\gamma\colon Y_\alpha \to X_\gamma$ are the limit projections of S_α. Obviously, X_α is naturally embedded as a dense subset of Y_α and each $\widetilde{p}^\alpha_\gamma$ restricted on X_α is p^α_γ. So, there exists $\gamma < \alpha$ and an open set $U_\gamma \subset X_\gamma$ containing $x_\gamma = p_\gamma(x)$ such that $(\widetilde{p}^\alpha_\gamma)^{-1}(U_\gamma) \subset \mathrm{Int}_{Y_\alpha}\overline{G_\alpha}^{Y_\alpha}$. Consequently, $(p^\alpha_\gamma)^{-1}(U_\gamma) \subset G_\alpha$. We can suppose that $U_\gamma = \mathrm{Int}\overline{U_\gamma}$. Then, according to the inductive assumption, there is an open set $W \subset X$ such that $q(W) \cap [1,\gamma]$ is finite and $x \in W \subset p_\gamma^{-1}(U_\gamma) \cap G$. So, $x_\gamma \in p_\gamma(W) \subset W_\gamma = \mathrm{Int}\overline{p_\gamma(W)}$ and $W_\gamma \subset U_\gamma$. Hence, $x \in p_\gamma^{-1}(W_\gamma) \cap G \subset G$. The next claim completes the induction.

Claim 7. $q\big(p_\gamma^{-1}(W_\gamma) \cap G\big) \cap [1,\alpha] = q(W) \cap [1,\gamma]$.

Indeed, for every $\beta \le \gamma$ we have $\overline{p_\beta\big(p_\gamma^{-1}(W_\gamma) \cap G\big)} = \overline{p_\beta(W)}$. This implies

$$q(W) \cap [1,\gamma] = q\big(p_\gamma^{-1}(W_\gamma) \cap G\big) \cap [1,\gamma]. \tag{6}$$

Moreover, since $(p^\alpha_\gamma)^{-1}(W_\gamma) \subset (p^\alpha_\gamma)^{-1}(U_\gamma) \subset \overline{p_\alpha(G)}$, we have

$$\overline{p_\beta\big(p_\gamma^{-1}(W_\gamma) \cap G\big)} = \overline{p_\beta\big(p_\gamma^{-1}(W_\gamma)\big)}$$

for each $\beta \in [\gamma,\alpha]$. Hence,

$$q\big(p_\gamma^{-1}(W_\gamma) \cap G\big) \cap [\gamma,\alpha] = q\big(p_\gamma^{-1}(W_\gamma)\big) \cap [\gamma,\alpha]. \tag{7}$$

Note that, by Lemma 4.1(2), $q\big(p_\gamma^{-1}(W_\gamma)\big) \cap [\gamma,\alpha] = \varnothing$. Then the combination of (1) and (2) provides the proof of the claim.

Therefore, for every $\alpha < \tau$ the open sets $W \subset X$ with $q(W) \cap [1,\alpha]$ being finite form a base for X. Now, we can finish the proof of the proposition. If $V \subset X$ is open and $x \in V$ we find a set $G \subset V$ with $x \in G = p_\beta^{-1}(G_\beta)$, where G_β is open in X_β for some $\beta < \tau$. Then there exists an open set $W \subset G$ containing x such that $q(W) \cap [1,\beta]$ is finite. Let $W_\beta = \mathrm{Int}\overline{p_\beta(W)}$ and $U = p_\beta^{-1}(W_\beta \cap G_\beta)$. It is easily seen that $x \in U$ and $\overline{p_\nu(U)} = \overline{p_\nu(W)}$ for all $\nu \le \beta$. This yields $q(U) \cap [1,\beta] = q(W) \cap [1,\beta]$. On the other hand, by Lemma 4.1(2), $q(U) \cap [\beta,\tau) = \varnothing$. Hence U is a neighborhood of x which is contained in V and $q(U)$ is finite. □

Similar to the previous proposition, the next was also announced in [14, Proposition 2.3] without a proof.

Proposition 4.3. [14] *Let $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ be an almost continuous inverse system with nearly open bonding maps such that $X = \mathrm{a}-\varprojlim S$. Then:*

(1) *X is regularly embedded in $\prod_{\alpha<\tau} X_\alpha$;*

(2) *If, additionally, each* X_α *is regularly embedded in a space* Y_α*, then* X *is regularly embedded in* $\prod_{\alpha<\tau} Y_\alpha$.

Proof. (1) We consider the embedding of X in $\widetilde{X} = \prod_{\alpha<\tau} X_\alpha$ generated by the maps p_α. According to Proposition 4.2, X has a base $\mathcal{B}$ consisting of open sets $U \subset X$ with finite rank $q(U)$. As in Proposition 2.7, for every $U \in \mathcal{B}$ let $\Omega(U) = \{\alpha_0, \alpha, \alpha+1 : \alpha \in q(U)\}$, where $\alpha_0 < \tau$ is fixed. For all $U \in \mathcal{B}$ and $\alpha < \tau$ let $U_\alpha = \mathrm{Int}\overline{p_\alpha(U)}$ and $\Gamma(U) \subset \prod\{X_\alpha : \alpha < \tau\}$ be defined by

$$\Gamma(U) = \prod\{U_\alpha : \alpha \in \Omega(U)\} \times \prod\{X_\alpha : \alpha \notin \Omega(U)\}.$$

Since $p_\alpha(U) \subset U_\alpha$ for each α, U is contained in $\Gamma(U)$.

Using the arguments from the proof of Proposition 2.7, one can show that $\Gamma(U) \cap X \subset \overline{U}$. Finally, we define the required regular operator $\mathrm{e} : \mathcal{T}_X \to \mathcal{T}_{\widetilde{X}}$ by $\mathrm{e}(V) = \bigcup\{\Gamma(U) : U \in \mathcal{B}, \overline{U} \subset V\}$.

(2) For each $\alpha < \tau$ let $\mathrm{e}_\alpha : \mathcal{T}_{X_\alpha} \to \mathcal{T}_{Y_\alpha}$ be a regular operator. Define a function $\theta_1 : \mathcal{B} \to \mathcal{T}_{\widetilde{Y}}$, where $\widetilde{Y} = \prod_{\alpha<\tau} Y_\alpha$, by

$$\theta_1(U) = \prod_{\alpha \notin \Omega(U)} \mathrm{e}_\alpha(U_\alpha) \times \prod_{\alpha \notin \Omega(U)} Y_\alpha.$$

Consider $\theta : \mathcal{T}_X \to \mathcal{T}_{\widetilde{Y}}$, $\theta(G) = \bigcup\{\theta_1(U) : U \in \mathcal{B} \text{ and } \overline{U} \subset G\}$. Since $\theta_1(U) \cap X = \Gamma(U)$ and $U \subset \Gamma(U) \subset \overline{U}$ for any $U \in \mathcal{B}$, $\theta(G) \cap X = G$. Moreover, Claim 4 implies that $\theta(G_1) \cap \theta(G_2) = \varnothing$ provided $G_1 \cap G_2 = \varnothing$. Thus, θ is a regular operator. □

Acknowledgments. The author would like to express his gratitude to A. Kucharski for several discussions.

References

[1] A. Arhangel'skii and M. Tkachenko, *Topological groups and related structures*, Atlantis Studies in Mathematics, vol. 1, Atlantis Press, Paris, World Scientific, 2008.

[2] P. Daniels, K. Kunen and H. Zhou *On the open-open game*, Fund. Math.**145** (1994), no. 3, 205–220.

[3] A. Kucharski, A private communication, May 2017.

[4] ———, *On open-open Games of Uncountable Length*, Int. J. Math. Math. Sci. **2012** (2012), Art. ID 208693, 1-11.

[5] A. Kucharski and S. Plewik, *Skeletal maps and I-favorable spaces*, Acta Univ. Carolin. Math. Phys. **51** (2010), 67–72.

[6] ———, *Inverse systems and I-favorable spaces*, Topology Appl. **156** (2008), no. 1, 110–116.

[7] ———, *Game approach to universally Kuratowski-Ulam spaces*, Topology Appl. **154** (2007), no. 2, 421–427.

[8] K. Kuratowski, *Topology, vol. I*, Academic Press, New York; PWN-Polish Scientific Publishers, Warsaw 1966.

[9] E. Shchepin, *Topology of limit spaces of uncountable inverse spectra*, Russian Math. Surveys **315** (1976), 155–191.

[10] ———, *Functors and uncountable degrees of compacta*, Uspekhi Mat. Nauk **36** (1981), no. 3, 3–62 (in Russian).

[11] L. Shirokov, *An external characterization of Dugundji spaces and k-metrizable compacta*, Dokl. Akad. Nauk SSSR **263** (1982), no. 5, 1073–1077 (in Russian).

[12] M. Tkachenko, *Some results on inverse spetra II*, Comment. Math. Univ. Carol. **22** (1981), no. 4, 819–841.

[13] V. Valov, *External characterization of* I*-favorable spaces*, Mathematica Balkanica **25** (2011), no. 1-2, 61–78.

[14] ———, *Some characterizations of the spaces with a lattice of d-open mappings*, C. R. Acad. Bulgare Sci **39** (1986), no. 9, 9–12.

Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada

E-mail address: veskov@nipissingu.ca