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LOCAL STRUCTURE OF HOMOGENEOUS ANR-SPACES

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Dedicated to Prof. Georgi Dimov on the occasion of his 75th birthday

Abstract. We investigate to what extend finite-dimensional homogeneous locally compact ANR-spaces have common properties with topological manifolds. Specially, the local structure of homogeneous ANR-spaces is described. Using that description, we provide a positive solution of the problem whether every finite-dimensional homogeneous metric ANR-compactum X is dimensionally full-valued, i.e. $\dim X \times Y = \dim X + \dim Y$ for any metric compactum Y.

1. Introduction

By a *space* we mean a locally compact separable metric space, and *maps* are continuous mappings. Reduced Čech homology $H_n(X;G)$ and cohomology groups $H^n(X;G)$ with coefficient from an abelian group G are considered everywhere below. A space is said to be *homogeneous* provided that, for every two points $x, y \in X$ there exists a homeomorphism h mapping X onto itself with $h(x) = y$. Homogeneity of X implies *local homogeneity*, that is for every $x, y \in X$ there exists a homeomorphism h mapping a neighborhood of x onto a neighborhood of y such that $h(x) = y$.

One of the motivations to investigate the homogeneous ANR-spaces is the well-known Bing-Borsuk [1] conjecture that every finite-dimensional homogeneous metric ANR-compactum is a topological manifold. According to Jakobsche [14], a positive solution of that conjecture in dimension three implies the celebrated Poincaré conjecture. J. Bryant and S. Ferry [4] announced last year the existence of infinitely many topologically different counter-examples to that conjecture for every $n \geq 6$ (the first announcement of Bryant-Ferry results was in 2018). For an additional background on the Bing-Borsuk conjecture one can see the survey [15].

Despite the existence of counterexamples to the Bing-Borsuk conjecture, it is still interesting to investigate the extend to which finite-dimensional homogeneous ANR-spaces have common properties with topological manifolds. We show that homogeneous ANR-spaces have indeed some properties typical for topological manifolds. The local structure of homogeneous ANR-compacta X with a finite cohomological dimension $\dim_G X = n$ was established in [29, Theorem 1.1], where G is a countable principal ideal domain with unity and $n \geq 2$. In the present paper

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we improve the results in [29] by considering locally compact homogeneous ANR-spaces and replacing the principal ideal domains with countable groups. Moreover, we also provide some new properties of homogeneous ANRs or locally homogeneous ANRs.

Theorem 1.1 describes the local structure of homogeneous ANR-spaces and shows their similarity with topological manifolds:

Theorem 1.1. *Let X be a homogeneous connected ANR-space with $\dim_G X = n \geq 2$, where G is a countable group. Then every point $x \in X$ has a base $\mathcal{B}_x$ of connected open sets $U \subset X$ each having a compact closure and satisfying the following conditions:*

(1) $\mathrm{Int}\overline{U} = U$*, the boundary* $\mathrm{bd}\,\overline{U}$ *of* $\overline{U}$ *is connected and its complement in* X *has exactly two components;*
(2) $H^{n-1}(\mathrm{bd}\,\overline{U}; G) \neq 0$, $H^n(\overline{U}; G) = H^{n-1}(\overline{U}; G) = 0$ *and* $\overline{U}$ *is an* $(n-1)$*-cohomology membrane spanned on* $\mathrm{bd}\,\overline{U}$ *for any non-zero*

$$\gamma \in H^{n-1}(\mathrm{bd}\,\overline{U}; G);$$

(3) $\mathrm{bd}\,\overline{U}$ *is an* $(n-1, G)$*-bubble.*

Recall that for any nontrivial abelian group G the Čech cohomology group $H^n(X; G)$ is isomorphic to the group of pointed homotopy classes of maps from X to $K(G, n)$, where $K(G, n)$ is a CW-complex of type (G, n), see [17]. The cohomological dimension $\dim_G X$ is the largest number n such that there exists a closed set $A \subset X$ with $H^n(X, A; G) \neq 0$. Equivalently, for a metric space X we have $\dim_G X \leq n$ if and only if for any closed pair $A \subset B$ in X the homomorphism $j^n_{B,A} : H^n(B; G) \to H^n(A; G)$, generated by the inclusion $A \hookrightarrow B$, is surjective, see [10]. This means that every map from A to $K(G, n)$ can be extended over B. If X is a finite-dimensional space, then for every G we have $\dim_G X \leq \dim_{\mathbb{Z}} X = \dim X$ [22]. On the other hand, there is a compactum X [8] with $\dim X = \infty$ and $\dim_{\mathbb{Z}} X = 3$. A space A is a (k, G)-*bubble* if $H^k(A; G) \neq 0$ but $H^k(B; G) = 0$ for every closed proper subset B of A.

The following property of homogeneous ANRs is essential, see [1], [2]: If (K, A) is a compact pair of subsets of X such that K is an $(n-1)$-cohomology membrane for some $\gamma \in H^{n-1}(A; G)$, then $(K \setminus A) \cap \overline{X \setminus K} = \varnothing$. We call this property the *n-cohomology membrane property.* Recall that K is said to be an *k-cohomology membrane spanned on a closed set $A \subset K$ for an element $\gamma \in H^k(A; G)$* if γ is not extendable over K, but it is extendable over every proper closed subset of K containing A. Here, $\gamma \in H^k(A; G)$ is not extendable over K means that γ is not contained in the image $j^k_{K,A}\big(H^k(K; G)\big)$.

Everywhere below, $\mathcal{B}_x$ stands for a local base at a point $x \in X$. We say that a space X has an *n-dimensional G-obstruction* at a point $x \in X$ [22] if there is $W \in \mathcal{B}_x$ such that the homomorphism $j^n_{U,W} : H^n(X, X\backslash U; G) \to H^n(X, X\backslash W; G)$ is nontrivial for every $U \in \mathcal{B}_x$ with $U \subset W$. Kuzminov [22], [21], proved that every compactum X with $\dim_G X = n$ contains a compact set Y with $\dim_G Y = n$ such that X has an n-dimensional G-obstruction at any point of a dense subset of Y. According to Theorem 1.2, n-dimensional homogeneous spaces have an obstruction at every point and the corresponding homomorphisms are surjective.

Theorem 1.2. *Let X be a locally homogeneous ANR-space such that* $\dim_G X = n$, *where G is a countable group. Then the following holds:*

(i) *X has the n-cohomology membrane property;*
(ii) *X has an n-dimensional G-obstruction at every $x \in X$. Moreover, there is $W \in \mathcal{B}_x$ such that the homomorphism $j^n_{U,V}$ is surjective for any $U, V \in \mathcal{B}_x$ with $\overline{U} \subset V \subset \overline{V} \subset W$.*

Here is another common property of locally homogeneous ANRs and topological manifolds.

Corollary 1.3. *Let X be as in Theorem* 1.2*. If $U \subset X$ is open and $f : U \to X$ is an injective map, then $f(U)$ is also open in X.*

One of the important questions concerning homogeneous ANRs is whether every finite-dimensional homogeneous ANR-compactum is dimensionally full-valued. Recall that a locally compact space X is *dimensionally full-valued* if $\dim X \times Y = \dim X + \dim Y$ for any compact space Y. Let us note that there are metric ANR-compacta which are not dimensionally full-valued, see [7] and [9]. The question for the dimensional full-valuedness of homogeneous ANR-compacta goes back to [3] and was also discussed in [5] and [13]. A positive answer of that question for 3-dimensional spaces was given in [29]. Now, we provide a complete solution of that problem.

Theorem 1.4. *Let X be a finite-dimensional locally homogeneous ANR-space. Then the following holds:*

(i) *X is dimensionally full-valued;*
(ii) *Providing X is homogeneous and connected, every $x \in X$ has a neighborhood W_x such that* $\operatorname{bd}\overline{U}$ *is dimensionally full-valued for all $U \in \mathcal{B}_x$ with $\overline{U} \subset W_x$.*

2. The cohomology membrane property

If, in the definition of the n-cohomology membrane property, we additionally require K to be a compact set contractible in a proper subset of X, then we say that X has the *weak n-cohomology membrane property.* We will see that the proof of Theorem 1.1 is based mainly on that property. In this section we prove that any locally homogeneous space X with $\dim_G X = n$, where G is countable, has the weak n-cohomology membrane property. We also provide some implications of that property.

A space X is called a *(k, G)-carrier* of a nontrivial element $\gamma \in H^k(X; G)$ if $j^k_{X,B}(\gamma) = 0$ for every closed proper subset $B \subset X$.

Lemma 2.1. *Let $A \subset X$ be a compact set and γ be a non-zero element of $H^n(A; G)$.*

(i) *If γ is not extendable over a compact set $P \subset X$ containing A, then there exists an n-cohomology membrane $K \subset P$ for γ spanned on A;*
(ii) *There is a closed set $B \subset A$ such that B is a carrier of $j^n_{A,B}(\gamma)$.*

Proof. (i) Consider the family $\mathcal{F}$ of all closed subsets F of P containing A such that γ is not extendable over F. Let $\{F_\tau\}$ be a decreasing subfamily of $\mathcal{F}$ and $F_0 = \bigcap_\tau F_\tau$. Suppose γ is extendable to $\widetilde{\gamma} \in H^n(F_0; G)$. Then, considering $\widetilde{\gamma}$ as a map from F_0 into $K(G, n)$ and having in mind that $K(G, n)$ is an absolute neighborhood for metrizable spaces [19], we can extend $\widetilde{\gamma}$ over a neighborhood W

of F_0 in P. But that is impossible since W contains some F_τ. Hence, by Zorn's lemma, $\mathcal{F}$ has a minimal element K which is an n-cohomology membrane for γ spanned on A.

(ii) Now, let $\mathcal{F}$ be the family of closed subsets F of A such that $j^n_{A,F}(\gamma) \neq 0$ and $\{F_\tau\}$ be a decreasing subfamily of $\mathcal{F}$ with $F_0 = \bigcap_\tau F_\tau$. If $\gamma_0 = j^n_{A,F_0}(\gamma) = 0$, then there is a homotopy $H : F_0 \times [0,1] \to K(G,n)$ connecting the constant map and γ_0. Using again that $K(G,n)$ is an absolute neighborhood extensor for metric spaces, we find a closed neighborhood W of F_0 in A and a homotopy $\widetilde{H} : W \times [0,1] \to K(G,n)$ connecting $j^n_{A,W}(\gamma)$ and the constant map. This is a contradiction because W contains some F_τ. Hence, $\mathcal{F}$ has a minimal element B which is a carrier of $j^n_{A,B}(\gamma)$. □

We also need the following version of Effros' theorem [11] for locally compact spaces, see [31, Theorem 2.5]):

Theorem 2.2. *Let X be a homogeneous space and ρ be a metric on the one-point compactification of X. Then for any $a \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ with $\rho(x,a) < \delta$ there exists a homeomorphism $h\colon X \to X$ with $h(x) = a$ and $\rho(h(y), y) < \varepsilon$ for all $y \in X$.*

Proposition 2.3. *Let X be a homogeneous ANR-space with $\dim_G X = n$, where G is an arbitrary group. Then $H^n(P;G) = 0$ for any proper compact set $P \subset X$.*

Proof. This was established in [28] in case X is compact using the following proposition, see [28, Proposition 2.3]: If X is a metric compactum with $\dim_G X = n$, $A \subset X$ is carrier of a nontrivial element of $H^n(X;G)$ and there is a map $f : X \to X$ homotopic to the identity id_X of X, then $A \subset f(A)$. This statement remains true if X is an ANR-space with $\dim_G X = n$, A is a carrier of a nontrivial element of $H^n(Q;G)$, where $Q \subset X$ is compact, and there exists a homotopy $F : A\times[0,1] \to X$ with $F(x,0) = x$ and $F(x,1) = f(x)$, $x \in A$.

Fix a metric ρ on X generating its topology and suppose there exists a proper compact set $P \subset X$ with $H^n(P;G) \neq 0$. So, by Lemma 2.1(ii), P contains a compact set K such that K is a carrier for $j^n_{P,K}(\alpha)$, where $\alpha \in H^n(P;G)$ is nontrivial. Take two open sets $U, V \subset X$ containing P and having compact closures in X with $\overline{U} \subset V$. Since $X \in ANR$, there is an open cover ω of X such that any two ω-close maps $f_1, f_2 : \overline{U} \to X$ are homotopic. Restricting ω on $\overline{V}$, we find $\varepsilon > 0$ such that any maps $f_1, f_2 : \overline{U} \to X$ with $\rho(f_1(x), f_2(x)) < \varepsilon$, $x \in \overline{U}$, are homotopic. Moreover, we can assume $\varepsilon < \min\{\rho(P, X\backslash U), \rho(\overline{U}, X\backslash V)\}$. Next, take a point a from the boundary $\mathrm{bd}\, K$ of K in X and a point $b \notin K$ with $\rho(a,b) < \delta$, where $\delta > 0$ is the Effros' number corresponding to ε and the point a. Accordingly, there is homeomorphism $h : X \to X$ such that $h(a) = b$ and $\rho(x, h(x)) < \varepsilon$ for all $x \in X$. So, $h(K) \subset U$ and $h(\overline{U}) \subset V$. Since h generates an isomorphism $h^* : H^n(h(P);G) \to H^n(P;G)$, the set $A = h(K)$ is a carrier for $j^n_{h(P),h(K)}(\beta)$, where $\beta = (h^*)^{-1}(\alpha)$. Consider the map $g = h^{-1}|\overline{U} : \overline{U} \to h^{-1}(\overline{U})$. Observe that $\rho(g(x), x) < \varepsilon$ for all $x \in \overline{U}$. Hence, g and the identity $\mathrm{id}_{\overline{U}}$ on $\overline{U}$ are homotopic. In particular, there is a homotopy $F : g(A) \times [0,1] \to X$ such that $F(x,0) = x$ and $F(x,1) = g(x)$, $x \in g(A)$. Therefore, we can apply the modification of [28, Proposition 2.3] stated above to conclude that $A \subset g(A)$. Because $b = h(a) \in A$, the last inclusion implies $b \in g(A) = K$, a contradiction. □

Remark. Proposition 2.3 remains true without homogeneity of X provided P is a closed set contractible in X.

This is true if $H^n(X;G) = 0$ because $\dim_G X \leq n$. If $H^n(X;G) \neq 0$ this is also true. Indeed, since $\dim_G X \leq n$, every $\alpha \in H^n(P;G)$ is extendable to $\widetilde{\alpha} \in H^n(X;G)$. On the other hand P is contractible in X, so every such $\widetilde{\alpha}$ is zero.

A compact pair $A \subset K$ is called a *k-homology membrane spanned on* A for a nontrivial element $\gamma \in H_k(A;G)$ if $i^k_{A,K}(\gamma) = 0$ but $i^k_{A,B}(\gamma) \neq 0$ for any proper closed subset B of K containing A. Here, H_* denotes the Čech homology and $i^k_{A,K} : H_k(A;G) \to H_k(K;G)$ is the homomorphism induced be the inclusion $A \hookrightarrow K$. According to Bing-Borsuk [1], if $A \subset K$ is a compact pair and $\gamma \in H_k(A;G)$ is a nontrivial element homologous to zero in K, then there is a closed set $B \subset K$ containing A such that $A \subset B$ a k-homology membrane for γ spanned on A. Bing-Borsuk [1] considered the Vietoris homology which is isomorphic with the Čech homology in the realm of metric compacta. So, homology membranes with respect to Čech homology always exist in the class of metric compacta.

Proposition 2.4. *Any locally homogeneous ANR-space with $\dim_G X = n$, where G is a countable group, has the weak n-cohomology membrane property.*

Proof. Suppose there exists a compact pair $A \subset K$ such that K is contractible in a proper subset of X and K is an $(n-1)$-cohomology membrane for some nontrivial $\gamma \in H^{n-1}(A;G)$, but $(K \setminus A) \cap \overline{X \setminus K} \neq \varnothing$. Since G is countable, we use the following result, see [18, viii 4G]: The homology group $H_{n-1}(Y;G^*)$ is isomorphic to $H^{n-1}(Y;G)^*$ for any metric compactum Y. Here, G^* and $H^{n-1}(Y;G)^*$ are the character groups of G and $H^{n-1}(Y;G)$, considered as discrete groups. This implies that K is an $(n-1)$-homology membrane spanned on A for some nontrivial $\beta \in H_{n-1}(A;G^*)$, see the proof of [29, Proposition 2.1]. Hence, following the proof of [1, Theorem 8.1], we can find a compact set $P \subset X$ contractible in X with $H_n(P;G^*) \neq 0$. So, $H^n(P;G) \neq 0$, which contradicts the remark after Proposition 2.3. □

We also need the following properties of cohomology membranes, see Corollary 2.2 and Lemma 2.4 from [29], respectively.

Lemma 2.5. *For any space X with the weak $(n-1)$-cohomology membrane property the following conditions hold:*

(i) *If K is an $(n-1)$-cohomology membrane spanned on a set $A \subset K$ for some $\gamma \in H^{n-1}(A;G)$, where K is a compactum contractible in a proper subset of X, then $K \setminus A$ is a connected open subset of X.*
(ii) *Let $A \subset P$ be a compact pair such that P is contractible in a proper subset of X and there exists a non-zero $\gamma \in H^{n-1}(A;G)$ not extendable over P. Then A separates every set $\Gamma \subset X$ containing P as a proper subset.*

Let (X, ρ) be a metric space. We say that ρ is *convex* if for each $x, y \in X$ there exists an arc $A \subset X$ with end-points x and y such that A with the restriction of the metric ρ is isometric to the interval $[0, \rho(x,y)]$ in the real line (where the real line is considered with its usual metric). According to [27], every connected locally connected space admits a convex metric.

Proposition 2.6. *Let G be a countable group and X be a locally homogeneous ANR-space with $\dim_G X = n$. Then we have:*

(1) *Every closed set $P \subset X$ with $\dim_G P = n$ has a non-empty interior;*
(2) *Every $x \in X$ has a neighborhood W such that any connected $U \in \mathcal{B}_x$ with $U = \mathrm{int}\overline{U} \subset W$ satisfies the following conditions:*
   (i) *$H^n(\overline{U}; G) = 0$, $H^{n-1}(\mathrm{bd}\,\overline{U}; G) \neq 0$ and it contains an element not extendable over $\overline{U}$;*
   (ii) *$\overline{U}$ is an $(n-1)$-cohomology membrane spanned on $\mathrm{bd}\,\overline{U}$ for any non-trivial $\alpha \in H^{n-1}(\mathrm{bd}\,\overline{U}; G)$ not extendable over $\overline{U}$.*

Proof. (1) This was established in [29, Corollary 2.3] in case X is compact homogeneous ANR. For the covering dimension dim and locally homogeneous ANRs it was established in [25, Theorem A] (see also [24] for a property stronger than locally homogeneity). Suppose $P \subset X$ is a closed set with $\dim_G P = n$. Then P is the union of countably many compact sets F_j each contractible in a proper subset of X. By the countable sum theorem for $\dim_G$, we have $\dim_G F_j = n$ for at least one j. So, we can assume that P is a compact set contractible in a proper subset of X. Since $\dim_G P = n$, there is a closed set $A \subset P$ and an element $\gamma \in H^{n-1}(A; G)$ not extendable over P. Then, according to Lemma 2.1(i), there exists a closed set $K \subset P$ such that K is an $(n-1)$-cohomology membrane for γ spanned on A. Because X has the weak n-cohomology membrane property, $(K \setminus A) \cap \overline{X \setminus K} = \varnothing$. This implies $K \setminus A$ is open in X. Finally, the inclusion $K \setminus A \subset P$ completes the proof.

(2) Since X is a countable union of compact sets each contractible in a proper subset of X, as in the previous paragraph, there exists a compact pair $A \subset K$ such that $\dim_G K = n$, K is an $(n-1)$-cohomology membrane for some $\gamma \in H^{n-1}(A; G)$ spanned on A and K is contractible in a proper subset of X. Then each $K \setminus A$ is a connected open set in X, see Lemma 2.5(i). Let $x \in K \setminus A$ and let $W \in \mathcal{B}_x$ with $\overline{W} \subset K \setminus A$.

Let $U \in \mathcal{B}_x$ be connected with $U = \mathrm{int}\overline{U} \subset W$. Such sets U exist. Indeed, since X is locally connected, each of its component of connectedness X_c is open, and according to [27], there is a convex metric generating the topology of X_c. On the other hand, if d is a convex metric, then every open ball $B(x, \delta) = \{y \in X_c : d(x, y) < \delta\}$ is connected and $\mathrm{int}\overline{B(x, \delta)} = B(x, \delta)$.

Since each $\overline{U}$ is contractible in a proper subset of X, $H^n(\overline{U}; G) = 0$. We claim that $H^{n-1}(\mathrm{bd}\,\overline{U}; G) \neq 0$ and it contains an element not extendable over $\overline{U}$. Indeed, $K \setminus U$ is a proper closed subset of K containing A. So, γ can be extended to $\widetilde{\gamma} \in H^{n-1}(K \setminus U; G)$. Then $\gamma_U = j^{n-1}_{K \setminus U, \mathrm{bd}\,\overline{U}}(\widetilde{\gamma})$ is a non-zero element of $H^{n-1}(\mathrm{bd}\,\overline{U}; G)$ (otherwise γ would be extendable over K). So, γ_U is not extendable over $\overline{U}$. Let's show that $\overline{U}$ is an $(n-1)$-cohomology membrane spanned on $\mathrm{bd}\,\overline{U}$ for every $\alpha \in H^{n-1}(\mathrm{bd}\,U; G)$ not extendable over $\overline{U}$. By Lemma 2.1, for any such α there is an $(n-1)$-cohomology membrane $K_\alpha \subset \overline{U}$ for α spanned on $\mathrm{bd}\,\overline{U}$. Hence, $(K_\alpha \setminus \mathrm{bd}\,\overline{U}) \cap \overline{X \setminus K_\alpha} = \varnothing$. In particular, $K_\alpha \setminus \mathrm{bd}\,\overline{U}$ is open in U. Thus, $K_\alpha = \overline{U}$, otherwise U would be the union of the non-empty disjoint open sets $U \setminus K_\alpha$ and $K_\alpha \setminus \mathrm{bd}\,\overline{U}$. Finally, since X is locally homogeneous, every $x \in X$ has a neighborhood W satisfying condition (2). □

Lemma 2.7. *Let X be a locally homogeneous ANR-space and $x \in X$. If G is a countable group and $H^{n-1}(\mathrm{bd}\,\overline{U};G) \neq 0$ for all sufficiently small neighborhoods $U \in \mathcal{B}_x$, then $\dim_G X \geq n$.*

Proof. Suppose $\dim_G X \leq n-1$. Since the interior of $\mathrm{bd}\,\overline{U}$ is empty, $\dim_G \mathrm{bd}\,\overline{U} \leq n-2$ for all $U \in \mathcal{B}_x$. This, according to the definition of cohomological dimension, implies $H^{n-1}(\mathrm{bd}\,\overline{U};G) = 0$, a contradiction. □

3. Proof of Theorem 1.1

Lemma 3.1 was established in [6, Theorem 8] for locally connected continua. The same proof works for locally connected and connected spaces X by passing to the one-point compactification of X.

Lemma 3.1. *Let X be a connected and locally connected space and $\{K_\alpha : \alpha \in \Lambda\}$ be an uncountable collection of disjoint continua such that for each α the set $X \setminus K_\alpha$ has more than one component. Then there exists $\alpha_0 \in \Lambda$ such that $X \setminus K_{\alpha_0}$ has exactly two components.*

The proof of Theorem 1.1 follows from Proposition 2.6 and Proposition 3.2.

Proposition 3.2. *Let X be a homogeneous connected ANR-space with $\dim_G X = n \geq 2$, where G is a countable group. Then every point $x \in X$ has a basis $\mathcal{B}_x$ of open connected sets U each with a compact closure satisfying the following conditions:*

(i) $H^{n-1}(\overline{U};G) = 0$ *and $X \setminus \mathrm{bd}\,\overline{U}$ has exactly two components;*
(ii) $\mathrm{bd}\,\overline{U}$ *is an $(n-1,G)$-bubble.*

Proof. By Proposition 2.6, every point $x \in X$ has a neighborhood W_x with a compact closure such any connected neighborhood $U = \mathrm{int}\overline{U} \subset W_x$ of x satisfies the following condition: $H^n(\overline{U};G) = 0$, $H^{n-1}(\mathrm{bd}\,\overline{U};G)$ contains elements γ not extendable over $\overline{U}$ and for each such γ the set $\overline{U}$ is an $(n-1)$-cohomology membrane for γ spanned on $\mathrm{bd}\,\overline{U}$. We assume also that $\overline{W}_x$ is a connected set contractible in a proper subset of X and there is $\alpha_x \in H^{n-1}(\mathrm{bd}\,\overline{W}_x;G)$ such that $\overline{W}_x$ is an $(n-1)$-cohomology membrane for α_x spanned on $\mathrm{bd}\,\overline{W}_x$. We fix $x \in X$ and let $\mathcal{B}'_x$ be the family of all open connected neighborhoods U of x such that $U = \mathrm{int}\overline{U}$ and $\overline{U}$ is contractible in W_x.

Claim 1. For every $U \in \mathcal{B}'_x$ there exists a non-zero $\gamma_U \in H^{n-1}(\mathrm{bd}\,\overline{U};G)$ such that γ_U is extendable over $\overline{W}_x \setminus U$ and $\overline{U}$ is an $(n-1)$-cohomology membrane for γ_U spanned on $\mathrm{bd}\,\overline{U}$.

Indeed, since $\overline{W}_x$ is an $(n-1)$-cohomology membrane for α_x spanned on $\mathrm{bd}\,\overline{W}_x$, α_x is not extendable over $\overline{W}_x$, but it is extendable over every closed proper subset of $\overline{W}_x$ containing $\mathrm{bd}\,\overline{W}_x$. So, α_x is extendable to an element $\widetilde{\alpha}_x \in H^{n-1}(\overline{W}_x \setminus U;G)$ and the element $\gamma_U = j^{n-1}_{\overline{W}_x\setminus U,\mathrm{bd}\,\overline{U}}(\widetilde{\alpha}_x) \in H^{n-1}(\mathrm{bd}\,\overline{U};G)$ is not extendable over $\overline{U}$ (otherwise α_x would be extendable over $\overline{W}_x$), in particular $\gamma_U \neq 0$. Therefore, by [29, Lemma 2.6], $\overline{U}$ is an $(n-1)$-cohomology membrane for γ_U spanned on $\mathrm{bd}\,\overline{U}$.

Let $\mathcal{B}''_x$ be the family of all $U \in \mathcal{B}'_x$ satisfying the following condition: $\mathrm{bd}\,\overline{U}$ contains a continuum F_U such that $X \setminus F_U$ has exactly two components and F_U is an $(n-1,G)$-carrier of $j^{n-1}_{\mathrm{bd}\,\overline{U},F_U}(\gamma_U)$.

Claim 2. $\mathcal{B}''_x$ is a local base at x.

Since X is arc connected and locally arc connected, there is a convex metric d generating the topology of X, see [27]. We fix $W_0 \in \mathcal{B}'_x$ and for every $\delta > 0$ denote by $B(x,\delta)$ the open ball in X with a center x and a radius δ. There exists $\varepsilon_x > 0$ such that $B(x,\delta) \subset W_0$ for all $\delta \leq \varepsilon_x$. We already observed in the proof of Proposition 2.6(2) that any $B(x,\delta)$ is connected and $\mathrm{int}(\overline{B(x,\delta)}) = B(x,\delta)$. Moreover, $\overline{B(x,\delta)}$ is contractible in W_x. Hence, all $U_\delta = B(x,\delta)$, $\delta \leq \varepsilon_x$, belong to $\mathcal{B}'_x$. Consequently, by Claim 1, for every δ there exists a non-zero $\gamma_\delta \in H^{n-1}(\mathrm{bd}\,\overline{U}_\delta; G)$ such that $\overline{U}_\delta$ is an $(n-1)$-cohomology membrane for γ_δ spanned on $\mathrm{bd}\,\overline{U}_\delta$ and γ_δ is extendable over $\overline{W}_x \setminus U_\delta$. By Lemma 2.1(ii), there exists a closed subset F_δ of $\mathrm{bd}\,\overline{U}_\delta$ which is a carrier of $\gamma^*_\delta = j^{n-1}_{\mathrm{bd}\,\overline{U}_\delta, F_\delta}(\gamma_\delta)$. Since $n \geq 2$ and F_δ is a carrier of $\gamma^*_\delta \in H^{n-1}(F_\delta; G)$, F_δ is a continuum, see [29, Lemma 2.7]. Let us show that the family $\{F_\delta : \delta \leq \varepsilon_x\}$ is uncountable. Since the function $f\colon X \to \mathbb{R}$, $f(y) = d(x,y)$, is continuous and W_0 is connected, $f(W_0)$ is an interval containing $[0,\varepsilon_x]$ and $f^{-1}([0,\varepsilon_x)) = B(x,\varepsilon_x) \subset W_0$. So, $f^{-1}(\delta) = \mathrm{bd}\,\overline{U}_\delta \neq \varnothing$ for all $\delta \leq \varepsilon_x$. Hence, the family $\{F_\delta : \delta \leq \varepsilon_x\}$ is uncountable and consists of disjoint continua. Moreover, γ^*_δ is a non-zero element of $H^{n-1}(F_\delta; G)$ not extendable over $\overline{W}_x$ because F_U is contractible in $\overline{W}_x$. Thus, by Lemma 2.5(ii), F_δ separates X. So, each $X \setminus F_\delta$ has at least two components. Then, by Lemma 3.1, there exists $\delta_0 \leq \varepsilon_x$ such that $X \setminus F_{\delta_0}$ has exactly two components. Therefore, $U_{\delta_0} = B(x,\delta_0) \in \mathcal{B}''_x$ and it is contained in W_0. This completes the proof of Claim 2.

Now, let $\widetilde{\mathcal{B}}_x$ be the family of all $U \in \mathcal{B}''_x$ with $H^{n-1}(\mathrm{bd}\,\overline{U}; G) \neq 0$ such that both U and $X \setminus \overline{U}$ are connected.

Claim 3. $\widetilde{\mathcal{B}}_x$ is a local base at x.

Let U_0 be an arbitrary neighborhood of x such that $\overline{U}_0$ is contractible in W_x. We are going to find a member of $\widetilde{\mathcal{B}}_x$ contained in U_0. To this end, let ρ be a metric generating the topology of the one-point compactification of X and let $\varepsilon = \rho(x, X \setminus U_0)$. According to Theorem 2.2 there is $\eta > 0$ corresponding to $\varepsilon/2$ and the point x (i.e., for every $y \in X$ with $\rho(y,x) < \eta$, there exists a homeomorphism $h : X \to X$ with $h(y) = x$ and $\rho(h(z),z) < \varepsilon/2$ for all $z \in X$). Now, choose a connected neighborhood W of x with $\overline{W} \subset B_\rho(x,\varepsilon/2)$ and $\rho - \mathrm{diam}(\overline{W}) < \eta$. Finally, take $U \in \mathcal{B}''_x$ such that $\overline{U}$ is contractible in W. By Claim 2, there exists a continuum $F_U \subset \mathrm{bd}\,\overline{U}$ such that $X \setminus F_U$ has exactly two components and F_U is an $(n-1,G)$-carrier for $\gamma^*_U = j^{n-1}_{\mathrm{bd}\,\overline{U}, F_U}(\gamma_U)$. If $F_U = \mathrm{bd}\,\overline{U}$, then U is the desired member of $\widetilde{\mathcal{B}}_x$. Indeed, since $X \setminus bd\overline{U} = U \cup X \setminus \overline{U}$ with $U \cap (X \setminus \overline{U}) = \varnothing$ and U is connected, then $X \setminus \overline{U}$ should be also connected (recall that $X \setminus F_U$ has exactly two components).

Suppose that F_U is a proper subset of $\mathrm{bd}\,\overline{U}$. Because F_U (as a subset of $\overline{U}$) is contractible in a compact set $\Gamma \subset W$, γ^*_U is not extendable over Γ. Thus, we can apply Lemma 2.5(ii) to conclude that F_U separates $\overline{W}$. So, $\overline{W} \setminus F_U = V_1 \cup V_2$ for some open, non-empty disjoint subsets $V_1, V_2 \subset \overline{W}$. Since U is a connected subset of $\overline{W} \setminus F_U$, U is contained in one of the sets V_1, V_2, say $U \subset V_1$. Hence, $F_U \cup \overline{V}_2 \subset \overline{W}_x \setminus U$. Since γ_U is extendable over $\overline{W}_x \setminus U$ (see Claim 1), γ^*_U is also extendable over $\overline{W}_x \setminus U$, in particular γ^*_U is extendable over $F_U \cup \overline{V}_2$. On the other hand, γ^*_U is not extendable over $\overline{W}$ because F_U is contractible in $\overline{W}$. The last fact, together with the equality $(F_U \cup \overline{V}_1) \cap (F_U \cup \overline{V}_2) = F_U$, yields that γ^*_U is not extendable over $F_U \cup \overline{V}_1$. Let $\beta = j^{n-1}_{F_U, F'}(\gamma^*_U)$, where $F' = \overline{V}_1 \cap F_U$ (observe that

$F' \neq \varnothing$ because $\overline{W}$ is connected). If F' is a proper subset of F_U, then $\beta = 0$ since F_U is a carrier for γ_U^*. So, β would be extendable over $\overline{V}_1$, which implies γ_U^* is extendable over $F_U \cup \overline{V}_1$, a contradiction. Therefore, $F' = F_U \subset \overline{V}_1$ and γ_U^* is not extendable over $\overline{V}_1$. Consequently, there exists an $(n-1)$-cohomology membrane $P_\beta \subset \overline{V}_1$ for γ_U^* spanned on F_U. By Lemma 2.5(i), $V = P_\beta \setminus F_U$ is a connected open set in X whose boundary is the set $F'' = \overline{X \setminus P_\beta} \cap \overline{P_\beta \setminus F_U} \subset F_U$. As above, using that γ_U^* is not extendable over P_β and $j^{n-1}_{F_U,Q}(\gamma_U^*) = 0$ for any proper closed subset $Q \subset F_U$, we can show that $F'' = F_U$ and $\mathrm{bd}\,\overline{V} = F_U$. Summarizing the properties of V, we have that $\overline{V}$ is contractible in W_x (because so is $\overline{U}_0$), $V = \mathrm{int}(\overline{V})$ (because $F_U = \mathrm{bd}\,\overline{V}$) and V is connected. Moreover, since $X \setminus F_U$ is the union of the open disjoint non-empty sets V and $X \setminus P_\beta$ such that V is connected and $X \setminus F_U$ has exactly two components, $X \setminus \overline{V}$ is also connected. Finally, because F_U is an $(n-1, G)$-carrier for the nontrivial γ_U^*, $H^{n-1}(\mathrm{bd}\,\overline{V}; G) \neq 0$. Thus, if V contains x, then V is a member of $\widetilde{\mathcal{B}}_x$.

If V does not contain x, we take a point $y \in V$ with $\rho(x, y) < \eta$. This is possible because $V \subset \overline{W}$ and $\rho - \mathrm{diam}(\overline{W}) < \eta$. So, according to the choice of η, there is a homeomorphism h on X such that $h(y) = x$ and $\rho(z, h(z)) < \varepsilon/2$ for all $z \in X$. Then $h(V) \subset U_0$ (this inclusion follows from the choice of ε and the fact that h is $(\varepsilon/2)$-close to the identity on X). So, $\overline{h(V)}$ is contractible in W_x. Since the remaining properties from the definition of $\widetilde{\mathcal{B}}_x$ are invariant under homeomorphisms, $h(V)$ is the required member of $\widetilde{\mathcal{B}}_x$, which provides the proof of Claim 3.

Claim 4 completes the proof of Proposition 3.2.

Claim 4. $H^{n-1}(\overline{U}; G) = 0$ and $\mathrm{bd}\,\overline{U}$ is an $(n-1, G)$-bubble for every $U \in \widetilde{\mathcal{B}}_x$.

Because each $\overline{U}$, $U \in \widetilde{\mathcal{B}}_x$, is contractible in $\overline{W}_x$, any nontrivial $\gamma \in H^{n-1}(\overline{U}; G)$ cannot be extendable over $\overline{W}_x$. Since $\overline{W}_x$ is contractible in a proper subset of X, by Lemma 2.5(ii), $\overline{U}$ would separate X provided $H^{n-1}(\overline{U}; G) \neq 0$. On the other hand, each $X \setminus \overline{U}$ is connected. Therefore, $H^{n-1}(\overline{U}; G) = 0$ for all $U \in \widetilde{\mathcal{B}}_x$. Suppose there exists a proper closed subset $F \subset \mathrm{bd}\,\overline{U}$ and a nontrivial element $\alpha \in H^{n-1}(F; G)$. Since $H^{n-1}(\overline{U}; G) = 0$, α is not extendable over $\overline{U}$. Hence, there is an $(n-1)$-cohomology membrane $K_\alpha \subset \overline{U}$ for α spanned on F. Therefore, $(K_\alpha \setminus F) \cap \overline{X \setminus K_\alpha} = \varnothing$. In particular, $K_\alpha \setminus F$ is open in $\overline{U} \setminus F$. On the other hand, $K_\alpha \setminus F$ is also closed in $\overline{U} \setminus F$. Because $\overline{U} \setminus F$ is connected (recall that U is a dense connected subset of $\overline{U} \setminus F$), we obtain $K_\alpha = \overline{U}$. Finally, observe that any point from $\mathrm{bd}\,\overline{U} \setminus F$ belongs to $(K_\alpha \setminus F) \cap \overline{X \setminus K_\alpha}$, a contradiction. Therefore, $\mathrm{bd}\,\overline{U}$ is an $(n-1, G)$-bubble. □

4. Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. (i) Suppose K is an $(n-1)$-cohomology membrane spanned on A for some $\gamma \in H^{n-1}(A; G)$ and there is a point $a \in (K \setminus A) \cap \overline{X \setminus K}$. Take $V \in \mathcal{B}_a$ with $\overline{V} \cap A = \varnothing$, where $\mathcal{B}_a$ is a local base at a satisfying the hypotheses of Proposition 2.6(2). Since $K \setminus V$ is a proper subset of K containing A, γ is extendable to $\gamma^* \in H^{n-1}(K \setminus V; G)$. Then $\alpha_V = j^{n-1}_{K \setminus V, K \cap \mathrm{bd}\,\overline{V}}(\gamma^*)$ is not extendable over $K \cap \overline{V}$ (otherwise γ would be extendable over K). In particular, α_V a non-zero element of $H^{n-1}(K \cap \mathrm{bd}\,\overline{V}; G)$. Because $\mathrm{bd}\,\overline{V}$ has an empty interior in X, $\dim_G \mathrm{bd}\,\overline{V} \leq n-1$, see Proposition 2.6(1). Consequently, α_V can be extended to $\widetilde{\alpha}_V \in H^{n-1}(\mathrm{bd}\,\overline{V};G)$. Moreover, since α_V is not extendable over $K\cap\overline{V}$, $\widetilde{\alpha}_V$ is not extendable over $\overline{V}$. Finally, using that $\overline{V}$ is an $(n-1)$-cohomology membrane spanned on $\mathrm{bd}\,\overline{V}$ for $\widetilde{\alpha}_V$ and, since $a \in (K\setminus A)\cap\overline{X\setminus K}$ implies that $\mathrm{bd}\,\overline{V}\cup(K\cap\overline{V})$ is a proper closed subset of $\overline{V}$ containing $\mathrm{bd}\,\overline{V}$, we can find $\beta \in H^{n-1}(\mathrm{bd}\,\overline{V}\cup(K\cap\overline{V});G)$ extending $\widetilde{\alpha}_V$. Thus, α_V is extendable over $K\cap\overline{V}$, a contradiction. Therefore, X has the n-cohomology membrane property.

(ii) We need the following result [21, Theorem 1]: For every metric compactum K with $\dim_G K = n$ there exist a closed set $Y\subset K$ and a dense set $D\subset Y$ such that $\dim_G Y = n$ and K has an n-dimensional G-obstruction at every $y\in D$. In our situation we take a compact set $K\subset X$ with $\dim_G K = n$ and find corresponding sets $D\subset Y\subset K$. Since $\dim_G Y = n$, $\mathrm{int}Y \neq \varnothing$ (Proposition 2.6(1)) and there is $y\in D\cap\mathrm{int}Y$. Because K has an n-dimensional G-obstruction at y, we can find a neighborhood $W\subset K$ of y such that for any open in K neighborhood $V\subset W$ of y the homomorphism $H^n(K,K\setminus V;G)\to H^n(K,K\setminus W;G)$ is not trivial. This implies that for every open in K neighborhoods U,V of y with $\overline{U}\subset V$ the homomorphism $H^n(K,K\setminus U;G)\to H^n(K,K\setminus V;G)$ is also nontrivial. Since $y\in\mathrm{int}Y$, we can suppose that $W\in\mathcal{B}_y$ such that $\overline{W}$ is contractible in X and satisfying the hypotheses of Proposition 2.6(2). Then, by the excision axiom, for every $U,V\in\mathcal{B}_y$ with $\overline{U}\subset V\subset\overline{V}\subset W$, the groups $H^n(K,K\setminus U;G)$ and $H^n(K,K\setminus V;G)$ are isomorphic to $H^n(X,X\setminus U;G)$ and $H^n(X,X\setminus V;G)$, respectively. So, $j_{U,V}: H^n(X,X\setminus U;G)\to H^n(X,X\setminus V;G)$ is a nontrivial homomorphism for all $U,V\in\mathcal{B}_y$ with $\overline{U}\subset V\subset\overline{V}\subset W$. Finally, because X is locally homogeneous, it has an n-dimensional G-obstruction at every $x\in X$.

To prove the second half of condition (ii), let $U,V\in\mathcal{B}_x$ with $\overline{U}\subset V\subset\overline{V}\subset W$.

Claim 5. The homomorphism $j^{n-1}_{\overline{W}\setminus U,\overline{W}\setminus V}: H^{n-1}(\overline{W}\setminus U;G)\to H^{n-1}(\overline{W}\setminus V;G)$ is surjective and $H^{n-1}(\overline{W}\setminus V;G)\neq 0$.

Indeed, consider the Mayer-Vietoris exact sequence below, where the coefficient group G is suppressed,

$$H^{n-1}(\overline{W}\setminus U)\xrightarrow{\ \varphi\ } H^{n-1}(\overline{V}\setminus U)\oplus H^{n-1}(\overline{W}\setminus V)\xrightarrow{\ \psi\ } H^{n-1}(\mathrm{bd}\,\overline{V})\dots$$

The maps φ and ψ are defined by $\varphi(\gamma) = (j^{n-1}_{\overline{W}\setminus U,\overline{V}\setminus U}(\gamma), j^{n-1}_{\overline{W}\setminus U,\overline{W}\setminus V}(\gamma))$ and $\psi((\beta,\alpha)) = j^{n-1}_{\overline{V}\setminus U,\mathrm{bd}\,\overline{V}}(\beta) - j^{n-1}_{\overline{W}\setminus V,\mathrm{bd}\,\overline{V}}(\alpha)$. For every $\alpha\in H^{n-1}(\overline{W}\setminus V;G)$ the element $\beta'_\alpha = j^{n-1}_{\overline{W}\setminus V,\mathrm{bd}\,\overline{V}}(\alpha)\in H^{n-1}(\mathrm{bd}\,\overline{V};G)$ is extendable to an element $\beta_\alpha\in H^{n-1}(\overline{V}\setminus U;G)$. Indeed, there are two possibilities: either β'_α is extendable over $\overline{V}$ or it is not extendable over $\overline{V}$. The first case obviously implies that β'_α is extendable over $\overline{V}\setminus U$. In the second case, by Proposition 2.6(2), $\overline{V}$ is an $(n-1)$-cohomological membrane spanned on $\mathrm{bd}\,\overline{V}$ for β'_α. Then, since $\overline{V}\setminus U$ is a proper closed subset of $\overline{V}$, β'_α is extendable over $\overline{V}\setminus U$. Hence, $\psi(\beta_\alpha,\alpha)=0$ for any $\alpha\in H^{n-1}(\overline{W}\setminus V;G)$. Consequently, there is $\gamma_\alpha\in H^{n-1}(\overline{W}\setminus U;G)$ with $\varphi(\gamma_\alpha) = (\beta_\alpha,\alpha)$. In particular, $j^{n-1}_{\overline{W}\setminus U,\overline{W}\setminus V}(\gamma_\alpha)=\alpha$, which shows the surjectivity of $j^{n-1}_{\overline{W}\setminus U,\overline{W}\setminus V}$. The nontrivially of $H^{n-1}(\overline{W}\setminus V;G)$ follows from the Mayer-Vietoris exact sequence

$$\to H^{n-1}(\overline{V};G)\oplus H^{n-1}(\overline{W}\setminus V;G)\to H^{n-1}(\mathrm{bd}\,\overline{V};G)\to H^n(\overline{W};G)\to .$$

Indeed, $\overline{W}$ being contractible in X implies that $H^n(\overline{W};G) = 0$ (see the remark after Proposition 2.3). Hence, the homomorphism $j^{n-1}_{\overline{V},\mathrm{bd}\,\overline{V}}$ would be surjective provided $H^{n-1}(\overline{W}\backslash V;G) = 0$. This would imply that every nontrivial element of $H^{n-1}(\mathrm{bd}\,\overline{V};G)$ is extendable over $\overline{V}$, which contradicts Proposition 2.6(2).

To complete the proof, consider the commutative diagram whose rows are parts of exact sequences

$$\begin{array}{ccccc} H^{n-1}(\overline{W}\backslash U;G) & \xrightarrow{\delta_U} & H^n(\overline{W},\overline{W}\backslash U;G) & \xrightarrow{i_U} & H^n(\overline{W};G) \\ \downarrow j^{n-1}_{\overline{W}\backslash U,\overline{W}\backslash V} & & \downarrow j'_{U,V} & & \downarrow \mathrm{id} \\ H^{n-1}(\overline{W}\backslash V;G) & \xrightarrow{\delta_V} & H^n(\overline{W},\overline{W}\backslash V;G) & \xrightarrow{i_V} & H^n(\overline{W};G). \end{array}$$

Since $H^n(\overline{W};G) = 0$, both δ_U and δ_V are surjective. This, combined with the surjectivity of $j^{n-1}_{\overline{W}\backslash U,\overline{W}\backslash V}$ and non-trivially of $H^{n-1}(\overline{W} \setminus V;G)$, implies that $j'_{U,V}$ is also surjective. Finally, by the excision axiom the groups $H^n(\overline{W},\overline{W}\backslash U;G)$ and $H^n(\overline{W},\overline{W}\backslash V;G)$ are isomorphic to $H^n(X,X\backslash U;G)$ and $H^n(X,X\backslash V;G)$, respectively. Therefore, the homomorphism $j^n_{U,V} : H^n(X,X\backslash U;G) \to H^n(X,X\backslash V;G)$ is surjective.

Proof of Corollary 1.3. For homogeneous ANR-compacta X with $\dim_G X < \infty$, where G is a countable principal ideal domain, Corollary 1.3 was established in [29]. The arguments from [29] also work in our situation.

This corollary implies the well-known invariance of domains property, which was established in [23] and [25], respectively, for compact homogeneous and locally homogeneous ANR-spaces X with $\dim X < \infty$.

5. Proof of Theorem 1.4

(i) It suffices to show that the one-point compactification $bX = X \cup \{b\}$ of X is dimensionally full-valued. To this end, we use the following result of Dranishnikov [7, Theorem 12.3-12.4]: If Y is a finite-dimensional ANR-compactum, then $\dim_{\mathbb{Z}_{(p)}} Y = \dim_{\mathbb{Z}_p} Y$ for all prime p. Moreover, $\dim_{\mathbb{Q}} Y \leq \dim_G Y$ for any group $G \neq 0$ and there exists a prime number p with $\dim Y = \dim_{\mathbb{Z}_{(p)}} Y = \dim_{\mathbb{Z}_p} Y$. The proof presented in [7] works also when Y is the one-point compactification of an ANR-space. Here $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is the cyclic group and $\mathbb{Z}_{(p)} = \{m/n : n \text{ is not divisible by } p\} \subset \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. We also consider the quotient group $\mathbb{Z}_{p^\infty} = \mathbb{Q}/\mathbb{Z}_{(p)}$. It is well known [21] that the so called Bockstein basis consists of the groups $\sigma = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_{(p)}, \mathbb{Z}_{p^\infty} : p \in \mathcal{P}\}$, $\mathcal{P}$ is the set of all primes. For any group (not necessarily countable) there exists a collection $\sigma(G) \subset \sigma$ such that $\dim_G X = \sup\{\dim_H X : H \in \sigma(G)\}$ for any space X.

Let X be a locally homogeneous ANR-space with $\dim X = n$. According to the mentioned above Dranishnikov's result, there exists $p \in \mathcal{P}$ with $\dim_{\mathbb{Z}_{(p)}} bX = \dim_{\mathbb{Z}_p} bX = n$. If $\dim_{\mathbb{Z}_{p^\infty}} bX = n$, we are done. Indeed, according to [7, Lemma 2.6], bX is p-regular, i.e.

$$\dim_{\mathbb{Z}_{(p)}} bX = \dim \mathbb{Z}_{p^\infty} bX = \dim_{\mathbb{Z}_p} bX = \dim_{\mathbb{Q}} bX = n.$$

Then, applying again Dranishnikov's result [7, Theorem 12.3], we obtain $\dim_{\mathbb{Q}} bX \leq \dim_G bX \leq n$ for any group $G \neq 0$. Hence, $\dim_G bX = \dim bX = n$ for all nontrivial groups G, and by [22, Theorem 11], bX is dimensionally full-valued. Therefore, Claim 6 completes the proof of Theorem 1.4(i).

Claim 6. If $\dim_{\mathbb{Z}_{(p)}} bX = \dim_{\mathbb{Z}_p} bX = n$ for some prime number p, then $\dim \mathbb{Z}_{p^\infty} bX = n$.

Suppose $\dim \mathbb{Z}_{p^\infty} bX \leq n-1$. According to the Bockstein inequalities (see, for example [7], or [22]), we have $\dim_{\mathbb{Z}_p} bX \leq \dim \mathbb{Z}_{p^\infty} bX + 1$, which in our case implies $\dim \mathbb{Z}_{p^\infty} bX = \dim \mathbb{Z}_{p^\infty} X = n-1$. Since, by [22, Theorem 7(1)], $\dim_{\mathbb{Z}_p} (bX)^2 = 2\dim_{\mathbb{Z}_p} X = 2n$, the next equality (see [22, Theorem 7(3)])

$$\dim_{\mathbb{Z}_{p^\infty}} (bX)^2 = \max\{2\dim_{\mathbb{Z}_{p^\infty}} bX, \dim_{\mathbb{Z}_p} (bX)^2 - 1\}$$

implies $\dim \mathbb{Z}_{p^\infty} X^2 = \dim \mathbb{Z}_{p^\infty} (bX)^2 = 2n-1$. Then, by Proposition 2.6(2), for every $z = (x,y) \in X^2$ there is a neighborhood W_z of z in X^2 with a compact closure such that if $W = U \times V \subset W_z$ with $W \in \mathcal{B}_z$ then $H^{2n-2}(\mathrm{bd}\,\overline{W}; \mathbb{Z}_{p^\infty}) \neq 0$. Since $\mathrm{bd}\,\overline{W} = (\overline{U} \times \mathrm{bd}\,\overline{V}) \cup (\mathrm{bd}\,\overline{U} \times \overline{V})$, we have the Meyer-Vietoris exact sequence (in all exact sequences below the coefficient groups Z_{p^∞} are suppressed)

$$H^{2n-3}(\Gamma_U \times \Gamma_V) \to H^{2n-2}(\Gamma_W) \to H^{2n-2}(\overline{U} \times \Gamma_V) \oplus H^{2n-2}(\Gamma_U \times \overline{V}) \to .$$

Here, $\Gamma_U = \mathrm{bd}\,\overline{U}$, $\Gamma_V = \mathrm{bd}\,\overline{V}$ and $\Gamma_W = \mathrm{bd}\,\overline{W}$. The Künneth formulas provide the following exact sequences

$$\sum_{i+j=2n-2} H^i(\overline{U}) \otimes H^j(\Gamma_V) \to H^{2n-2}(\overline{U} \times \Gamma_V) \to \sum_{i+j=2n-1} H^i(\overline{U}) * H^j(\Gamma_V),$$

$$\sum_{i+j=2n-2} H^i(\Gamma_U) \otimes H^j(\overline{V}) \to H^{2n-2}(\Gamma_U \times \overline{V}) \to \sum_{i+j=2n-1} H^i(\Gamma_U) * H^j(\overline{V})$$

and

$$\sum_{i+j=2n-3} H^i(\Gamma_U) \otimes H^j(\Gamma_V) \to H^{2n-3}(\Gamma_U \times \Gamma_V) \to \sum_{i+j=2n-2} H^i(\Gamma_U) * H^j(\Gamma_V).$$

Since $\dim_{Z_{p^\infty}} \overline{U} \leq n-1$ and $\dim_{Z_{p^\infty}} \overline{V} \leq n-1$, for all $i \geq 1$ the groups $H^{n-1+i}(\overline{U}; Z_{p^\infty})$ and $H^{n-1+i}(\overline{V}; Z_{p^\infty})$ are trivial. On the other hand, by Proposition 2.6(2), we can assume that the groups $H^{n-1}(\overline{U}; Z_{p^\infty})$ and $H^{n-1}(\overline{V}; Z_{p^\infty})$ are also trivial. Moreover, Proposition 2.6(1) implies that $\dim_{Z_{p^\infty}} \Gamma_U \leq n-2$ and $\dim_{Z_{p^\infty}} \Gamma_V \leq n-2$. Hence, the groups $H^{n-1+i}(\Gamma_U; Z_{p^\infty})$ and $H^{n-1+i}(\Gamma_V; Z_{p^\infty})$ are trivial for all $i \geq 0$. Therefore, all groups $H^{2n-2}(\overline{U} \times \Gamma_V; Z_{p^\infty})$, $H^{2n-2}(\Gamma_U \times \overline{V}; Z_{p^\infty})$ and $H^{2n-3}(\Gamma_U \times \Gamma_V; Z_{p^\infty})$ are trivial, which implies the triviality of the group $H^{2n-2}(\Gamma_W; Z_{p^\infty})$, a contradiction. Therefore, $\dim \mathbb{Z}_{p^\infty} bX = n$.

(ii) To prove the second half of Theorem 1.4, suppose $\dim X = n$. Since bX is dimensionally full-valued, by [22, Theorem 11], $\dim_{\mathbb{Q}} X = \dim_{\mathbb{Q}} bX = \dim bX = n$. Denote by $\widehat{H}_*$ the exact homology developed in [26] for locally compact spaces. The homological dimension $h\dim_G Y$ of a space Y is the largest number n such that $\widehat{H}_n(Y, \Phi; G) \neq 0$ for some closed set $\Phi \subset Y$. According to [16] and [26], $h\dim_G Y$ is the greatest m such that the local homology group $\widehat{H}_m(Y, Y \setminus y; G) = \varinjlim_{y \in U} \widehat{H}_m(Y, Y \setminus U; G)$ is not trivial for some $y \in Y$. Moreover, for any field F we have $h\dim_F Y = \dim_F Y$, see [16]. Therefore, $h\dim_{\mathbb{Q}} X = \dim_{\mathbb{Q}} X = n$ and $\widehat{H}_n(X, X \setminus x; \mathbb{Q}) \neq 0$ for all $x \in X$. This means that every $x \in X$ has a neighborhood W such that $\widehat{H}_n(X, X \setminus U; \mathbb{Q}) \neq 0$ for all open sets $U \subset W$ containing x. We assume that W satisfies conditions (1)–(3) from Theorem 1.1 with $G = \mathbb{Z}$ such that $\overline{W}$ is contractible in a proper set in X and $H^{n-1}(\mathrm{bd}\,\overline{U}; \mathbb{Z}) \neq 0$ for all neighborhoods U of x with $U \subset W$, see Proposition 2.6(2). Let V be a neighborhood of x with $\overline{V} \subset W$.

Claim 7. The pair $\overline{W}\backslash V \subset \overline{W}$ is an $(n-1)$-homology membrane spanned on $\overline{W}\backslash V$ for any nontrivial $\gamma \in H_{n-1}(\overline{W}\backslash V;\mathbb{Q})$.

Since $\widehat{H}_n(X, X\backslash V;\mathbb{Q}) \neq 0$, by the excision axiom $\widehat{H}_n(\overline{W},\overline{W}\backslash V;\mathbb{Q}) \neq 0$. Consider the exact sequences (see [26] for the existence of such sequences)

$$\to \widehat{H}_n(\overline{W};\mathbb{Q}) \to \widehat{H}_n(\overline{W},\overline{W}\backslash V;\mathbb{Q}) \to \widehat{H}_{n-1}(\overline{W}\backslash V;\mathbb{Q}) \to \widehat{H}_{n-1}(\overline{W};\mathbb{Q})$$

and

$$0 \to \mathrm{Ext}(H^{n+1}(\overline{W};\mathbb{Z}),\mathbb{Q}) \to \widehat{H}_n(\overline{W};\mathbb{Q}) \to \mathrm{Hom}(H^n(\overline{W};\mathbb{Z}),\mathbb{Q}) \to 0$$
$$0 \to \mathrm{Ext}(H^n(\overline{W};\mathbb{Z}),\mathbb{Q}) \to \widehat{H}_{n-1}(\overline{W};\mathbb{Q}) \to \mathrm{Hom}(H^{n-1}(\overline{W};\mathbb{Z}),\mathbb{Q}) \to 0.$$

Since $H^{n+1}(\overline{W};\mathbb{Z}) = H^n(\overline{W};\mathbb{Z}) = H^{n-1}(\overline{W};\mathbb{Z}) = 0$, both $\widehat{H}_n(\overline{W};\mathbb{Q})$ and $\widehat{H}_{n-1}(\overline{W};\mathbb{Q})$ are trivial. Therefore, the group $\widehat{H}_{n-1}(\overline{W}\backslash V;\mathbb{Q})$ is nontrivial. Moreover, the exact sequence

$$\mathrm{Ext}(H^n(\overline{W}\backslash V;\mathbb{Z}),\mathbb{Q}) \to \widehat{H}_{n-1}(\overline{W}\backslash V;\mathbb{Q}) \to \mathrm{Hom}(H^{n-1}(\overline{W}\backslash V;\mathbb{Z}),\mathbb{Q}) \to 0$$

shows that $\widehat{H}_{n-1}(\overline{W}\backslash V;\mathbb{Q})$ is isomorphic to $\mathrm{Hom}(H^{n-1}(\overline{W}\backslash V;\mathbb{Z}),\mathbb{Q})$ because $H^n(\overline{W}\backslash V;\mathbb{Z}) = 0$ according to the remark after Proposition 2.3.

To complete the proof of Claim 7, we need to show that if B is a proper closed subset of $\overline{W}$ containing $\overline{W}\backslash V$, then $i^{n-1}_{\overline{W}\backslash V,B}(\gamma) \neq 0$ for every nontrivial $\gamma \in H_{n-1}(\overline{W}\backslash V;\mathbb{Q})$. To this end, consider the exact sequence

$$0 \to \mathrm{Ext}(H^n(B;\mathbb{Z}),\mathbb{Q}) \to \widehat{H}_{n-1}(B;\mathbb{Q}) \to \mathrm{Hom}(H^{n-1}(B;\mathbb{Z}),\mathbb{Q}) \to 0.$$

Since B is contractible in X (as a subset of W), $H^n(B;\mathbb{Z}) = 0$. Hence, $\widehat{H}_{n-1}(B;\mathbb{Q})$ is isomorphic to $\mathrm{Hom}(H^{n-1}(B;\mathbb{Z}),\mathbb{Q})$. Because $B = \overline{W}\backslash\Gamma$ for some open set $\Gamma \subset V$, passing to a smaller subset of Γ, we can assume that Γ is a neighborhood of some $y \in V$ such that $\overline{\Gamma} \subset V$ and Γ satisfies conditions (1)–(3) from Theorem 1.1. Then, the arguments from the proof of Claim 5 show that $H^{n-1}(B;\mathbb{Z}) = H^{n-1}(\overline{W}\backslash\Gamma;\mathbb{Z})$ is not trivial and there is a surjective homomorphism $H^{n-1}(B;\mathbb{Z}) \to H^{n-1}(\overline{W}\backslash V;\mathbb{Z})$. Therefore, there exists an injective homomorphism from $\widehat{H}_{n-1}(\overline{W}\backslash V;\mathbb{Q})$ into $\widehat{H}_{n-1}(B;\mathbb{Q})$ because $\widehat{H}_{n-1}(\overline{W}\backslash V;\mathbb{Q})$ is isomorphic to $\mathrm{Hom}(H^{n-1}(\overline{W}\backslash V;\mathbb{Z}),\mathbb{Q})$ and $\widehat{H}_{n-1}(B;\mathbb{Q})$ is isomorphic to $\mathrm{Hom}(H^{n-1}(B;\mathbb{Z}),\mathbb{Q})$. We already proved that $\widehat{H}_{n-1}(\overline{W};\mathbb{Q}) = 0$. Hence, $\overline{W}\backslash V \subset \overline{W}$ is an $(n-1)$-homology membrane (with respect to the exact homology $\widehat{H}_*$) spanned on $\overline{W}\backslash V$ for any nontrivial $\gamma \in \widehat{H}_{n-1}(\overline{W}\backslash V;\mathbb{Q})$. Next, we need the following result [26, Theorem 4]: For every pair $A \subset Y$ of a space Y and its closed set A, any integer k and a group G there is an epimorphism $T^k_{Y,A} : \widehat{H}_k(Y,A;G) \to H_k(Y,A;G)$, which is an isomorphism when G is a field. Hence, in our situation, we have the commutative diagram

$$\begin{array}{ccc}
\widehat{H}_{n-1}(\overline{W}\backslash V;\mathbb{Q}) & \xrightarrow{\widehat{i}^{n-1}_{\overline{W}\backslash V,B}} & \widehat{H}_{n-1}(B;\mathbb{Q}) \\
\downarrow T^{n-1}_{\overline{W}\backslash V} & & \downarrow T^{n-1}_{B} \\
H_{n-1}(\overline{W}\backslash V;\mathbb{Q}) & \xrightarrow{i^{n-1}_{\overline{W}\backslash V,B}} & H_{n-1}(B;\mathbb{Q})
\end{array}$$

such that $T^{n-1}_{\overline{W}\setminus V}$ and T^{n-1}_{B} are isomorphisms and $\widehat{i}^{n-1}_{\overline{W}\setminus V,B}$ is injective. Therefore $i^{n-1}_{\overline{W}\setminus V,B}$ is also injective, which implies $i^{n-1}_{\overline{W}\setminus V,B}(\gamma) \neq 0$. This completes the proof of Claim 7.

Suppose U is a neighborhood of x with $\overline{U} \subset V$. Since the pair $\overline{W}\setminus V \subset \overline{W}$ is an $(n-1)$-homology membrane spanned on $\overline{W}\setminus V$ for any nontrivial $\gamma \in H_{n-1}(\overline{W}\setminus V;\mathbb{Q})$, according to [1], $H_{n-1}(\operatorname{bd}\overline{U};\mathbb{Q}) \neq 0$ and $\overline{U}$ is an $(n-1)$-homology membrane spanned on $\operatorname{bd}\overline{U}$ for some nontrivial $\gamma' \in H_{n-1}(\operatorname{bd}\overline{U};\mathbb{Q})$. The nontriviality of $H_{n-1}(\operatorname{bd}\overline{U};\mathbb{Q})$ implies $H_{n-1}(\operatorname{bd}\overline{U};\mathbb{Z}) \neq 0$, see [30, Proposition 4.5]. Observe that $H^{n-1}(\operatorname{bd}\overline{U};\mathbb{Z}) \neq 0$ yields $\dim \operatorname{bd}\overline{U} = n-1$. Finally, according to [20, Corollary], $\operatorname{bd}\overline{U}$ is dimensionally full-valued. □

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