ON UNIFORMLY CONTINUOUS SURJECTIONS BETWEEN FUNCTION SPACES

ALİ EMRE EYSEN AND VESKO VALOV

Abstract. We consider uniformly continuous surjections between $C_p(X)$ and $C_p(Y)$ (resp, $C_p^*(X)$ and $C_p^*(Y)$) and show that if X has some dimensional-like properties, then so does Y. In particular, we prove that if $T : C_p(X) \to C_p(Y)$ is a continuous linear surjection, then $\dim Y = 0$ if $\dim X = 0$. This provides a positive answer to a question raised by Kawamura-Leiderman [10, Problem 3.1].

1. Introduction

All spaces in this paper, if not said otherwise, are Tychonoff spaces and all maps are continuous. By $C(X)$ (resp., $C^*(X)$) we denote the set of all continuous (resp., continuous and bounded) real-valued functions on a space X. We write $C_p(X)$ (resp., $C_p^*(X)$) for the spaces $C(X)$ (resp., $C^*(X)$) endowed with the pointwise topology. More information about the spaces $C_p(X)$ can be found in [19]. By dimension we mean the *covering dimension* dim defined by finite functionally open covers, see [3]. According to that definition, we have $\dim X = \dim \beta X$, where βX is the Čech-Stone compactification of X. We also denote by $\overline{\mathbb{R}}$ the extended real line $[-\infty, \infty]$.

After the striking result of Pestov [17] that $\dim X = \dim Y$ provided $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, and Gul'ko's [7] generalization of Pestov's theorem that the same is true for uniformly continuous homeomorphisms, a question arose whether $\dim Y \leq \dim X$ if there is continuous linear surjection from $C_p(X)$ onto $C_p(Y)$, see [1]. This was answered negatively by Leiderman-Levin-Pestov [14] and Leiderman-Morris-Pestov [15]. On the other hand, it was shown in [14] that if there is a linear continuous surjection $C_p(X) \to C_p(Y)$ for compact metric spaces, then $\dim Y = 0$ provided $\dim X = 0$. The

---

2010 *Mathematics Subject Classification.* Primary 54C35; Secondary 54F45.

Key words and phrases. $C_p(X)$-space, C-space, 0-dimensional spaces, strongly countable-dimensional spaces, uniformly continuous surjections.

The first author was partially supported by TUBITAK-2219.

The second author was partially supported by NSERC Grant 261914-19.

last result was extended for arbitrary compact spaces by Kawamura-Leiderman [10] who also raised the question whether this is true without compactness of X and Y. In this paper we provide a positive answer to that question for continuous linear surjections $T : C_p(X) \to C_p(Y)$ and discuss the situation when linearity of T is replaced by uniformly continuity.

Suppose $E_p(X) \subset C(X)$ and $E_p(Y) \subset C(Y)$ are subspaces containing the zero functions on X and Y, respectively. Recall that a map $\varphi : E_p(X) \to E_p(Y)$ is *uniformly continuous* if for every neighborhood U of the zero function in $E_p(Y)$ there is a neighborhood V of the zero function in $E_p(X)$ such that $f, g \in E_p(X)$ and $f - g \in V$ implies $\varphi(f) - \varphi(g) \in U$. Evidently, if $E_p(X)$ and $E_p(Y)$ are linear spaces, then every linear continuous map between $E_p(X)$ and $E_p(Y)$ is uniformly continuous. If $f \in C_p(X)$ is a bounded function, then $||f||$ stands for the supremum norm of f. The notion of c-good maps was introduced in [6] (see also [5]), where c is a positive number. A map $\varphi : E_p(X) \to E_p(Y)$ is *c-good* if for every bounded function $g \in E_p(Y)$ there exists a bounded function $f \in E_p(X)$ such that $\varphi(f) = g$ and $||f|| \le c.||g||$.

Everywhere below, by $D(X)$ we denote either $C^*(X)$ or $C(X)$. Here is one of our main results.

Theorem 1.1. *Let $T : D_p(X) \to D_p(Y)$ be a c-good uniformly continuous surjection for some $c > 0$. Then Y is 0-dimensional provided so is X.*

Corollary 1.2. *Suppose there is a linear continuous surjection from $C_p^*(X)$ either onto $C_p^*(Y)$ or onto $C_p(Y)$. Then Y is 0-dimensional provided so is X.*

We consider properties $\mathcal{P}$ of σ-compact separable metric spaces such that:

(a) If $X \in \mathcal{P}$ and $F \subset X$ is closed, then $F \in \mathcal{P}$;
(b) $\mathcal{P}$ is closed under finite products;
(c) If X is a countable union of closed subsets each having the property $\mathcal{P}$, then $X \in \mathcal{P}$;
(d) If $f : X \to Y$ is a perfect map with countable fibers and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$.

For example, 0-*dimensionality*, *strongly countable-dimensionality* and *C-space property* satisfy conditions $(a) - (d)$. Another two properties of this type for σ-compact spaces X are: all finite powers of X to be *weakly infinite-dimensional*, or *$m - C$-spaces* in the sense of Fedorchuk

[4]. The definitions of all these dimensional invariants can be found in [3] and [4].

For σ-compact separable metric spaces we have the following version of Theorem 1.1:

Theorem 1.3. *Suppose X and Y are σ-compact separable metric spaces and let $T : D_p(X) \to D_p(Y)$ be a c-good uniformly continuous surjection for some $c > 0$. If X has a property $\mathcal{P}$ satisfying conditions $(a) - (d)$ above, then Y has the same property.*

Theorem 1.3 was established in [6] in the special case when X, Y are compact metric spaces, $T : C_p(X) \to C_p(Y)$ is a c-good uniformly continuous surjection and $\mathcal{P}$ being either 0-dimensionality or strongly countable-dimensionality.

Corollary 1.4. *If in Theorem 1.3 X is a C-space or all finite powers of X are weakly infinite-dimensional or $m - C$-spaces, then Y also has the same property.*

Similar results were established by Krupski [12] under the assumption that $T : C_p(X) \to C_p(Y)$ is a continuous open surjection.

The notion of c-good maps is crucial in the proof of the above results. We don't know if every linear continuous surjection $T : C_p(X) \to C_p(Y)$ is c-good for some c, but the following theorem is true and provides a positive answer to [10, Problem 3.1] in case $\dim X = 0$:

Theorem 1.5. *Let $T : C_p(X) \to C_p(Y)$ be a linear continuous surjection. If $\dim X = 0$, then $\dim Y = 0$.*

2. Preliminary results

In this section we prove a proposition which is used in the proofs of Theorem 1.1 and Theorem 1.3. Our proof is based on the idea of support introduced by Gul'ko [7] and the extension of this notion introduced by Krupski [11].

Let $\mathbb{Q}$ be the set of rational numbers. A subspace $E(X) \subset C(X)$ is called a *QS-algebra* [7] if it satisfies the following conditions: (i) If $f, g \in E(X)$ and $\lambda \in \mathbb{Q}$, then all functions $f + g$, $f \cdot g$ and λf belong to $E(X)$; (ii) For every $x \in X$ and its neighborhood U in X there is $f \in E(X)$ such that $f(x) = 1$ and $f(X \backslash U) = 0$.

We are using the following facts from [7]:

(2.1) If X has a countable base and $\Phi \subset C(X)$ is a countable set, then there is a countable QS-algebra $E(X) \subset C(X)$ containing Φ. Moreover, it follows from the proof of [7, Proposition 1.2] that that $E(X) \subset C^*(X)$ providing $\Phi \subset C^*(X)$;

(2.2) If U is an open set in X, $x_1, x_2, .., x_k \in U$ and $\lambda_1, \lambda_2, .., \lambda_k \in \mathbb{Q}$, then there exists $f \in E(X)$ such that $f(x_i) = \lambda_i$ for each i and $f(X \backslash U) = 0$.

(2.3) If $\mathcal{B}$ is a countable finitely additive base of X, then by (2.1) we can always suppose that for any $U, V \in \mathcal{B}$ with $\overline{V} \subset U$ there is $f_{V,U} \in E(X)$ such that $f_{V,U}|\overline{V} = 1$, $f_{V,U}|(X \backslash U) = 0$ and $f_{V,U}(x) \in [0,1]$ for all $x \in X$. In particular, for every compact set $K \subset V$ we have $f_{V,U}|K = 1$.

Proposition 2.1. *Let $\overline{X}$ and $\overline{Y}$ be metric compactifications of X and Y, and $H \subset \overline{X}$ be a σ-compact space containing X. Suppose $E(H)$ is a QS-algebra on H, $E(X) = \{\overline{f}|X : \overline{f} \in E(H)\}$ and $E(Y) \subset C(Y)$ is a family such that every $g \in E(Y)$ is extendable to a map $\overline{g} : \overline{Y} \to \overline{\mathbb{R}}$ and $E(\overline{Y}) = \{\overline{g} : g \in E(Y)\}$ containing a QS-algebra Γ on $\overline{Y}$. Let also $\varphi : E_p(X) \to E_p(Y)$ be an uniformly continuous surjection which is c-good for some $c > 0$.*

If H has a property $\mathcal{P}$ satisfying conditions $(a)-(d)$, then there exists a σ-compact set $Y_\infty \subset \overline{Y}$ containing Y with $Y_\infty \in \mathcal{P}$.

Proof. For every $\overline{f} \in E(H)$ denote by f the restriction $\overline{f}|X$. For every $y \in \overline{Y}$ there is a map $\alpha_y : E(H) \to \overline{\mathbb{R}}$, $\alpha_y(\overline{f}) = \overline{\varphi(f)}(y)$. Since φ is uniformly continuous, so is each $\alpha_y|E_p(X)$, $y \in Y$. Suppose $H = \bigcup_k H_k$ is the union of an increasing sequence $\{H_k\}$ of compact sets. Following Krupski [11], for every $y \in \overline{Y}$ and every $p, k \in \mathbb{N}$ we define the families

$$\mathcal{A}^k(y) = \{K \subset H_k : K \text{ is closed and } a(y,K) < \infty\}$$

and

$$\mathcal{A}^k_p(y) = \{K \subset H_k : K \text{ is closed and } a(y,K) \leq p\},$$

where

$$a(y,K) = \sup\{|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| : \overline{f}, \overline{g} \in E(H), |\overline{f}(x) - \overline{g}(x)| < 1 \ \forall x \in K\}.$$

Possibly, some or both of the values $\alpha_y(\overline{f}), \alpha_y(\overline{g})$ from the definition of $a(y,K)$ could be $\pm\infty$. That's why we use the following agreements:

(*) $\infty + \infty = \infty, \infty - \infty = -\infty + \infty = 0, -\infty - \infty = -\infty$.

Note that $a(y, \varnothing) = \infty$ since φ is surjective.

Using that $E(X)$ and $E(H)$ are QS-algebras on X and H, and following the arguments from Krupski's paper [11] (see also the proofs of [7, Proposition 1.4] and [16, Proposition 3.1]), one can establish the following claims (for the sake of completeness we provide the proofs):

Claim 1. *For every $y \in Y$ there is $p, k \in \mathbb{N}$ such that $\mathcal{A}^k_p(y)$ contains a finite nonempty subset of X.*

This claim follows from the proof of [11, Proposition 2.1]. Indeed, since φ is uniformly continuous there is $p \in \mathbb{N}$ and a finite set $K \subset X$ such that if $f, g \in E(X)$ and $|f(x) - g(x)| < 1/p$ for every $x \in K$, then $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| < 1$. Take arbitrary $\overline{f}, \overline{g} \in E(H)$ with $|\overline{f}(x) - \overline{g}(x)| < 1$ for every $x \in K$ and consider the functions $\overline{f}_m = \overline{f} + \frac{m}{p}(\overline{g} - \overline{f}) \in E(H)$ for each $m = 0, 1, .., p$. Obviously $|f_m(x) - f_{m+1}(x)| < 1/p$ for all $x \in K$, so $|\alpha_y(\overline{f}_m) - \alpha_y(\overline{f}_{m+1})| < 1$. Consequently, $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| \leq \sum_{m=0}^{p-1} |\alpha_y(\overline{f}_m) - \alpha_y(\overline{f}_{m+1})| < p$. Because K is finite, there is $k \in \mathbb{N}$ with $K \subset H_k$. Hence $K \in \mathcal{A}_p^k(y)$.

Consider the sets $Y_p^k = \{y \in \overline{Y} : \mathcal{A}_p^k(y) \neq \varnothing\}$, $p, k \in \mathbb{N}$.

Claim 2. *Each Y_p^k is a closed subset of $\overline{Y}$.*

We use the proof of [11, Lemma 2.2]. Suppose $y \notin Y_p^k$. Since $y \in Y_p^k$ iff $H_k \in \mathcal{A}_p^k(y)$, $H_k \notin \mathcal{A}_p^k(y)$. So, there exist $\overline{f}, \overline{g} \in E(H)$ with $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in H_k$ and $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| > p$. Then $V = \{z \in \overline{Y} : |\alpha_z(\overline{f}) - \alpha_z(\overline{g})| > p\}$ is a neighborhood of y with $V \cap Y_p^k = \varnothing$.

Claim 3. *Each set $Y_{p,q}^k = \{y \in Y_p^k : \exists K \in \mathcal{A}_p^k(y)$ with $|K| \leq q\}$, $p, q, k \in \mathbb{N}$, is closed in Y_p^k.*

Following the proof of [11, Lemma 2.3], we first show that the set $Z = \{(y, K) \in Y_p^k \times [H_k]^{\leq q} : K \in \mathcal{A}_p^k(y)\}$ is closed in $Y_p^k \times [H_k]^{\leq q}$, where $[H_k]^{\leq q}$ denotes the space of all subsets $K \subset H_k$ of cardinality $\leq q$ endowed with the Vietoris topology. Indeed, if $(y, K) \in Y_p^k \times [H_k]^{\leq q} \setminus Z$, then $K \notin \mathcal{A}_p^k(y)$. Hence, $a(y, K) > p$ and there are $\overline{f}, \overline{g} \in E(H)$ such that $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in K$ and $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| > p$. Let $U = \{z \in Y_p^k : |\alpha_z(\overline{f}) - \alpha_z(\overline{g})| > p\}$ and $V = \{x \in H_k : |\overline{f}(x) - \overline{g}(x)| < 1\}$. The set $U \times < V >$ is a neighborhood of (y, K) in $Y_p^k \times [H_k]^{\leq q}$ disjoint from Z (here $< V > = \{F \in [H_k]^{\leq q} : F \subset V\}$). Since $Y_p^k \times [H_k]^{\leq q}$ is compact and $Y_{p,q}^k$ is the image of Z under the projection $Y_p^k \times [H_k]^{\leq q} \to Y_p^k$, $Y_{p,q}^k$ is closed in Y_p^k.

For every k let $Y_k = \bigcup_{p,q} Y_{p,q}^k$. Obviously, $Y_k \subset \{y \in \overline{Y} : \mathcal{A}^k(y) \neq \varnothing\}$. Since $H_k \subset H_{k+1}$ for all k, the sequence $\{Y_k\}$ is increasing. It may happen that $Y_k = \varnothing$ for some k, but Claim 1 implies that $Y \subset \bigcup_k Y_k$.

Claim 4. *For every $y \in Y_k$ the family $\mathcal{A}^k(y)$ is closed under finite intersections and $a(y, K_1 \cap K_2) \leq a(y, K_1) + a(y, K_2)$ for all $K_1, K_2 \in \mathcal{A}^k(y)$.*

We follow the proof of [11, Lemma 2.5] to show that $K_1 \cap K_2 \in \mathcal{A}^k(y)$ for any $K_1, K_2 \in \mathcal{A}^k(y)$. Let $\overline{f}, \overline{g} \in E(H)$ with $|\overline{f}(x) - \overline{g}(x)| < 1$ for all

$x \in K_1 \cap K_2$ and $U = \{x \in H : |\overline{f}(x) - \overline{g}(x)| < 1\}$. Take a base open set W in H containing K_1 with $W \cap (K_2 \backslash U) = \varnothing$ and choose $\overline{u} \in E(H)$ such that $\overline{u}|K_1 = 1$, $\overline{u}|(H \backslash W) = 0$ and $\overline{u}(x) \in [0, 1]$ for all $x \in H$. Then $\overline{h} = \overline{u}.(\overline{f} - \overline{g}) + \overline{g} \in E(H)$, $\overline{h}|K_1 = \overline{f}|K_1$, $\overline{h}|(K_2 \backslash U) = \overline{g}|(K_2 \backslash U)$ and $|\overline{h}(x) - \overline{g}(x)| < 1$ for $x \in K_2$. Since $K_1 \in \mathcal{A}^k(y)$ and $\overline{h}|K_1 = \overline{f}|K_1$, we have $|\alpha_y(\overline{f}) - \alpha_y(\overline{h})| \leq a(y, K_1) < \infty$. Similarly, $K_2 \in \mathcal{A}^k(y)$ and $|\overline{h}(x) - \overline{g}(x)| < 1$ for $x \in K_2$ imply $|\alpha_y(\overline{h}) - \alpha_y(\overline{g})| \leq a(y, K_2) < \infty$. Therefore,

$$|\alpha_y(\overline{f}) - \alpha_y(\overline{g}) \leq |\alpha_y(\overline{f}) - \alpha_y(\overline{h})| + |\alpha_y(\overline{h}) - \alpha_y(\overline{g})| \leq a(y, K_1) + a(y, K_2).$$

Note that the last inequality is true if some of $\alpha_y(\overline{f}), \alpha_y(\overline{h}), \alpha_y(\overline{g})$ are $\pm\infty$. Indeed, if $\alpha_y(\overline{f}) = \pm\infty$, then $|\alpha_y(\overline{f}) - \alpha_y(\overline{h})| < \infty$ implies $\alpha_y(\overline{h}) = \pm\infty$. Consequently, $\alpha_y(\overline{g}) = \pm\infty$ because $|\alpha_y(\overline{h}) - \alpha_y(\overline{g})| < \infty$. Therefore, $a(y, K_1 \cap K_2) \leq a(y, K_1) + a(y, K_2)$, which means that $K_1 \cap K_2 \neq \varnothing$ (otherwise $a(y, K_1 \cap K_2) = \infty$) and $K_1 \cap K_2 \in \mathcal{A}^k(y)$.

Since each family $\mathcal{A}^k(y)$, $y \in Y_k$, consists of compact subsets of H_k, $K(y, k) = \bigcap \mathcal{A}^k(y)$ is nonempty and compact.

Claim 5. *For every $y \in Y_k$ the set $K(y, k)$ is a nonempty finite subset of H_k with $K(y, k) \in \mathcal{A}^k(y)$. Moreover, if $y \in Y$ then there exists k such that $y \in Y_k$ and $K(y, k) \subset X$.*

Let $y \in Y_k$. We already observed that $K(y, k)$ is compact and nonempty. Since $y \in Y^k_{p,q}$ for some p, q, $\mathcal{A}^k(y)$ contains finite sets. Hence, $K(y, k)$ is also finite and $K(y, k) \in \mathcal{A}^k(y)$ because it an intersection of finitely many elements of $\mathcal{A}^k(y)$. If $y \in Y$, then by Claim 1, there is k such that $\mathcal{A}^k(y)$ contains a finite subset of X. Since $K(y, k)$ is the minimal element of $\mathcal{A}^k(y)$, it is also a subset of X.

Following [7], for every k we define $M^k(p, 1) = Y^k_{p,1}$ and $M^k(p, q) = Y^k_{p,q} \backslash Y^k_{2p,q-1}$ if $q \geq 2$.

Claim 6. $Y_k = \bigcup\{M^k(p, q) : p, q = 1, 2, ..\}$ *and for every $y \in M^k(p, q)$ there exists a unique set $K_{kp}(y) \in \mathcal{A}^k(y)$ of cardinality q such that $a(y, K_{kp}(y)) \leq p$.*

Since $M^k(p, q) \subset Y^k_{p,q} \subset Y_k$, $\bigcup\{M^k(p, q) : p, q = 1, 2, ..\} \subset Y_k$. If $y \in Y_k$, then $K(y, k) \in \mathcal{A}^k(y)$ is a finite subset of H_k. Suppose $|K(y, k)| = q$ and $a(y, K(y, k)) \leq p$ for some p, q. So, $y \in Y^k_{p,q}$. Moreover $y \notin Y^k_{2p,q-1}$, otherwise there would be $K \in \mathcal{A}^k(y)$ with $a(y, K) \leq 2p$ and $|K| \leq q-1$. The last inequality contradicts the minimality of $K(y, k)$. Hence, $y \in M^k(p, q)$ which shows that $Y_k = \bigcup\{M^k(p, q) : p, q = 1, 2, ..\}$.

Suppose $y \in M^k(p, q)$. Then there exists a set $K \in \mathcal{A}^k(y)$ with $a(y, K) \leq p$ and $|K| \leq q$. Since $y \notin Y^k_{2p,q-1}$, $|K| = q$. If there exists another $K' \in \mathcal{A}^k(y)$ with $a(y, K') \leq p$ and $|K'| = q$, then $K \cap K' \neq \varnothing$, $|K \cap K'| \leq q-1$ and, by Claim 4, $a(y, K \cap K') \leq a(y, K) + a(y, K') \leq 2p$. This means that $y \in Y^k_{2p,q-1}$, a contradiction. Hence, there exists a unique $K_{kp}(y) \in \mathcal{A}^k(y)$ such that $a(y, K_{kp}(y)) \leq p$ and $|K_{kp}(y)| = q$.

For every q let $[H_k]^q$ denote the set of all q-points subsets of H_k endowed with the Vietoris topology.

Claim 7. *The map* $\Phi_{kpq} : M^k(p, q) \to [H_k]^q$, $\Phi_{kpq}(y) = K_{kp}(y)$, *is continuous.*

Because $K_{kp}(y) \subset H_k$ consists of q points for all $y \in M^k(p, q)$, it suffices to show that if $K_{kp}(y) \cap U \neq \varnothing$ for some open $U \subset H$, then there is a neighborhood V of y in $\overline{Y}$ with $K_{kp}(z) \cap U \neq \varnothing$ for all $z \in V \cap M^k(p, q)$. We can assume that $K_{kp}(y) \cap U$ contains exactly one point x_0.

Suppose $q \geq 2$, so $K_{kp}(y) = \{x_0, x_1, .., x_{q-1}\}$. Since $y \notin Y^k_{2p,q-1}$ we have $a(y, K) > 2p$, where $K = \{x_1, .., x_{q-1}\}$. Hence, there are $\overline{f}, \overline{g} \in E(H)$ such that $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in K$ and $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| > 2p$. The last inequality implies $\overline{f}(x_0) \neq \overline{g}(x_0)$, otherwise $a(y, K_{kp}(y))$ would be greater than $2p$ (recall that $y \in M^k(p, q)$ implies $a(y, K_{kp}(y)) \leq p$). So, at least one of the numbers $\overline{f}(x_0), \overline{g}(x_0)$ is not zero. Without loss of generality we can suppose $\overline{f}(x_0) > 0$, and let r be a rational number with $\frac{-1+\delta}{\overline{f}(x_0)} < r < \frac{1+\delta}{\overline{f}(x_0)}$, where $\delta = \overline{f}(x_0) - \overline{g}(x_0)$. Then $-1 < (1-r)\overline{f}(x_0) - \overline{g}(x_0) < 1$, and choose $\overline{h}_1 \in E(H)$ such that $\overline{h}_1(x_0) = r$ and $\overline{h}_1(x) = 0$ for all $x \notin U$. Consider the function $\overline{h} = (1 - \overline{h}_1)\overline{f}$. Clearly, $\overline{h}(x_0) = (1-r)\overline{f}(x_0)$ and $\overline{h}(x) = \overline{f}(x)$ if $x \notin U$. Hence, $\overline{h} \in E(H)$ and $|\overline{h}(x) - \overline{g}(x)| < 1$ for all $x \in K_{kp}(y)$. This implies $|\alpha_y(\overline{h}) - \alpha_y(\overline{g})| \leq p$. Then

$$|\alpha_y(\overline{f}) - \alpha_y(\overline{h})| \geq |\alpha_y(\overline{f}) - \alpha_y(\overline{g})| - |\alpha_y(\overline{h}) - \alpha_y(\overline{g})| > 2p - p = p.$$

Observe that it is not possible $\alpha_y(\overline{f}) = \alpha_y(\overline{h}) = \pm\infty$ because $|\alpha_y(\overline{h}) - \alpha_y(\overline{g})| \leq p$ would imply $\alpha_y(\overline{g}) = \pm\infty$. Then $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| = 0$, a contradiction.

The set $V = \{z \in \overline{Y} : |\alpha_z(\overline{f}) - \alpha_z(\overline{h})| > p\}$ is a neighborhood of y. Since $\overline{h}(x) = \overline{f}(x)$ for all $x \notin U$, $K_{kp}(z) \cap U = \varnothing$ for some $z \in V \cap M^k(p, q)$ would imply $|\alpha_z(\overline{h}) - \alpha_z(\overline{f})| \leq p$, a contradiction. Therefore, $K_{kp}(z) \cap U \neq \varnothing$ for $z \in V \cap M^k(p, q)$.

If $q = 1$, then $K_{kp}(y) = \{x_0\}$ and $K(y, k) = K_{kp}(y)$. So, $H_k \backslash U \notin \mathcal{A}^k(y)$ (otherwise $K(y, k) \subset H_k \backslash U$). Hence, there exist $\overline{f}, \overline{g} \in E(H)$ such that $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in H_k \backslash U$ and $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| > 2p$.

Define $\overline{h} \in E(H)$ as in the previous case and use the same arguments to complete the proof.

Since $Y^k_{p,q}$ are compact subsets of $\overline{Y}$, each $M^k(p,q)$ is a countable union of compact subsets $\{F^k_n(p,q) : n = 1,2,..\}$ of $\overline{Y}$. So, by Claim 6, $Y_k = \bigcup\{F^k_n(p,q) : n,p,q = 1,2,..\}$. According to Claim 7, all maps $\Phi^n_{kpq} = \Phi_{kpq}|F^k_n(p,q) : F^k_n(p,q) \to [H_k]^q$ are continuous. Moreover, since $Y \subset \bigcup_k Y_k$, $Y \subset \bigcup\{F^k_n(p,q) : n,p,q,k = 1,2,..\}$.

Claim 8. *The fibers of $\Phi^n_{kpq} : F^k_n(p,q) \to [H_k]^q$ are finite.*

We follow the arguments from the proof of [6, Theorem 4.2]. Fix $z \in F^k_n(p,q)$ for some n,p,q,k and let $A = \{y \in F^k_n(p,q) : K_{kp}(y) = K_{kp}(z)\}$. Since Φ^n_{kpq} is a perfect map, A is compact. Suppose A is infinite, so it contains a convergent sequence $S = \{y_m\}$ of distinct points. Because $E(\overline{Y})$ contains a QS-algebra Γ on $\overline{Y}$, for every y_m there exist its neighborhood U_m in $\overline{Y}$ and a function $\overline{g}_m \in \Gamma$, $\overline{g}_m : \overline{Y} \to [0,2p]$ such that: $U_m \cap S = \{y_m\}$, $\overline{g}_m(y_m) = 2p$ and $\overline{g}_m(y) = 0$ for all $y \notin U_m$. Since φ is c-good, for each m there is $f_m \in E(X)$ with $\varphi(f_m) = \overline{g}_m|Y = g_m$ and $||f_m|| \leq c||\overline{g}_m||$. So, $||\overline{f}_m|| \leq 2pc$, $m = 1,2,..$ and the sequence $\{\overline{f}_m\}$ is contained in the compact set $[-2pc,2pc]^H$. Hence, $\{\overline{f}_m\}$ has an accumulation point in $[-2pc,2pc]^H$. This implies the existence of $i \neq j$ such that $|\overline{f}_i(x) - \overline{f}_j(x)| < 1$ for all $x \in K_{kp}(z)$. Consequently, since $K_{kp}(y_j) = K_{kp}(z)$, $|\alpha_{y_j}(\overline{f}_j) - \alpha_{y_j}(\overline{f}_i)| \leq p$. On the other hand, $\alpha_{y_j}(\overline{f}_j) = \overline{\varphi(f_j)}(y_j) = \overline{g}_j(y_j) = 2p$ and $\alpha_{y_j}(\overline{f}_i) = \overline{\varphi(f_i)}(y_j) = \overline{g}_i(y_j) = 0$, so $|\alpha_{y_j}(\overline{f}_j) - \alpha_{y_j}(\overline{f}_i)| = 2p$, a contradiction.

Now, we can complete the proof of Proposition 2.1. Suppose H has a property $\mathcal{P}$ satisfying conditions $(a)-(d)$. Then so does H^q_k for each k,q because H_k is closed in H. Hence, each set $\Phi^n_{kpq}(F^k_n(p,q))$ also has the property $\mathcal{P}$ because it is a compact subset of H^q_k. Finally, since the maps $\Phi^n_{kpq} : F^k_n(p,q) \to \Phi^n_{kpq}(F^k_n(p,q))$ are perfect and have finite fibers, each $F^k_n(p,q)$ has the property $\mathcal{P}$. Therefore, by condition (c), $Y_\infty = \bigcup\{F^k_n(p,q) : n,p,q,k = 1,2,..\}$ has the property $\mathcal{P}$. □

All definitions below, except that one of $m-C$-spaces, can be found in [3]. A normal space X is called strongly countable-dimensional if X can be represented as a countable union of closed finite-dimensional subspaces. Recall that a normal space X is weakly infinite-dimensional if for every sequence $\{(A_i,B_i)\}$ of pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2,..$ such that L_i is a partition between A_i and B_i and $\bigcap_i L_i = \varnothing$. A normal space X is a C-space if for every sequence $\{\mathcal{G}_i\}$ of open covers of X there exists a sequence $\{\mathcal{H}_i\}$ of families of pairwise disjoint open subsets of X such that for $i = 1,2,..$

each member of $\mathcal{H}_i$ is contained in a member of $\mathcal{G}_i$ and the union $\bigcup_i \mathcal{H}_i$ is a cover of X. The $m-C$-spaces, where $m \geq 2$ is a natural number were introduced by Fedorchuk [4]: A normal space X is a $m-C$-space if for any sequence $\{\mathcal{G}_i\}$ of open cowers of X such that each $\mathcal{G}_i$ consists of at most m elements, there is a sequence of disjoint open families $\{\mathcal{H}_i\}$ such that each $\mathcal{H}_i$ refines $\mathcal{G}_i$ and $\bigcup_i \mathcal{H}_i$ is a cover of X. The $2-C$-spaces are exactly the weakly infinite-dimensional spaces and for every m we have the inclusion $(m+1)-C \subset m-C$. Moreover, every C-space is $m-C$ for all m.

It is well known that the class of metric strongly countable-dimensional spaces contains all finite-dimensional metric spaces and is contained in the class of metric C-spaces. The last inclusion follows from the following two facts: (i) every finite-dimensional paracompact space is a C-space [3, Theorem 6.3.7]; (ii) every paracompact space which is a countable union of its closed C-spaces is also a C-space [8, Theorem 4.1]. Moreover, every C-space is weakly infinite-dimensional [3, Theorem 6.3.10].

In the class of σ-compact separable metric spaces the 0-dimensionality satisfies all conditions $(a)-(d)$, see following from [3] Theorems 1.5.16, 1.2.2 and 1.12.4. The strongly countable-dimensionality also satisfies all these conditions, condition (d) follows easily from [3, Theorem 1.12.4]. For the C-space property this follows from mentioned above fact that a countable union of closed compact C-spaces is a C-space [8, Theorem 4.1] and the following results of Hattori-Yamada [9]: the class of compact C-spaces is closed under finite products any perfect preimage of a C-space with C-space fibers is a C-space. Finally, if every finite power of a σ-compact space X are weakly infinite-dimensional, then obviously every closed subset of X, as well as any countable union of closed subsets of X have the same property. Condition (d) follows from the following result of Pol [18, Theorem 4.1]: If $f : X \to Y$ is a continuous map between compact metric spaces such that Y is weakly infinite-dimensional and each fiber $f^{-1}(y)$, $y \in Y$, is at most countable, then X is weakly infinite-dimensional. The validity of conditions $(a)-(c)$ for $m-C$-spaces follows from the following results [4] in the class of σ-compact metrizable spaces: The $m-C$-space property is hereditary with respect to closed subsets, a countable union of closed $m-C$-spaces is also $m-C$. Moreover, any perfect preimage of an $m-C$-space under countable-to-one surjection is $m-C$ [12] which means that condition (d) also holds for the class of $m-C$-spaces.

3. Uniformly continuous surjections

In this section we provide the proofs of Theorem 1.1 and Theorem 1.3, as well as their corollaries.

For every space X let $\mathcal{F}_X$ be the set of all maps from X onto second countable spaces. For any two maps $h_1, h_2 \in \mathcal{F}_X$ we write $h_1 \succ h_2$ if there exists a continuous map $\theta : h_1(X) \to h_2(X)$ with $h_2 = \theta \circ h_1$. If $\Phi \subset C(X)$ we denote by $\triangle\Phi$ the diagonal product of all $f \in \Phi$. Clearly, $(\triangle\Phi)(X)$ is a subspace of the product $\prod\{\mathbb{R}_f : f \in \Phi\}$, and let $\pi_f : (\triangle\Phi)(X) \to \mathbb{R}_f$. Following [7], we call a set $\Phi \subset C(X)$ *admissible* if the family $\pi(\Phi) = \{\pi_f : f \in \Phi\}$ is a QS-algebra on $(\triangle\Phi)(X)$. We are using the following statements:

(3.1) $\dim X \leq n$ if and only if for every $h \in \mathcal{F}_X$ there exists a $h_0 \in \mathcal{F}_X$ such that $\dim h_0(X) \leq n$ and $h_0 \succ h$ [17];

(3.2) If $\dim X \leq n$ and $\Phi \subset C(X)$ is countable, then there exists a countable set $\Theta \subset C(X)$ containing Φ with $\dim(\triangle\Theta)(X) \leq n$. Moreover, it follows from the proof of [7, Lemma 2.2] that we can choose $\Theta \subset C^*(X)$ provided $\Phi \subset C^*(X)$;

(3.3) For every countable $\Phi' \subset C(X)$ there is a countable admissible set Φ containing Φ' such that $(\triangle\Phi)(X)$ is homeomorphic to $(\triangle\Phi')(X)$. According to the proof of [7, Lemma 2.4], Φ could be taken to be a subset of $C^*(X)$ if $\Phi' \subset C^*(X)$;

(3.4) If $\{\Psi_n\}$ is an increasing sequence of admissible subsets of $C(X)$, then $\Psi = \bigcup_n \Psi_n$ is also admissible, see [7, Lemma 2.5].

We also need the following well known lemmas:

Lemma 3.1. *Let X_0 be a 0-dimensional separable metric space and $E(X_0)$ be a countable subfamily of $C^*(X_0)$. Then there exists a metric 0-dimensional compactification $\overline{X}_0$ of X_0 such that each $f \in E(X_0)$ can be extended over $\overline{X}_0$.*

Lemma 3.2. *Let Y be a separable metric space and $E(Y)$ be a countable subfamily of $C(Y)$. Then there exists a metric compactification $\overline{Y}$ of Y such that each $f \in E(Y)$ can be extended to a map $\overline{f} : \overline{Y} \to \overline{\mathbb{R}}$. Moreover, if $\dim Y = 0$, then we can assume $\dim \overline{Y} = 0$.*

Proof. Take a metric compactification Y_1 of Y and let $\theta_1 : \beta Y \to Y_1$ be a map fixing the points of Y. Then each function $f \circ \theta_1$, $f \in E(Y)$, can be extended to a map $f' : \beta Y \to \overline{\mathbb{R}}$. Let $E' = \{f' : f \in E(Y)\}$ and $\theta = (\triangle E')\triangle\theta_1$. Because θ is a homeomorphism on Y, $Z = \theta(\beta Y)$ is a metric compactification of Y. Moreover, every $f \in E(Y)$ can be extended to a map $\overline{f} : Z \to \overline{\mathbb{R}}$. In case $\dim Y = 0$, we apply Mardešić's factorization theorem [3, Theorem 3.4.1], to obtain a metric compactum

P with $\dim P = 0$ and maps $\nu : \beta Y_0 \to P$ and $\eta : P \to Z$ with $\eta \circ \nu = \theta$. Obviously, ν is fixing the points of Y, so P is a compactification of Y. For every $f' \in E'$ there is a map $g' : P \to \overline{\mathbb{R}}$ such that $g' \circ \nu = f'$ and $g'|Y_0 = f$. □

Proof of Theorem 1.1. Let $T : D_p(X) \to D_p(Y)$ be an uniformly continuous c-good surjection. Everywhere below for $f \in C^*(X)$ let $\overline{f} \in C(\beta X)$ be its extension; similarly if $g \in C(Y)$ then $\overline{g} \in C(\beta Y, \overline{\mathbb{R}})$ is the extension of g. According to (3.1), it suffices to prove that for every $h \in \mathcal{F}_Y$ there is $h_0 \in \mathcal{F}_Y$ such that $\dim h_0(Y) = 0$ and $h_0 \succ h$. So, we fix $h \in \mathcal{F}_Y$ and let $\overline{h} : \beta Y \to \overline{h(Y)}$ be an extension of h, where $\overline{h(Y)}$ is a metric compactification of $h(Y)$. We will construct by induction two sequences $\{\Psi_n\}_{n\geq 1} \subset C(\beta X)$ and $\{\Phi_n\}_{n\geq 1} \subset C(\beta Y, \overline{\mathbb{R}})$ of countable sets , countable QS-algebras Λ_n on $Y'_n = (\triangle \Phi'_n)(\beta Y)$, where $\Phi'_n = \{\overline{T(f)} : \overline{f} \in \Psi_n\}$, satisfying the following conditions for every $n \geq 1$:

- (3.5) $\Phi_1 \subset C(\beta Y)$ is admissible and $\triangle \Phi_1 \succ \overline{h}$;
- (3.6) $\Phi_n \subset \Phi_{n+1} = \Phi'_n \cup \{\lambda \circ (\triangle \Phi'_n) : \lambda \in \Lambda_n\}$;
- (3.7) Each Ψ_n is admissible, $\dim(\triangle \Psi_n)(\beta X) = 0$ and $\Psi_n \subset \Psi_{n+1}$;
- (3.8) Λ_{n+1} contains $\{\lambda \circ \delta_n : \lambda \in \Lambda_n\}$, where $\delta_n : Y'_{n+1} \to Y'_n$ is the surjective map generated by the inclusion $\Phi'_n \subset \Phi'_{n+1}$;
- (3.9) For every $\overline{g} \in \Phi_n \cap C(\beta Y)$ there is $\overline{f}_g \in \Psi_n$ with $||f_g|| \leq c.||g||$ and $T(f_g) = g$.

Since $\overline{h}(\beta Y)$ is a separable metric space, by (3.3), there is a countable admissible set $\Phi_1 \subset C(\beta Y)$ with $\overline{h} = \triangle \Phi_1$. Choose a countable set $\Psi'_1 \subset C(\beta X)$ such that for every $\overline{g} \in \Phi_1$ there is $\overline{f}_g \in \Psi'_1$ with $||f_g|| \leq c.||g||$ and $T(f_g) = g$. Next, use (3.2) to find countable $\Theta_1 \subset C(\beta X)$ containing Ψ'_1 such that $\dim(\triangle \Theta_1)(\beta X) = 0$. Finally, by (3.3), we can extend Θ_1 to a countable admissible set $\Psi_1 \subset C(\beta X)$ such that $(\triangle \Psi_1)(\beta X)$ is homeomorphic to $(\triangle \Theta_1)(\beta X)$. Hence, $\dim(\triangle \Psi_1)(\beta X) = 0$.

Suppose Φ_k and Ψ_k are already constructed for all $k \leq n$. Then $\Phi'_n = \{\overline{T(f)} : \overline{f} \in \Psi_n\}$ is a countable set in $C(\beta Y, \overline{\mathbb{R}})$. Because $\Psi_{n-1} \subset \Psi_n$, $\Psi'_{n-1} \subset \Psi'_n$. So, there is a surjective map $\delta_{n-1} : Y'_n \to Y'_{n-1}$. Choose a countable QS-algebra Λ_n on Y'_n containing the family $\{\lambda \circ \delta_{n-1} : \lambda \in \Lambda_{n-1}\}$ and let $\Phi_{n+1} = \Phi'_n \cup \{\lambda \circ (\triangle \Phi'_n) : \lambda \in \Lambda_n\}$. Next, choose a countable set $\Psi'_{n+1} \subset C(\beta X)$ containing Ψ_n such that for every $\overline{g} \in \Phi_{n+1} \cap C(\beta Y)$ there is $\overline{f}_g \in \Psi'_{n+1}$ with $||f_g|| \leq c.||g||$ and $T(f_g) = g$. Then, by (3.2) there is countable $\Theta_{n+1} \subset C(\beta X)$ containing Ψ'_{n+1} with $\dim(\triangle \Theta_{n+1})(\beta X) = 0$. Finally, according to (3.3), we extend Θ_{n+1} to a countable admissible set $\Psi_{n+1} \subset C(\beta X)$ such that $(\triangle\Psi_{n+1})(\beta X)$ is homeomorphic to $(\triangle\Theta_{n+1})(\beta X)$. This completes the induction.

Let $\Psi = \bigcup_n \Psi_n$, $X_0 = (\triangle\Psi)(X)$ and $\overline{X}_0 = (\triangle\Psi)(\beta X)$. Similarly, let $\Phi = \bigcup_n \Phi_n$, $Y_0 = h_0(Y)$ and $\overline{Y}_0 = (\triangle\Phi)(\beta Y)$, where $h_0 = (\triangle\Phi)|Y$. Both Ψ and Φ are countable and Ψ is an admissible subset of $C(\beta X)$, see (3.4). Hence, the family $E(\overline{X}_0) = \{\pi_{\overline{f}} : \overline{f} \in \Psi\}$ is a countable QS-algebra on $\overline{X}_0$. Moreover, the family $E(Y_0) = \{\pi_{\overline{g}}|Y_0 : \overline{g} \in \Phi\}$ is extendable over $\overline{Y}_0$. Since $\Psi_n \subset \Psi_{n+1}$ for every n, there are maps $\theta_n : (\triangle\Psi_{n+1})(\beta X) \to (\triangle\Psi_n)(\beta X)$. Because $\Psi = \bigcup_n \Psi_n$, the space $\overline{X}_0$ is the limit of the inverse sequence $S_X = \{(\triangle\Psi_n)(\beta X), \theta_n\}$ with $\dim(\triangle\Psi_n)(\beta X) = 0$ for all n. Hence, $\overline{X}_0$ is also 0-dimensional, see [3, Theorem 3.4.11].

It follows from the construction that $\Phi = \{\overline{T(f)} : \overline{f} \in \Psi\}$ and for every $\overline{g} \in \Phi \cap C(\beta Y)$ there is $\overline{f}_g \in \Psi$ with $T(f_g) = g$ and $||f_g|| \leq c.||g||$. Observe that for all $\overline{f} \in \Psi$ and $\overline{g} \in \Phi$ we have $f = \pi_{\overline{f}} \circ (\triangle\Psi)|X$ and $g = \pi_{\overline{g}} \circ (\triangle\Phi)|Y$. Therefore, there is a surjective map $\varphi : E_p(X_0) \to E_p(Y_0)$ defined by $\varphi(\pi_f) = \pi_{T(f)}$, where π_f and π_g denote, respectively, the functions $\pi_{\overline{f}}|X_0$ and $\pi_{\overline{g}}|Y_0$. Moreover $||f|| = ||\pi_f||$ and $||g|| = ||\pi_g||$ for all $\overline{f} \in \Psi$ and $\overline{g} \in \Phi \cap C(\beta Y)$. This implies that φ is a c-good surjection.

Let's show that φ is uniformly continuous. Suppose

$$V = (y_1, y_2, ..y_k, \varepsilon) = \{\pi_g : |\pi_g(y_i)| < \varepsilon\ \forall i\}$$

is a neighborhood of the zero function in $E_p(Y_0)$. Take points $\overline{y}_i \in Y$ with $h_0(\overline{y}_i) = y_i$, $i = 1, 2, .., k$, and let $\overline{V} = \{g \in D_p(Y) : |g(\overline{y}_i)| < \varepsilon\ \forall i\}$. Since T is uniformly continuous, there is a neighborhood

$$\overline{U} = (\overline{x}_1, .., \overline{x}_p, \delta) = \{f \in D_p(X) : |f(\overline{x}_j)| < \delta\ \forall j\}$$

of the zero function in $D_p(X)$ such that $f - f' \in \overline{U}$ implies $T(f) - T(f') \in \overline{V}$ for all $f, f' \in D_p(X)$. Let $x_j = (\triangle\Psi)(\overline{x}_j)$ and

$$U = (x_1, x_2, .., x_p, \delta) = \{\pi_f : |\pi_f(x_j)| < \delta\ \forall j\}.$$

Obviously, $\pi_f - \pi_{f'} \in U$ implies $f - f' \in \overline{U}$. Hence, $T(f) - T(f') \in \overline{V}$, which implies $\varphi(\pi_f) - \varphi(\pi_{f'}) \in V$.

Finally, we can show that $E(\overline{Y}_0)$ contains a QS-algebra on $\overline{Y}_0$. Since $\Lambda_n \subset C(Y'_n)$ is a QS-algebra on Y'_n, it separates the points and the closed sets in Y'_n. So, Y'_n is homeomorphic to $(\triangle\Lambda_n)(Y'_n)$. This implies that $Y_{n+1} = (\triangle\Phi_{n+1})(\beta Y)$ is homeomorphic to Y'_n. Therefore, Λ_n can be considered as a QS-algebra on Y_{n+1}. On the other hand $\overline{Y}_0$ is the limit of the inverse sequence $S_Y = \{Y_{n+1}, \gamma^{n+2}_{n+1}, n \geq 1\}$, where $\gamma^{n+2}_{n+1} : Y_{n+2} \to Y_{n+1}$ is the surjective map generated by the inclusion

$\Phi_{n+1} \subset \Phi_{n+2}$. According to condition (3.8), we can also suppose that Λ_{n+1} contains the family $\{\lambda \circ \gamma_{n+1}^{n+2} : \lambda \in \Lambda_n\}$. Denote by $\gamma_{n+1} : \overline{Y}_0 \to Y_{n+1}$ the projections in S_Y, and let $\Gamma_n = \{\lambda \circ \gamma_{n+1} : \lambda \in \Lambda_n\}$. Then $\{\Gamma_n\}_{n\geq 1}$ is an increasing sequence of countable families and $\Gamma_n \subset E(\overline{Y}_0) = \{\pi_{\overline{g}} : \overline{g} \in \Phi\}$ for every n. We claim that $\Gamma = \bigcup_n \Gamma_n$ is a QS-algebra on $\overline{Y}_0$. Indeed, Γ is closed under addition, multiplication and multiplication by rational numbers. Because the subbase of $\overline{Y}_0$ consists of open subsets of the form $\gamma_{n+1}^{-1}(V)$, where V is a basic open set in Y_{n+1} for some n, it follows that for every $y \in \overline{Y}_0$ and its neighborhood U there is $\overline{g} \in \Gamma$ with $\overline{g}(y) = 1$ and $\overline{g}(\overline{Y}_0 \backslash U) = 0$.

To prove Theorem 1.1, we apply Proposition 2.1 with $H = \overline{X}_0$ to find a σ-compact set $Y_\infty \subset \overline{Y}_0$ containing Y_0 with $\dim Y_\infty = 0$. Therefore, by [3, Proposition 1.2.2], $\dim Y_0 = 0$. □

Proof of Corollary 1.2. If $T : C_p^*(X) \to C_p^*(Y)$ is a continuous linear surjection, Corollary 1.2 follows from Theorem 1.1 and Proposition 3.3 below. If T maps $C_p^*(X)$ onto $C_p(Y)$, we consider the composition $T \circ i$, where $i : C_p(\beta X) \to C_p^*(X)$ is the restriction map. Then, by [20], Y is pseudocompact and we can apply the previous case.

Proposition 3.3. [13] *For every linear continuous surjective map $T : C_p^*(X) \to C_p^*(Y)$ there is $c > 0$ such that T is c-good.*

Proof. By the Closed Graph Theorem, T considered as a map between the Banach spaces $C_u^*(X)$ and $C_u^*(Y)$, both equipped with the sup-norm, is continuous. Then T induced a linear isomorphism T_0 between $C_u^*(X)/K$ and $C_u^*(Y)$, where K is the kernel of T. So, for every $g \in C_u^*(Y)$ we have $||T_0^{-1}(g)|| \leq ||T_0^{-1}||.||g||$. Because

$$||T_0^{-1}(g)|| = \inf\{||f - h|| : h \in K\},$$

where $f \in C_u^*(X)$ with $T(f) = g$, there exists $h_g \in K$ such that $||f - h_g|| \leq 2||T_0^{-1}(g)||$. Hence, $||f - h_g|| \leq 2||T_0^{-1}||.||g||$. Since $T(f - h_g) = T(f) = g$, we obtain that T is c-good with $c = 2||T_0^{-1}||$. □

Proof of Theorem 1.3. Following the proof of Theorem 1.1, we construct two sequences $\{\Psi_n\}_{n\geq 1} \subset C(\beta X)$ and $\{\Phi_n\}_{n\geq 1} \subset C(\beta Y, \overline{\mathbb{R}})$ of countable sets and countable QS-algebras Λ_n on $Y_n' = (\triangle\Phi_n')(\beta Y)$ satisfying the conditions (3.5) – (3.9) except (3.7). Because X and Y are separable metric spaces, we can choose countable sets Ψ_1 and Φ_1 such that $(\triangle\Phi_1)|Y$ and $(\triangle\Psi_1)|X$ are homeomorphisms.

Then, following the notations from the proof of Theorem 1.1, we have that X_0 and Y_0 are homeomorphic to X and Y, respectively. Moreover, there exists a uniformly continuous c-good surjection $\varphi : E_p(X_0) \to E_p(Y_0)$ such that $E(\overline{X}_0) \subset C(\overline{X}_0)$ and $E(\overline{Y}_0) \subset C(\overline{Y}_0, \overline{\mathbb{R}})$

such that $E(\overline{Y}_0)$ contains a countable QS-algebra Γ on $\overline{Y}_0$. So, we can apply Proposition 2.1 with $H = X_0$ to find a σ-compact set $Y_\infty \subset \overline{Y}_0$ containing Y_0 with $Y_\infty \in \mathcal{P}$. Let $Y_0 = \bigcup_m Y_m$ with each Y_m being compact. Because each Y_m is a closed subset of Y_∞, $Y_m \in \mathcal{P}$. So, $Y_0 \in \mathcal{P}$. □

Proof of Corollary 1.4. Corollary 1.4 follows directly from Theorem 1.3. □

4. Proof of Theorem 1.5

We consider topological properties $\mathcal{P}$ of separable metric spaces satisfying conditions $(b), (c)$ from the introduction section plus the following two:

- (a') If $X \in \mathcal{P}$, then $F \in \mathcal{P}$ for every subset $F \subset X$;
- (d') If $f : X \to Y$ is a perfect map between metric spaces with 0-dimensional fibers and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$.

The 0-dimensionality, the countable-dimensionality and the strong countable dimensionality satisfy these conditions.

Proposition 4.1. *Let X and Y be separable metric spaces and $E(X) \subset C(X)$ be a countable QS-algebra on X and $E(Y) \subset C(Y)$ be a countable family. Suppose there are metric compactifications $\overline{X}$ and $\overline{Y}$ of X and Y such that:*

- *Every $f \in E(X)$ can be extended to a map $\overline{f} : \overline{X} \to \overline{\mathbb{R}}$ and for every finite open cover $\gamma = \{U_i : i = 1, 2, .., k\}$ of basic open subsets of $\overline{X}$ there exists a partition of unity $\{\overline{f}_i : i = 1, 2, ., k\}$ subordinated to γ with $f_i \in E(X)$;*
- *Every $g \in E(Y)$ can be extended to a map $\overline{g} : \overline{Y} \to \overline{\mathbb{R}}$ and the set of all real-valued elements of $E_p(\overline{Y}) = \{\overline{g} : g \in E(Y)\}$ is dense in $C_p(\overline{Y})$;*
- *For every basic open subsets V, U of $\overline{X}$ with $\overline{V} \subset U$ there is a function $h_{V,U} \in E(X)$ such that $\overline{h}_{V,U}(\overline{V}) = 1$, $\overline{h}_{V,U}(\overline{X} \backslash U) = 0$ and $h_{V,U}(x) \in [0, 1]$ for all $x \in X$.*

If $\overline{X}$ has a property $\mathcal{P}$ satisfying conditions $(b), (c)$, (a') and (d'), and $\varphi : E_p(X) \to E_p(Y)$ is a linear continuous surjection, then $Y \in \mathcal{P}$.

Proof. Let $E(\overline{X}) = \{\overline{f} : f \in E(X)\}$. Every $y \in \overline{Y}$ generates a map $l_y : E(\overline{X}) \to \overline{\mathbb{R}}$, $l_y(\overline{f}) = \overline{\varphi(f)}(y)$. Assuming the equalities from $(*)$, for any $\overline{f}_1, \overline{f}_2 \in E(\overline{X})$ and $x \in \overline{X}$, we can write $\overline{f}_1(x) + \overline{f}_2(x)$ but not always $\overline{f}_1 + \overline{f}_2 = \overline{f_1 + f_2}$. Also, if $\overline{f}_1 + \overline{f}_2 \in E(\overline{X})$, it is possible $l_y(\overline{f}_1 + \overline{f}_2) \neq l_y(\overline{f}_1) + l_y(\overline{f}_2)$. If $\lambda \in \mathbb{R} \backslash \{0\}$ and $\overline{\lambda . f} \in E(\overline{X})$, then $l_y(\overline{\lambda . f}) = \lambda . l_y(\overline{f})$ (here, $\lambda . (\pm\infty) = \pm\infty$ if $\lambda > 0$ and $\lambda . (\pm\infty) = \mp\infty$ if

$\lambda < 0$). In case $\lambda = 0$, we have $l_y(\overline{0.f}) = 0$. More general, if $\overline{h} \in C(\overline{X})$ such that $h.f \in E(X)$ for some $f \in E(X)$, then $l_y(\overline{h.f})$ is well defined.

The *support of* l_y, $y \in \overline{Y}$, is defined to be the set $supp(l_y)$ of all $x \in \overline{X}$ satisfying the following condition [21]: for every neighborhood $U \subset \overline{X}$ of x there is $f \in E(X)$ such that $\overline{f}(\overline{X}\backslash U) = 0$ and $l_y(\overline{f}) \neq 0$. Obviously, $supp(l_y)$ is closed in $\overline{X}$, possibly empty.

Claim 9. *Let $U \subset \overline{X}$ be a neighborhood of $supp(l_y)$ and $f, g \in E(X)$ with $\overline{f}|U = \overline{g}|U$. Then $l_y(\overline{f}) = l_y(\overline{g})$. In particular, $\overline{f}(U) = 0$ implies $l_y(\overline{f}) = 0$.*

Suppose first that $\overline{f}(U) = 0$ for some $\overline{f} \in E(\overline{X})$. We can assume that U is a finite union of basic open sets V_i, $i = 1,..,k$. For every $x \in \overline{X}\backslash supp(l_y)$ there is a neighborhood V_x such that for each $g \in E(X)$ with $\overline{g}(\overline{X}\backslash V_x) = 0$ we have $l_y(\overline{g}) = 0$. Clearly, we can assume that each V_x is from the base of $\overline{X}$ and $\mathrm{Int}\overline{V}_x = V_x$. So, there is a finite open cover $\gamma = \{V_1,..,V_k,V_{x_1},..,V_{x_m}\}$ of $\overline{X}$ and a partition of unity $\{\overline{h}_1,.,\overline{h}_k,\overline{\theta}_1,..,\overline{\theta}_m\} \subset E(\overline{X})$ subordinated to γ. Then $h_i.f, \theta_j.f \in E(X)$ for all i, j and $f = \sum_{i=1}^k h_i.f + \sum_{j=1}^m \theta_j.f$. Take a sequence $\{y_n\} \subset Y$ with $\lim_n y_n = y$. Hence, $\varphi(f)(y_n) = \sum_{i=1}^k \varphi(h_i.f)(y_n) + \sum_{j=1}^m \varphi(\theta_j.f)(y_n)$. Observe that $(h_i.f)(x) = 0$ for all $x \in X$, so $\varphi(h_i.f)$ is the zero function on Y and $\varphi(h_i.f)(y_n) = 0$, $i = 1,..,k$. Since, $l_y(\overline{f}) = \lim_n \varphi(f)(y_n)$, we have

$$l_y(\overline{f}) = \sum_{j=1}^m \lim_n \varphi(\theta_j.f)(y_n) = \sum_{j=1}^m \lim_n l_{y_n}(\overline{\theta_j.f}) = \sum_{j=1}^m l_y(\overline{\theta_j.f}).$$

On the other hand, $(\theta_j.f)(x) = 0$ for all $x \in X\backslash V_{x_j}$. So, $(\overline{\theta_j.f})(x) = 0$ for all $x \in \overline{X}\backslash V_{x_j}$ because $X\backslash V_{x_j}$ is dense in $\overline{X}\backslash V_{x_j}$. This implies that $l_y(\overline{\theta_j.f}) = 0$ for all $j = 1,..,m$. Therefore, $l_y(\overline{f}) = 0$.

Suppose now that $\overline{f}|U = \overline{g}|U$ for some $f, g \in E(X)$. Then $f(x) - g(x) = 0$ for all $x \in U \cap X$. Consequently, $(\overline{f-g})(x) = 0$ for all $x \in U$ and, according to the previous paragraph, $l_y(\overline{f-g}) = 0$. Hence for every sequence $\{y_n\} \subset Y$ converging to y we have

$$l_y(\overline{f-g}) = \lim \varphi(f-g)(y_n) = \lim \varphi(f)(y_n) - \lim \varphi(g)(y_n) = l_y(\overline{f}) - l_y(\overline{g}).$$

Claim 10. *If $supp(l_{y_0}) \cap U \neq \varnothing$ for some open $U \subset \overline{X}$ and $y_0 \in \overline{Y}$, then y_0 has a neighborhood $V \subset \overline{Y}$ such that $supp(l_y) \cap U \neq \varnothing$ for every $y \in V$.*

Let $x_0 \in supp(l_{y_0}) \cap U$ and $\overline{f} \in E(\overline{X})$ be such that $\overline{f}(\overline{X}\backslash W) = 0$ and $l_{y_0}(\overline{f}) \neq 0$, where W is a neighborhood of x_0 with $\overline{W} \subset U$. Assuming the claim is not true we can find a sequence $\{y_n\} \subset \overline{Y}$ converging to y_0 such that $supp(l_{y_n}) \cap U = \varnothing$ for every n. Since $\overline{X} \backslash \overline{W}$ is a neighborhood of each $supp(l_{y_n})$, by Claim 9, $l_{y_n}(\overline{f}) = 0$. Because $\lim l_{y_n}(\overline{f}) = l_{y_0}(\overline{f})$, we have $l_{y_0}(\overline{f}) = 0$, a contradiction.

Following the notations from the proof of Proposition 2.1, for every $y \in \overline{Y}$ we define

$$a(y) = \sup\{|l_y(\overline{f})| : \overline{f} \in E(\overline{X}) \text{ and } |\overline{f}(x)| < 1 \ \forall x \in supp(l_y)\}.$$

Claim 11. *If $y \in Y$ then $supp(l_y) = \{x_1(y), x_2(y), .., x_q(y)\}$ is a finite subset of X. Moreover, there exist real numbers $\lambda_i(y)$, $i = 1, .., q$, such that $\sum_{i=1}^{q} |\lambda_i(y)| = a(y)$ and $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$ for all $\overline{f} \in E(\overline{X})$.*

Since φ is uniformly continuous, as in the proof of Claim 1 from Proposition 2.1, there exists a finite set $K = \{x_1(y), x_2(y), .., x_q(y)\} \subset X$ such that

$$\sup\{|l_y(\overline{f})| : \overline{f} \in E(\overline{X}) \text{ and } |\overline{f}(x)| < 1 \ \forall x \in K\} < \infty.$$

Because $l_y|E(X)$ is linear, we can show that $l_y(\overline{g}) = 0$ for any $g \in E(X)$ with $g(x_i(y)) = 0$, $i = 1, .., q$. Since $E(X)$ is a QS-algebra, for every i there is a function $g_i \in E(X)$ such that $g_i(x_i(y)) = 1$ and $g_i(x_j(y)) = 0$ with $j \neq i$. Now, for every $f \in E(X)$ the function $g = f - \sum_{i=1}^{q} g_i.f(x_i(y))$ belongs to $E(X)$ and $g(x_i(y)) = 0$ for all i, so $l_y(\overline{g}) = 0$. The last equality implies $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$ with $\lambda_i(y) = l_y(\overline{g}_i)$. Note that each $\lambda_i(y)$ is a real number because $l_y(\overline{g}_i) = \varphi(g_i)(y)$. The equality $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$ for all $\overline{f} \in E(\overline{X})$ shows that $K = supp(l_y)$.

To complete the proof of Claim 11, for every natural k take a function $f_k \in E(X)$ with $f_k(x_i(y)) = \varepsilon_i(1 - 1/k)$, where $\varepsilon_i = 1$ if $\lambda_i(y) > 0$ and $\varepsilon_i = -1$ if $\lambda_i(y) < 0$. Clearly, $|f_k(x_i(y))| < 1$ for all i, k and $\lim_k l_y(\overline{f}_k) = \sum_{i=1}^{q} |\lambda_i(y)|$. Hence $\sum_{i=1}^{q} |\lambda_i(y)| \leq a(y)$. The reverse inequality $a(y) \leq \sum_{i=1}^{q} |\lambda_i(y)|$ follows from the equality $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$, $\overline{f} \in E(\overline{X})$.

For every $p, q \in \mathbb{N}$ let $Y_{p,q} = \{y \in \overline{Y} : |supp(l_y)| \leq q \text{ and } a(y) \leq p\}$.

Claim 12. *Every set $Y_{p,q}$ is closed in $\overline{Y}$.*

Let $\{y_n\}$ be a sequence in $Y_{p,q}$ converging to $y \in \overline{Y}$. Suppose $y \notin Y_{p,q}$. Then either $supp(l_y)$ contains at least $q + 1$ points or $a(y) > p$. If $supp(l_y)$ contains at least $q + 1$ points $x_1, x_2, .., x_{q+1}$, we choose disjoint neighborhoods O_i of x_i, $i = 1, .., q + 1$. By Claim 10, there is a neighborhood V of y such that $supp(l_z) \cap O_i \neq \varnothing$ for all i and $z \in V$.

This implies that $supp(l_{y_n})$ has at least $q+1$ points for infinitely many n's, a contradiction.

If $y \notin Y_p$ then there exists $\overline{f} \in E(\overline{X})$ such that $|\overline{f}(x)| < 1$ for all $x \in supp(l_y)$ and $|l_y(\overline{f})| > p$. Take a neighborhood U of $supp(l_y)$ with $U \subset \{x \in \overline{X} : |\overline{f}(x)| < 1\}$ and $\mathrm{Int}\overline{U} = U$. Choose another neighborhood W of $supp(l_y)$ with $\overline{W} \subset U$. Next, there is $h \in E(X)$ such that $\overline{h}(\overline{W}) = 1$, $\overline{h}(\overline{X}\backslash U) = 0$ and $h(x) \in [0,1]$ for all $x \in X$. Then $\overline{g} = \overline{h.f} \in E(\overline{X})$ and $|\overline{g}(x)| < 1$ for all $x \in \overline{X}$. Moreover, $\overline{g}|W = \overline{f}|W$. So, by Claim 9, $|l_y(\overline{g})| = |l_y(\overline{f})| > p$. Therefore, $V = \{z \in \overline{Y} : |l_z(\overline{g})| > p\}$ is a neighborhood of y in $\overline{Y}$ with $V \cap Y_p = \varnothing$, a contradiction.

According to Claim 11, for every $y \in \overline{Y}$ then there exist $p, q \in \mathbb{N}$ and real numbers $\lambda_i(y)$ such that for all $\overline{f} \in E(\overline{X})$ we have $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$ with $\sum_{i=1}^{q} |\lambda_i(y)| \leq p$, where $\{x_1(y), .., x_q(y)\} = supp(l_y)$. Hence,$Y \subset \bigcup\{Y_{p,q} : p, q \in \mathbb{N}\}$. For every $p \geq 1$ and $q \geq 2$ we define

$$M(p,1) = Y_{p,1} \text{ and } M(p,q) = Y_{p,q}\backslash Y_{2p,q-1}.$$

Some of the sets $M(p,q)$ could be empty, but $Y \subset \bigcup\{M(p,q) : p, q = 1, 2, ..\}$. Since each $Y_{p,q}$ is closed, $M(p,q) = \bigcup_{n=1}^{\infty} F_n'(p,q)$ such that each $F_n'(p,q)$ is a compact subset of $\overline{Y}$. We define $F_n(p,q) = \overline{Y \cap F_n'(p,q)}$. Then $Y \subset \bigcup\{F_n(p,q) : n, p, q = 1, 2, ,\}$. Obviously, $supp(l_y)$ consists of q different points for any $y \in M(p,q)$. So, we have a map $S_{p,q} : M(p,q) \to [\overline{X}]^q$, $S_{p,q}(y) = supp(l_y)$. According to Claim 10, $S_{p,q}$ is continuous when $[\overline{X}]^q$ is equipped with the Vietoris topology. For every $y \in M(p,q)$ let $S_{p,q}(y) = \{x_i(y)\}_{i=1}^{q}$. Everywhere below we consider the restriction $S_{p,q}|F_n(p,q)$ and for every $z \in F_n(p,q)$ denote by $A(z) = \{y \in F_n(p,q) : S_{p,q}(y) = S_{p,q}(z)\}$ the fiber $S_{p,q}^{-1}(S_{p,q}(z))$ generated by z. Since $F_n(p,q)$ is compact and $S_{p,q}$ is continuous, each $A(z)$ is a compact subset of $F_n(p,q)$.

Claim 13. *Let $z \in Y \cap F_n(p,q)$. Then for every $y \in A(z)$ there are real numbers $\{\lambda_i(y)\}_{i=1}^{q}$ such that $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(z))$, where $S_{p,q}(z) = \{x_i(z)\}_{i=1}^{q}$, and $\sum_{i=1}^{q} |\lambda_i(y)| \leq p$ for all $\overline{f} \in E(\overline{X})$. Moreover, each λ_i is a continuous real-valued function on $A(z)$.*

Choose disjoint neighborhoods O_i of $x_i(z)$ in $\overline{X}$ and functions $\overline{g}_i \in E(\overline{X})$ with $\overline{g}_i|O_i = 1$ and $\overline{g}_i|O_j = 0$ if $j \neq i$ (this can be done by choosing $g_i \in E(X)$ with $g_i(O_i \cap X) = 1$ and $g_i(O_j \cap X) = 0$ when $j \neq i$). According to the proof of Claim 11, for every $y \in A(z) \cap Y$ and the real numbers $\lambda_i(y) = l_y(\overline{g}_i)$ we have $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(z))$ for all $\overline{f} \in E(\overline{X})$. Let's show this is true for all $y \in A(z)$. So, fix $y \in A(z)\backslash Y$ and take a sequence $\{y_m\} \subset Y \cap F_n(p,q)$ converging to

y. Then, by Claim 11, $S_{p,q}(y_m) = \{x_i(y_m)\}_{i=1}^q \subset X$ and there are real numbers $\{\lambda_i(y_m)\}_{i=1}^q$ such that $l_{y_m}(\overline{f}) = \sum_{i=1}^q \lambda_i(y_m)\overline{f}(x_i(y_m))$ for all $\overline{f} \in E(\overline{X})$. On the other hand, since the map $S_{p,q}$ is continuous, each sequence $\{x_i(y_m)\}_{m=1}^\infty$ converges to $x_i(y) = x_i(z)$, $i = 1, .., q$. So, we can assume that $\{x_i(y_m)\}_{m=1}^\infty \subset O_i$. Consequently, $l_{y_m}(\overline{g}_i) = \lambda_i(y_m)$ and $\lim_m \lambda_i(y_m) = l_y(\overline{g}_i)$. Since $\{y_m\} \subset Y \cap Y_{p,q}$, $\sum_{i=1}^q |\lambda_i(y_m)| \leq p$ for each m (see Claim 11). Hence, $\sum_{i=1}^q |l_y(\overline{g}_i)| \leq p$. Denoting $\lambda_i(y) = l_y(\overline{g}_i)$, we obtain $\sum_{i=1}^q |\lambda_i(y)| \leq p$. The last inequality means that all $\lambda_i(y)$ are real numbers. Since $\lim_m \overline{f}(x_i(y_m)) = \overline{f}(x_i(y))$ for all $\overline{f} \in E(\overline{X})$ and each $\overline{f}(x_i(y))$ is a real number (recall that $z \in Y$, so by Claim 11, $x_i(z) = x_i(y) \in X$), we have $l_y(\overline{f}) = \lim l_{y_m}(\overline{f}) = \sum_{i=1}^q \lambda_i(y)\overline{f}(x_i(y))$.

Finally, the equality $\lambda_i(y) = \overline{\varphi(g_i)}(y)$ implies that λ_i are continuous on $A(z)$.

Claim 14. *Let $A(z) = \{y \in F_n(p,q) : S_{p,q}(y) = S_{p,q}(z)\}$ with $z \in Y \cap F_n(p,q)$. Then there is a linear continuous map $\varphi_z : C_p(S_{p,q}(z)) \to C_p(A(z))$ such that $\varphi_z(C(S_{p,q}(z)))$ is dense in $C_p(A(z))$.*

Following the previous notations, for every $h \in C(S_{p,q}(z))$ and $y \in A(z)$ we define $\varphi_z(h)(y) = \sum_{i=1}^q \lambda_i(y)h(x_i(z))$. Because λ_i are continuous real-valued functions on $A(z)$, so is each $\varphi_z(h)$. Continuity of φ_z with respect to the pointwise convergence topology is obvious. Let's show that $\varphi_z(C(S_{p,q}(z)))$ is dense in $C_p(A(z))$. Indeed, take $\theta \in C_p(A(z))$ and its neighborhood $V \subset C_p(A(z))$. Then extend θ to a function $\overline{\theta} \in C(\overline{Y})$. Because the set of real-valued elements of $E_p(\overline{Y})$ is dense in $C_p(\overline{Y})$, there is $\overline{g} \in E_p(\overline{Y})$ with $\overline{g}|A(z) \in V$, so $\overline{g}(y) \in \mathbb{R}$ for all $y \in \overline{Y}$. Next, choose $f \in E(X)$ such that $\varphi(f) = g$. Since $z \in Y$, each $x_i(z) \in X$. So, all $\overline{f}(x_i(z))$ are real numbers. Then $h = \overline{f}|S_{p,q}(z) \in C(S_{p,q}(z))$ and, according to Claim 13, we have $\varphi_z(h) = \overline{g}|A(z)$.

Claim 15. *The fibers $A(z)$ of the map $S_{p,q} : F_n(p,q) \to [\overline{X}]^q$ are 0-dimensional for all $z \in Y \cap F_n(p,q)$.*

Since, by Claim 14, there is a linear continuous map $\varphi_z : C_p(S_{p,q}(z)) \to C_p(A(z))$ such that $\varphi_z(C(S_{p,q}(z)))$ is dense in $C_p(A(z))$, this claim follows from [10, Proposition 2.1].

Now, we can complete the proof of Proposition 4.1. Every set $F_n(p,q)$ is compact, so is $S_{p,q}(F_n(p,q)) = \overline{X}_{n,p,q}$ in $\overline{X}^q$. Because $\overline{X}^q \in \mathcal{P}$, $X_{n,p,q} = S_{p,q}(Y \cap F_n(p,q)) \in \mathcal{P}$. Since $S_{p,q} : S_{p,q}^{-1}(X_{n,p,q}) \to X_{n,p,q}$ is a perfect map having 0-dimensional fibers, $S_{p,q}^{-1}(X_{n,p,q}) \in \mathcal{P}$. Observe that $S_{p,q}^{-1}(X_{n,p,q}) \subset F_n(p,q)$ and contains $Y \cap F_n(p,q)$ (recall that $F_n(p,q)$ is the closure of $Y \cap F_n(p,q)$). Hence, $Y \cap F_n(p,q) \in \mathcal{P}$. Finally, since each $Y \cap F_n(p,q) \in \mathcal{P}$ is closed in Y and $Y = \bigcup_{n,p.q} Y \cap F_n(p,q)$, $Y \in \mathcal{P}$. $\square$

Lemma 4.2. *For every countable set $\Phi' \subset C(\beta Y, \overline{\mathbb{R}})$ there is a countable set $\Phi \subset C(\beta Y, \overline{\mathbb{R}})$ containing Φ' such that $(\triangle\Phi)(\beta Y)$ is homeomorphic to $(\triangle\Phi')(\beta Y)$ and the set of real-valued elements of $E_\Phi = \{\pi_{\overline{g}} : \overline{g} \in \Phi\}$ is dense in $C_p((\triangle\Phi)(\beta Y))$.*

Proof. Let $\phi' = \triangle\Phi'$. Since Φ' is countable, $\phi'(\beta Y)$ is a metric compactum. Hence, by [7, Proposition 1.2], there is a countable QS-algebra $E \subset C(\phi'(\beta Y))$. Let $\Phi = \Phi' \cup \{g \circ \phi' : g \in E\}$. Since the functions of E separate the points and closed subsets of $\phi'(\beta Y)$, $(\triangle\Phi)(\beta Y)$ is homeomorphic to $\phi'(\beta Y)$. Since E is a QS-algebra on $\phi'(\beta Y)$, E is a dense subset of $C_p(\phi'(\beta Y))$. Clearly E is a subset of E_Φ and consists of real-valued functions. $\square$

Lemma 4.3. *Let X be a 0-dimensional space and $\Psi' \subset C(X)$ be a countable set. Then there is a countable admissible set $\Psi \subset C(X)$ containing Ψ' and a 0-dimensional metric compactification $\overline{X}_\Psi$ of $X_\Psi = (\triangle\Psi)(X)$ such that:*

- *$\overline{X}_\Psi = (\triangle\overline{\Psi})(\beta X)$, where $\overline{\Psi} = \{\overline{f} : f \in \Psi\} \subset C(\beta X, \overline{\mathbb{R}})$;*
- *Each π_f, $f \in \Psi$, is extendable to a map $\overline{\pi}_f : \overline{X}_\Psi \to \overline{\mathbb{R}}$;*
- *$E(X_\Psi) = \{\pi_f : f \in \Psi\}$ is a countable QS-algebra on X_Ψ and $E(\overline{X}_\Psi) = \{\overline{\pi}_f : f \in \Psi\}$ contains a countable QS-algebra on $\overline{X}_\Psi$;*
- *For every finite open cover γ of $\overline{X}_\Psi$ of basic open sets, the family $E(\overline{X}_\Psi)$ contains a partition of unity subordinated to γ.*

Proof. We first choose a countable admissible set $\Psi_0 \subset C(X)$ such that $\dim(\triangle\Psi_0)(X) = 0$ and $\Psi' \subset \Psi_0$, see (3.2) – (3.3). Then, by Lemma 3.2 there is a metric compactification Z_0 of $X_0 = (\triangle\Psi_0)(X)$ with $\dim Z_0 = 0$ such that each π_f, $f \in \Psi_0$, is extendable to a map $\overline{\pi}_f : Z_0 \to \overline{\mathbb{R}}$. Choose a countable QS-algebra $C_0 \subset C(Z_0)$. For every finite open cover γ of Z_0 consisting of basic open sets fix a partition of unity α_γ subordinated to γ. Because the family Ω_0 of all finite open covers of Z_0 consisting of basic open sets is countable, so is the family $E(Z_0) = \{\overline{f} : f \in \Psi_0\} \cup \{\alpha_\gamma : \gamma \in \Omega_0\} \cup C_0$. The set $E_0 = \{h|X_0 : h \in E(Z_0)\}$ may not be a QS-algebra on X_0 but, according to [7, Proposition 1.2], there exists a countable QS-algebra Θ_1 on X_0 containing E_0 as a proper subset. Because $\{h|X_0 : h \in C_0\} \subset \Theta_1$ and it separates the points and the closed sets of X_0, there is a metric compactification Z_1 of X_0 such that $\dim Z_1 = 0$ and each $h \in \Theta_1$ is extendable to a map $\overline{h} : Z_1 \to \overline{\mathbb{R}}$, and a map $\theta_0^1 : Z_1 \to Z_0$ which is identity on X_0, see the proof of

Lemma 3.2. Next, consider the family Ω_1 of all finite open covers of Z_1 consisting of basic open sets, and for each $\gamma \in \Omega_1$ fix a partition of unity α_γ subordinated to γ. Let $E(Z_1) = \{\overline{h} : h \in \Theta_1\} \cup \{\alpha_\gamma : \gamma \in \Omega_1\} \cup C_1$ and $E_1 = \{f|X_0 : f \in E(Z_1)\}$, where C_1 is a countable QS-algebra of Z_1 such that $\{h \circ \theta_0^1 : h \in C_0\} \subset C_1$. Continuing in this way we construct by induction an increasing sequence $\{\Theta_n\}$ of countable QS-algebras on X_0, 0-dimensional metric compactifications Z_n of X_0 and countable families $E(Z_n) \subset C(Z_n, \overline{\mathbb{R}})$ such that:

- For every finite open cover γ of Z_n consisting of basic open sets, the family $E(Z_n)$ contains a partition of unity α_γ subordinated to γ;
- For every n there is a map $\theta_{n-1}^n : Z_n \to Z_{n-1}$ fixing the points of X_0;
- Each $h \in \Theta_n$ is extendable to a map $\overline{h} \in C(Z_n, \overline{\mathbb{R}})$ and let $E(Z_n) = \{\overline{h} : h \in \Theta_n\} \cup \{\alpha_\gamma : \gamma \in \Omega_n\} \cup C_n$. Here, Ω_n is the family of all finite open covers of Z_n consisting of basic open sets and $C_n \subset C(Z_n)$ is a countable QS-algebra on Z_n with $\{h \circ \theta_{n-1}^n : h \in C_{n-1}\} \subset C_n$;
- The family $E_n = \{f|X_0 : f \in E(Z_n)\}$ is contained in Θ_{n+1}.

Clearly, $\Theta = \bigcup_n \Theta_n$ is a countable QS-algebra on X_0 and the limit space Z of the inverse sequence $S = \{Z_n, \theta_{n-1}^n\}$ is a 0-dimensional compactification of X_0. Moreover, each $h \in \Theta$ is extendable to a map $\overline{h} : Z \to \overline{\mathbb{R}}$. Indeed, if $h \in \Theta_n$, then h can be extended to a map $\overline{h} : Z_n \to \overline{\mathbb{R}}$. So, $\overline{h} \circ \theta_n$ is an extension of h over Z, where $\theta_n : Z \to Z_n$ denotes the n-th limit projection in S (recall that θ_n is fixing the points of X_0). Denote $C'_n = \{h \circ \theta_n : h \in C_n\}$, $n \geq 0$. Because $\{h \circ \theta_{n-1}^n : h \in C_{n-1}\} \subset C_n$, the sequence $\{C'_n\}$ is increasing and $C = \bigcup_n C'_n$ is a countable QS-algebra on Z. Let's show that for any finite open cover γ of Z consisting of basic open sets, the set $E(Z) = \{\overline{h} : h \in \Theta\}$ contains a partition of unity subordinated to γ. Indeed, for any such a cover $\gamma = \{U_1, .., U_k\}$ there is n and a cover $\gamma_n = \{U_1^n, .., U_k^n\} \in \Omega_n$ such that $U_i = \theta_n^{-1}(U_i^n)$. So, there is a partition of unity $\alpha_{\gamma_n} = \{h_i^n : i = 1, .., k\}$ subordinated to γ_n with $\alpha_{\gamma_n} \subset E(Z_n)$. Since $\alpha_{\gamma_n} \subset E(Z_n)$, $h_i^n|X_0 \in \Theta_{n+1}$ for each i. Then $\{h_i^n \circ \theta_n : i = 1, ..., k\}$ is a partition of unity subordinated to γ and it is contained in $E(Z)$. Finally, let $\Psi = \{h \circ \triangle\Psi_0 : h \in \Theta\}$, $\overline{\Psi} = \{\overline{f} : f \in \Psi\} \subset C(\beta X, \overline{\mathbb{R}})$ and $\overline{X}_\Psi = (\triangle\overline{\Psi})(\beta X)$. Since C is a QS-algebra on Z, it separates the points and the closed sets in Z. Moreover, $C \subset E(Z)$, which means that $\overline{X}_\Psi$ is homeomorphic to Z. □

Proof of Theorem 1.5. Let $T : C_p(X) \to C_p(Y)$ be a continuous linear surjection and $\dim X = 0$. We fix $h \in \mathcal{F}_Y$ and let find $h_0 \in \mathcal{F}_Y$ with

$h_0 \succ h$ and $\dim h_0(Y) = 0$. Following the proof of Theorem 1.1 and using Lemmas 4.2-4.3, we are constructing two increasing sequences of countable sets $\{\Psi_n\} \subset C(X)$, $\{\Phi_n\} \subset C(Y)$ and metric compactifications $\overline{X}_n$ and $\overline{Y}_n$ of the spaces $X_n = (\triangle\Psi_n)(X)$ and $Y_n = (\triangle\Phi_n)(Y)$ satisfying the following conditions (everywhere below, if $f \in C(X)$ then $\overline{f} : \beta X \to \overline{\mathbb{R}}$ denotes its extension):

- (4.1) $\triangle\Phi_1 \succ h$, $\Phi_n \subset \{T(f) : f \in \Psi_n\} \subset \Phi_{n+1}$ and $\Psi_n \subset \Psi_{n+1}$;
- (4.2) Ψ_n is admissible, $\dim \overline{X}_n = 0$ and $\overline{X}_n = (\triangle\overline{\Psi}_n)(\beta X)$ with $\overline{\Psi}_n = \{\overline{f} : f \in \Psi_n\}$;
- (4.3) Each π_f, $f \in \Psi_n$, is extendable to a map $\overline{\pi}_f : \overline{X}_n \to \overline{\mathbb{R}}$;
- (4.4) $E(\overline{X}_n) = \{\overline{\pi}_f : f \in \Psi_n\}$ contains a countable QS-algebra $C_n \subset C(\overline{X}_n)$ on $\overline{X}_n$;
- (4.5) For every finite open cover γ of $\overline{X}_n$, consisting of basic open sets, there exists a partition of unity α_γ subordinated to γ with $\alpha_\gamma \subset E(\overline{X}_n)$;
- (4.6) Every π_g, $g \in \Phi_n$, is extendable to a map $\overline{\pi}_g : \overline{Y}_n \to \overline{\mathbb{R}}$;
- (4.7) The set of real-valued functions from $E(\overline{Y}_n) = \{\overline{\pi}_g : g \in \Phi_n\}$ is dense in $C_p(\overline{Y}_n)$.

Since $h(Y)$ is a separable metric space, there is a countable set $\Phi_1' \subset C(Y)$ with $h = \triangle\Phi_1'$. Let $\overline{\Phi_1'} = \{\overline{g} : g \in \Phi_1'\} \subset C(\beta Y, \overline{\mathbb{R}})$. By Lemma 4.2, there is a countable set $\overline{\Phi}_1 \subset C(\beta Y, \overline{\mathbb{R}})$ containing $\overline{\Phi_1'}$ such that $(\triangle\overline{\Phi}_1)(\beta Y)$ is homeomorphic to $(\triangle\overline{\Phi_1'})(\beta Y)$ and $\{\pi_{\overline{g}} : \overline{g} \in \overline{\Phi}_1\}$ contains a dense subset of $C_p(\overline{Y}_1)$, where $\overline{Y}_1 = (\triangle\overline{\Phi}_1)(\beta Y)$. Let $\Phi_1 = \{g : \overline{g} \in \overline{\Phi}_1\}$, $Y_1 = (\triangle\Phi_1)(Y)$ and $E(\overline{Y}_1) = \{\overline{\pi}_g : g \in \Phi_1\}$. So, Φ_1 satisfies conditions $(4.6)-(4.7)$. Next, choose a countable set $\Psi_1' \subset C(X)$ with $T(\Psi_1') = \Phi_1$ and apply Lemma 4.3 to find a countable admissible set Ψ_1 containing Ψ_1' and a metric compactification $\overline{X}_1$ of $X_1 = \triangle\Psi_1(X)$ satisfying conditions $(4.2)-(4.5)$.

Suppose the construction is done for all $k \leq n$. Let $\Phi_{n+1}' \subset C(Y)$ be a countable set containing $T(\Psi_n)$ and denote $\overline{\Phi'}_{n+1} = \{\overline{g} : g \in \Phi_{n+1}'\} \subset C(\beta Y, \overline{\mathbb{R}})$. By Lemma 4.2, there is a countable set $\overline{\Phi}_{n+1} \subset C(\beta Y, \overline{\mathbb{R}})$ containing $\overline{\Phi'}_{n+1}$ such that $(\triangle\overline{\Phi}_{n+1})(\beta Y)$ is homeomorphic to $(\triangle\overline{\Phi'}_{n+1})(\beta Y)$ and $\{\pi_{\overline{g}} : \overline{g} \in \overline{\Phi}_{n+1}\}$ contains a dense subset of $C_p(\overline{Y}_{n+1})$, where $\overline{Y}_{n+1} = (\triangle\overline{\Phi}_{n+1})(\beta Y)$. Let $\Phi_{n+1} = \{g : \overline{g} \in \overline{\Phi}_{n+1}\}$ and $Y_{n+1} = (\triangle\Phi_{n+1})(Y)$. Note that $\Phi_n \subset \Phi_{n+1}$ because $\Phi_n \subset T(\Psi_n)$. Next, choose a countable set $\Psi_{n+1}' \subset C(X)$ with $T(\Psi_{n+1}') = \Phi_{n+1}$ and apply Lemma 4.3 to find a countable admissible set Ψ_{n+1} containing $\Psi_{n+1}' \cup \Psi_n$ and a metric compactification $\overline{X}_{n+1}$ of $X_{n+1} = (\triangle\Psi_{n+1})(X)$ satisfying conditions $(4.1)-(4.5)$. This completes the induction.

As in the proof of Theorem 1.1, we denote $\overline{X}_0 = (\triangle\overline{\Psi})(\beta X)$, $\overline{Y}_0 = (\triangle\overline{\Phi})(\beta Y)$ and $h_0 = \triangle\Phi$, where $\Psi = \bigcup_n \Psi_n$ and $\Phi = \bigcup_n \Phi_n$. Clearly, $\Phi = \{T(f) : f \in \Psi\}$. Since $\overline{X}_0$ is the limit space of the inverse sequence $\{\overline{X}_n, \theta_n^{n+1}\}$, where $\theta_n^{n+1} : \overline{X}_{n+1} \to \overline{X}_n$, and $\dim \overline{X}_n = 0$, $\dim \overline{X}_0 = 0$. Moreover, $E(X_0) = \{\pi_f : f \in \Psi\}$ is a countable QS-algebra on X_0 such that every π_f is extendable to a continuous map $\overline{\pi}_f : \overline{X}_0 \to \overline{\mathbb{R}}$. Denote $E(\overline{X}_0) = \{\overline{\pi}_f : f \in \Psi\}$. Let show that for every finite open cover γ of $\overline{X}_0$, consisting of basic open sets, there exists a partition of unity α_γ subordinated to γ with $\alpha_\gamma \subset E(\overline{X}_0)$. Indeed, if $\gamma = \{U_1, .., U_k\}$ is such a cover of $\overline{X}_0$, then there is n with $U_i = \theta_n^{-1}(V_i)$ such that each V_i is from the base of $\overline{X}_n$, where $\theta_n : \overline{X}_0 \to \overline{X}_n$ is the limit projection. So, $E(\overline{X}_n)$ contains a partition of unity α_{γ_n} subordinated to the cover $\gamma_n = \{V_1, .., V_k\}$ of $\overline{X}_n$. Since $E(\overline{X}_0) = \bigcup_n \theta_n^*(E(\overline{X}_n))$, where $\theta_n^* : C(\overline{X}_n) \to C(\overline{X}_0)$ is dual map generated by θ_n, $\alpha_\gamma = \{h \circ \theta_n : h \in \alpha_{\gamma_n}\}$ is a partition of unity subordinated to γ and $\alpha_\gamma \in E(\overline{X}_0)$.

We also need to show that for every basic open sets U, V in $\overline{X}_0$ with $\overline{V} \subset U$ there is a function $h_{V,U} \in E(\overline{X}_0)$ such that $h_{V,U}(\overline{V}) = 1$, $h_{V,U}(\overline{X}_0 \backslash U) = 0$ and $h_{V,U}(x) \in [0, 1]$ for all $x \in \overline{X}_0$. We can suppose that every $\overline{X}_n$, $n \geq 0$, has a finitely additive base. Then there is n such that $V = \theta_n^{-1}(V_n)$ and $U = \theta_n^{-1}(U_n)$ such that V_n, U_n are basic open subsets of $\overline{X}_n$ with $\overline{V}_n \subset U_n$. Because $E(\overline{X}_n)$ contains a QS-algebra on $\overline{X}_n$, there is $h_n \in E(\overline{X}_n)$ such that $h_n(\overline{V}_n) = 1$, $h_n(\overline{X}_n \backslash U_n) = 0$ and $h_n(x) \in [0, 1]$ for all $x \in \overline{X}_n$, see (2.3). Then $h_{V,U} = h_n \circ \theta_n$ is the required function.

Let $E(Y_0) = \{\pi_g : g \in \Phi\}$ and $E(\overline{Y}_0) = \{\overline{\pi}_g : g \in \Phi\}$. Because T is linear, so is the map $\varphi : E_p(X_0) \to E_p(Y_0)$ defined by $\varphi(\pi_f) = \pi_{T(f)}$. The arguments from the proof of Theorem 1.1 show that φ is continuous. We claim that the set of real-valued elements of $E_p(\overline{Y}_0)$ is dense in $C_p(\overline{Y}_0)$. Since $\Phi_n \subset \Phi_{n+1}$, for every n there is a continuous map $\delta_n^{n+1} : \overline{Y}_{n+1} \to \overline{Y}_n$ such that $\overline{Y}_0$ is the limit space of the inverse sequence $\{\overline{Y}_n, \delta_n^{n+1}\}$. Consider the natural projections $\delta_n : \overline{Y}_0 \to \overline{Y}_n$ and their duals $\delta_n^* : C_p(\overline{Y}_n) \to C_p(\overline{Y}_0)$. One can show that $\bigcup_n \delta_n^*(C_p(\overline{Y}_n))$ is dense in $C_p(\overline{Y}_0)$. Because the set of real-valued functions from $E(\overline{Y}_n) = \{\overline{\pi}_g : g \in \Phi_n\}$ is dense in $C_p(\overline{Y}_n)$ and $E_p(\overline{Y}_0) = \bigcup_n \delta_n^*(E(\overline{Y}_n))$, we conclude that the set of real-valued functions from $E_p(\overline{Y}_0)$ is dense in $C_p(\overline{Y}_0)$.

Therefore, the spaces X_0, $\overline{X}_0$, Y_0 and $\overline{Y}_0$ satisfy the hypothese of Proposition 4.1. Now, we apply Proposition 4.1 with $X = X_0$ and $Y = Y_0$ to conclude that $\dim Y_0 = 0$. That completes the proof of Theorem 1.5. $\square$

Acknowledgments. The authors express their gratitude to Prof. A. Leiderman for his support and providing us with the paper of Gorak-Krupski-Marciszewski [6].

References

[1] A. Arkhangel'skii, *Problems in C_p-theory*, in J. van Mill and G. M. Reed eds., Open Problems in Topology, North-Holland (1990), 601–615.

[2] R. Cauty, *Sur l'invariance de la dimension infinite forte par t-équivalence*, Fund.Math. 160 (1999), 95–100.

[3] R. Engelking, *Theory of dimensions finite and infinite*, Sigma Series in Pure Mathematics, 10. Heldermann Verlag, Lemgo, 1995.

[4] V. Fedorchuk, *Some classes of weakly infinite-dimensional spaces*, J. Math. Sciences 155 (2008), no 4, 523–570.

[5] P. Gartside and Z. Feng, *Spaces l-dominated by $\mathbb{I}$ or $\mathbb{R}$*, Topology Appl. 219 (2017), 1–8.

[6] R. Gorak, M. Krupski and W. Marciszewski, *On uniformly continuous maps between function spaces*, Fund. Math. 246 (2019), no. 3, 257–274.

[7] S. Gul'ko, *On uniform homeomorphisms of spaces of continuous functions*, Trudy Mat. Inst. Steklov, 193 (1992), 82–88 (in Russian): English translation: Proc. Steklov Inst. Math. 193 (1992), 87–93.

[8] V. Gutev and V. Valov, *Continuous selections and C-spaces*, Proc. Amer. Math. Soc. 130 (2002), 233–242.

[9] Y. Hattori and K. Yamada, *Closed pre-images of C-spaces*, Math. Japonica 34 (1989), 555–561.

[10] K. Kawamura and A. Leiderman, *Linear continuous surjections of C_p-spaces over compacta*, Topology Appl. 227 (2017), 135–145.

[11] M. Krupski, *On κ-pseudocompactess and uniform homeomorphisms of function spaces*, Results Math. 78 (2023), no. 4, Paper No. 154, 11 pp.

[12] M. Krupski, *On the t-equivalence relation*, Topology Appl. 160 (2013), 368–373.

[13] A. Leiderman, A private mail, December 2023.

[14] A. Leiderman, M. Levin and V. Pestov, *On linear continuous open surjections on the spaces $C_p(X)$*, Topology Appl. 81 (1997), 269–279.

[15] A. Leiderman, S. Morris and V. Pestov, *The free abelian topological group and the free locally convex space on the unit interval*, J. London Math. Soc. 56 (1997), 529–538.

[16] W. Marciszewski and J. Pelant, *Absolute Borel sets and function spaces*, Trans. Amer. Math. Soc. 349 (1887), 3585–3596.

[17] V. Pestov, *The coincidence of the dimension* dim *of l-equivalent topological spaces*, Soviet Math. Dokl. 26 (1982), 380–383.

[18] R. Pol, *On light mappings without perfect fibers on compacta*, Tsukuba J. Math. 20 (1996), 11–19.

[19] V. Tkachuk, *Cp-theory problem book. Topological and function spaces. Problem Books in Mathematics*, Springer, New York, 2011

[20] V. Uspenskii, *A characterization of compactness in terms of uniform structure in a function space*, Uspehi Mat.Nauk 37 (1982), 183-184 (in Russian).

[21] V. Valov and D. Vuma, *Function spaces and Diedonné completeness*, Quaestiones Math. 21 (1998), no 3-4, 303–309.

Department of Mathematics, Faculty of Science, Trakya University, Edirne, Turkey
Email address: aemreeysen@trakya.edu.tr

Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada
Email address: veskov@nipissingu.ca