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# PROPER ABSOLUTE EXTENSORS

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ABSTRACT. We describe the proper absolute (neighborhood) extensors for the class of at most $n$-dimensional spaces, notation $\mathrm{A(N)E_p(n)}$. For example, the unique locally compact $n$-dimensional separable metric space $X \in \mathrm{AE_p(n)}$ satisfiyng the $\mathrm{DD^nP}$-property is the space obtained by deleting a point from the $n$-dimensional Menger compactum. Non-metrizable $\mathrm{A(N)E_p(n)}$-spaces are also described.

## 1. Introduction and preliminary results

In this note we describe the proper absolute extensors for finite-dimensional spaces, see Theorem 2.4 and Theorem 3.2. Recall that a map $f : X \to Y$ is proper if $f^{-1}(K)$ is compact for every compact $K \subset Y$. Note that if $X, Y$ are locally compact, then $f$ is proper iff it is closed and all fibres $f^{-1}(y)$, $y \in Y$, are compact. Proper and closed extensions of maps were considered by different authors, see Michael [13], Nowiński [15]. Our results are closer to Chigogidze's ones from [4], where proper absolute (neighborhood) extensors were introduced and studied.

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We say that a locally compact space $X$ is a *proper absolute neighborhood extensor for the class of at most n-dimensional spaces* (notation $X \in \mathrm{ANE_p(n)}$) if every proper map $f : A \to X$, where $A$ is a closed subset of a locally compact Lindelöf-space $Y$ with $\dim Y \leq n$, admits a perfect extension $\widetilde{f}$ over a closed neighborhood of $A$ in $Y$. When $f$ admits a proper extension over $Y$, we say $X$ is a *proper absolute extensor for the class of at most n-dimensional spaces* (notation $X \in \mathrm{AE_p(n)}$). Since every space admitting a proper map into a compact space is compact, it follows from the definition that there is no compact $\mathrm{AE_p(n)}$-space. In particular, the $n$-sphere $\mathbb{S}^n$, which is an absolute extensor for the $n$-dimensional spaces, is not an $\mathrm{AE_p(n)}$.

If, in the above definition, $Y$ is metric and we drop the requirement for $f$ and $\widetilde{f}$ to be proper maps, we obtain the definition of absolute (neighborhood) extensors (notation A(N)E(n)) for the class of at most $n$-dimensional spaces. It is well known [3] that a metric space $X$ is an ANE(n) iff $X$ is $\mathrm{LC^{n-1}}$. Moreover, if in addition, $X$ is $\mathrm{C^{n-1}}$, then $X \in \mathrm{AE(n)}$. Recall that a space $X$ is $\mathrm{LC^n}$ if for every $x \in X$ and its neighborhood $U$ in $X$ there is another neighborhood $V$ of $x$ such that $V \overset{m}{\hookrightarrow} U$ for all $m \leq n$ (here $V \overset{m}{\hookrightarrow} U$ means that $V \subset U$ and every map from $m$-dimensional sphere $\mathbb{S}^m$ into $V$ can be extended to a map $\mathbb{B}^{m+1} \to U$ over the $(m+1)$-dimensional cub $\mathbb{B}^{m+1}$). We also say that a set $A \subset X$ is $k-\mathrm{LCC}$ in $X$ if for every point $x \in A$ and its neighborhood $U$ in $X$ there exists another neighborhood $V$ of $x$ with $V \setminus A \overset{k}{\hookrightarrow} U \setminus A$. If $A$ is $k-\mathrm{LCC}$ in $X$ for all $k \leq n$, then $A$ is said to be $\mathrm{LCC^n}$ in $X$.

## 2. Second countable $\mathrm{AE_p(n)}$-spaces

Everywhere in this section by a space, if not explicitly said otherwise, we mean a locally compact separable metric space. By $C(Z, X)$ we denote the set of all continuous maps from $Z$ to $X$ equipped with the compact-open topology. A close subset $A \subset X$ is said to be a *$Z_n$-set in $X$* [14] if the set $C(\mathbb{B}^n, X\backslash A$ is dense in $C(\mathbb{B}^n, X)$. The following description of $Z_n$-sets in metric spaces is well known, but we couldn't find a reference.

**Lemma 2.1.** *Let $(X, d)$ be a metric $\mathrm{LC^{n-1}}$-space and $A$ be a closed nowhere dense set in $X$. Then $A$ is a $Z_n$-set in $X$ iff $A$ is $\mathrm{LCC^{n-1}}$.*

*Proof.* The sufficiency follows from the properties of metric $\mathrm{LC^{n-1}}$-spaces and the definition of $Z_n$-sets. Suppose $A$ is $\mathrm{LCC^{n-1}}$, $f : \mathbb{B}^n \to X$ is a given map and $\eta > 0$. We consider the following property for every $x \in X$: if $U$ is a neighborhood of $x$ in $X$, then there is another neighborhood $V \subset U$ such that $V \setminus A \overset{n-1}{\hookrightarrow} U \setminus A$. Because $A$ is $\mathrm{LCC^{n-1}}$ and $X$ is $\mathrm{LC^{n-1}}$, every $x \in X$ has that property. So, by [9] we can assume that the metric

$d$ satisfies the following condition: To every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $B_\delta(x) \setminus A \overset{m}{\hookrightarrow} B_\varepsilon(x) \setminus A$ for every $x \in X$ and every $m \leq n-1$. Here $B_\delta(x)$ denotes the open ball in $X$ with center $x$ and a radius $\delta$. We write $\delta \overset{m}{\hookrightarrow} \varepsilon$ to denote that $B_\delta(x) \setminus A \overset{m}{\hookrightarrow} B_\varepsilon(x) \setminus A$ for every $x \in X$. We choose a finite sequence $\{\varepsilon_m\}_{m \leq n}$ with $\varepsilon_m \overset{m}{\hookrightarrow} \varepsilon_{m+1}$ for every $0 \leq m \leq n-1$ and $\varepsilon_n = \eta$. Next, define the set-valued maps $\varphi_m : \mathbb{B}^n \rightsquigarrow X$ by $\varphi_m(y) = B_{\varepsilon_m}(f(y)) \setminus A$, $0 \leq m \leq n$. Since $A$ is $\mathrm{LCC^{n-1}}$, $\varphi_m(y) \neq \varnothing$ for all $y \in \mathbb{B}^n$. It is easily seen that each $\varphi_m$ has the following property: If $K$ is a compact subset of $\varphi_m(y_0)$ for some $y_0 \in \mathbb{B}^n$, then there is a neighborhood $O(y_0) \subset \mathbb{B}^k$ with $K \subset \varphi_m(y)$ for all $y \in O(y_0)$. According to [10], there exists a map $g : \mathbb{B}^n \to X$ with $g(y) \in \varphi_n(y)$ for all $y \in \mathbb{B}^n$. This means that $g$ maps $\mathbb{B}^n$ into $X \setminus A$ and $d(f(y), g(y)) < \eta$ for every $y \in \mathbb{B}^n$. Hence, $A$ is a $Z_n$-set in $X$. □

**Remark 2.2.** The referee noted that Lemma 2.1 also follows from the filtration theorem of Shchepin-Brodsky [17].

For every locally compact space $X$ let $\omega X = X \cup \{\omega\}$ be the one-point compactification of $X$.

**Lemma 2.3.** *If $X$ is an $\mathrm{AE_p(n)}$-space, then $\{\omega\}$ is a $Z_n$-set in $\omega X$.*

*Proof.* Since every metric $\mathrm{AE_p(n)}$-space is an absolute extensor for compact spaces of dimension $\leq n$, $X$ is $\mathrm{LC^{n-1}}$. Hence, by Lemma 2.1, we need to show that $\{\omega\}$ is $\mathrm{LCC^{n-1}}$. Suppose there are $k \leq n-1$ and a neighborhood $U$ of $\{\omega\}$ in $\omega X$ such that for every neighborhood $V$ of $\{\omega\}$ there exists a map $g_V : \mathbb{S}^k \to V \backslash \{\omega\}$ which does not admit an extension from $\mathbb{B}^{k+1}$ into $U \backslash \{\omega\}$. Take a local base $\{V_m\}$ of neighborhoods of $\{\omega\}$ in $\omega X$ and corresponding maps $g_m : \mathbb{S}^k \to V_m \backslash \{\omega\}$ such that $V_m \subset U$ and each $g_m$ cannot be extended to a map $\widetilde{g}_m : \mathbb{B}^{k+1} \to U \backslash \{\omega\}$. Now, for each $m$ let $\mathbb{S}^k_m$ and $\mathbb{B}^{k+1}_m$ be copies of $\mathbb{S}^k$ and $\mathbb{B}^{k+1}$, respectively. Consider the disjoint unions $Y = \biguplus_{m=1}^\infty \mathbb{B}^{k+1}_m$, $A = \biguplus_{m=1}^\infty \mathbb{S}^k_m$ and their one-point compactification $\omega Y = Y \cup \{\omega_Y\}$. Obviously, $\omega A = A \cup \{\omega_Y\}$ and there is a map $g : A \to X$ with $g|\mathbb{S}^k_m = g_m$. Since the map $g$ is proper, it admits a proper extension $\widetilde{g} : Y \to X$. Hence, $\widetilde{g}$ is extended to a map $h : \omega Y \to \omega X$ such that $h(\{\omega_Y\}) = \{\omega\}$. Consequently, $h^{-1}(U)$ contains almost all $\mathbb{B}^{k+1}_m$. On the other hand, $h(\mathbb{B}^{k+1}_m) = \widetilde{g}(\mathbb{B}^{k+1}_m) \subset X$. Therefore, $\widetilde{g}|\mathbb{B}^{k+1}_m)$ is a map into $U \backslash \{\omega\}$ extending $g_m$ for every $\mathbb{B}^{k+1}_m$ contained in $h^{-1}(U)$, a contradiction. □

**Proposition 2.4.** *Every space $X$ is an $\mathrm{AE_p}(0)$.*

*Proof.* Let $A \subset Y$ be a closed set and $f : A \to X$ be a proper map, where $Y$ is a 0-dimensional locally compact and Lindelöf space. Denote by $\overline{A}$ the closure of $A$ in the Čech-Stone compactification $\beta Y$ of $Y$. Then $f$ can be extended to a map $f_1 : \overline{A} \to \omega X$ with $f_1(\overline{A}\backslash A) = \{\omega\}$. Next, consider the map $g : A \cup (\beta Y\backslash Y) \to \omega X$ such that $g(y) = f_1(y)$ for $y \in \overline{A}$ and $g(y) = \{\omega\}$ for $y \in \beta Y\backslash Y$. Since $\omega X \in \mathrm{AE}(0)$ (as a complete metric space), $g$ admits an extension $\widetilde{g} : \beta Y \to \omega X$. The set $\widetilde{g}(A) = f(A)$ is closed in $X$, so $A_1 = \widetilde{g}^{-1}(f(A))$ is closed and $G_\delta$ in $Y$ such that $\widetilde{g}|A_1$ is proper. Consider the function space $C(\beta Y, \omega X)$ with the uniform convergence topology and let $B = A_1 \cup (\beta Y\backslash Y)$. The set $C_B = \{h \in C(\beta Y, \omega X) : h|B = \widetilde{g}|B\}$ is a complete metric space. Choose a sequence $\{K_i\}$ of compact sets $K_i \subset Y$ with $\bigcup_{i\geq 1} K_i = Y\backslash A_1$ (this is possible because $Y$ is a locally compact Lindelöf-space and $A_1$ is a closed $G_\delta$-set in $Y$). Let $C_i$ be the set of all maps $h \in C_B$ such that $h(K_i) \subset X$. Since $\{\omega\}$ is nowhere dense set in $\omega X$ (it is actually a $Z_0$-set in $\omega X$) and $\omega X \in \mathrm{AE}(0)$, we can show that each $C_i$ is an open and dense subset of $C_B$. Hence, $\bigcap_{i\geq 1} C_i \neq \varnothing$. Then $h(Y) \subset X$ and $h(\beta Y\backslash Y) = \{\omega\}$ for every $h \in \bigcap_{i\geq 1} C_i$. Hence, $h|Y$ is a proper map into $X$ extending $f$. □

**Theorem 2.5.** *The following conditions are equivalent for any space $X$ and $n \geq 1$:*

(i) $X \in \mathrm{AE_p(n)}$;
(ii) *The one-point compactification $\omega X$ is an* AE(n) *and $\{\omega\}$ is a $Z_n$-set in $\omega X$;*
(iii) *There exists a metrizable compactification $\widetilde{X}$ of $X$ such that both $\widetilde{X}$ and the remainder $\widetilde{X}\backslash X$ are* AE(n)*spaces, and $\widetilde{X}\backslash X$ is an $Z_n$-set in $\widetilde{X}$.*

*Proof.* Suppose $X \in \mathrm{AE_p(n)}$ and embed $\omega X$ in the Hilbert cube $Q$. According to Dranishnikov [7] there exists a surjective, open $n$-invertible map $d_n : \mu^n \to Q$ such that $d_n^{-1}(z)$ is homeomorphic to $\mu^n$ for every $z \in Q$, where $\mu^n$ is the universal $n$-dimensional Menger compactum. Recall that the $n$-invertibility of $d_n$ means that for any paracompact space $Z$ of dimension $\dim Z \leq n$ and a map $g : Z \to Q$ there is a map $\widetilde{g} : Z \to \mu^n$ such that $d_n \circ \widetilde{g} = g$. Then $d_n^{-1}(\{\omega\})$ is nowhere dense in $\mu^n$ and $\mu^n$ is a compactification of $\mu^n\backslash d_n^{-1}(\{\omega\})$. Now, consider the restriction $d'_n = d_n|d_n^{-1}(X)$. Obviously, $d'_n$ is a proper map, so it admits a proper extension $h_n : \mu^n\backslash d_n^{-1}(\{\omega\}) \to X$. The properness of $h_n$ implies that $h_n$ can be extended to a continuous map $\widetilde{h}_n : \mu^n \to \omega X$ such that $\widetilde{h}_n(d_n^{-1}(\{\omega\})) = \{\omega\}$. Then $\widetilde{h}_n|(d_n^{-1}(\omega X)) = d_n|(d_n^{-1}(\omega X))$. Hence, $\widetilde{h}_n$ is an $n$-invertible map because so is $d_n$. This fact in combination with the well known fact that $\mu^n$ is an absolute extensor for all Lindelöf locally compact $n$-dimensional spaces, yields that $\omega X \in \mathrm{AE(n)}$. Finally, by Lemma 2.2, $\{\omega\}$ is a $Z_n$-set in $\omega X$. That completes the implication $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (iii)$ is trivial.

To prove the implication $(iii) \Rightarrow (i)$ we follow the proof of Proposition 2.3. Let $A \subset Y$ be a closed set and $f : A \to X$ be a proper map, where $Y$ is at most $n$-dimensional locally compact and Lindelöf space. Following the notations from the proof of Proposition 2.3, we first extend $f$ to a map $f_1 : \overline{A} \to \widetilde{X}$ with $f_1(\overline{A} \backslash A) \subset \widetilde{X} \backslash X$. Then, using that $\widetilde{X} \backslash X \in \mathrm{AE(n)}$, we extend $f_1$ to a map $g : \overline{A} \cup (\beta Y \backslash Y) \to \widetilde{X}$ such that $g(\beta Y \backslash Y) \subset \widetilde{X} \backslash X$. Next, since $\widetilde{X}$ is an AE(n), we find a map $\widetilde{g} : \beta Y \to \widetilde{X}$ extending $g$. Then $\widetilde{g}|A_1 : A_1 \to X$ is a proper extension of $f$ with $A_1 \subset Y$ being a closed $G_\delta$-subset of $Y$ containing $A$. Let $B = A_1 \cup (\beta Y \backslash Y)$ and consider a sequence $\{K_i\}$ of compact sets in $Y$ with $\bigcup_{i \geq 1} K_i = Y \backslash A_1$ and the corresponding sets $C_B = \{h \in C(\beta Y, \widetilde{X}) : h|B = \widetilde{g}|B\}$ and $C_i$. Now, since $\widetilde{X} \backslash X$ is $Z_n$-set in $\widetilde{X}$, all $C_i$ are open and dense in $C_B$. Therefore, $\bigcap_{i \geq 1} C_i \neq \varnothing$ and $h|Y$ is a proper map into $X$ extending $f$ for every $h \in \bigcap_{i \geq 1} C_i$. $\square$

We say that a space $X$ satisfies the *disjoint $n$-disks property* (br., $\mathrm{DD^nP}$-property) if any two maps $f, g : \mathbb{B}^n \to X$ can be approximated by maps $f', g' : \mathbb{B}^n \to X$ with $f'(\mathbb{B}^n) \cap g'(\mathbb{B}^n) = \varnothing$. Bestvina [2] characterized $\mu^n$ as the only $n$-dimensional metric AE(n)-compactum satisfying the $\mathrm{DD^nP}$-property. Since $\omega X$ satisfies the $\mathrm{DD^nP}$ provided $X \in \mathrm{DD^nP}$ and $\{\omega\}$ is a $Z_n$-set in $\omega X$, Bestvina's result implies the following one:

**Corollary 2.6.** *Let $X \in \mathrm{AE_p(n)}$ with $\dim X = n$. Then $X \in \mathrm{DD^nP}$ iff $\omega X$ is homeomorphic to $\mu^n$.*

Chigogidze [5] introduced the $n$-shape functor $(n-\mathrm{Sh})$ and proved that two $Z_n$-sets $X$ and $Y$ in $\mu^n$, $n \geq 1$, have the same $(n-1)$-shape if and only if $\mu^n \backslash X$ is homeomorphic to $\mu^n \backslash Y$. Surprisingly, Theorem 2.4 implies a particular case of Chigogidze's complement theorem.

**Corollary 2.7.** *Suppose $X$ and $Y$ are two $Z_n$-sets in $\mu^n$ such that $X, Y \in \mathrm{AE(n)}$. Then $\mu^n \backslash X$ is homeomorphic to $\mu^n \backslash Y$.*

*Proof.* Indeed, by Theorem 2.4 both $X' = \mu^n \backslash X$ and $Y' = \mu^n \backslash Y$ are $\mathrm{AE_p(n)}$. Moreover, $X'$ and $Y'$ satisfy the $\mathrm{DD^nP}$ because $X$ and $Y$ are $Z_n$-sets in $\mu^n$. Hence, by Corollary 2.5, both $\omega X'$ and $\omega Y'$ are homeomorphic to $\mu^n$. Finally, since $\mu^n$ is homogeneous [2], $X'$ is homeomorphic to $Y'$. $\square$

**Proposition 2.8.** *A space $X$ is an $\mathrm{AE_p(n)}$ if and only if $X$ is a proper $n$-invertible image of $\mu^n \backslash F$ for some $Z_n$-set $F \subset \mu^n$ with $F \in \mathrm{AE(n)}$.*

*Proof.* Let $X \in \mathrm{AE_p(n)}$ and embed $\omega X$ in $Q$. As in Theorem 2.4, considering Dranishnikov's resolution $d_n : \mu^n \to Q$ we obtain a proper map $h_n : \mu^n \backslash d_n^{-1}(\{\omega\}) \to X$ which extends the map $d_n | d_n^{-1}(X)$. Since $d_n$ is invertible, so is $h_n$. On the other hand, by [1], we can assume that $d_n^{-1}(K)$

is a $Z_n$-set in $\mu^n$ for every $Z$-set $K \subset Q$. Hence, $d_n^{-1}(\{\omega\}) \subset \mu^n$ is a $Z_n$-set (recall that a $Z$-set is a set which is $Z_n$-set for all $n$ and that every $z \in Q$ is a $Z$-set in $Q$). On the other hand, $d_n^{-1}(\{\omega\})$ is homeomorphic to $\mu^n$, so $d_n^{-1}(\{\omega\}) \in \mathrm{AE(n)}$.

Now, suppose there is a proper $n$-invertible map $g : \mu^n \backslash F \to X$ for some $Z_n$-set $F \subset \mu^n$. Since $g$ is $n$-invertible, every proper map $f : A \to X$, where $A$ is a closed subset of at most $n$-dimensional locally compact and Lindelöf space $Y$, can be lifted to a proper map $f' : A \to \mu^n \backslash F$. By Theorem 2.4, $\mu^n \backslash F \in \mathrm{AE_p(n)}$. So, $f'$ admits a proper extension $h : Y \to \mu^n \backslash F$. Finally, $g \circ h : Y \to X$ is a proper extension of $f$. Therefore, $X \in \mathrm{AE_p(n)}$. □

**Corollary 2.9.** *A space $X$ is an* $\mathrm{AE_p(n)}$ *if and only if $X$ is a proper $n$-invertible image of $\mu^n \backslash \{pt\}$.*

*Proof.* By Theorem 2.4, $\mu^n \backslash F$ is an $\mathrm{AE_p(n)}$ for every $F \in \mathrm{AE(n)}$ which is a $Z_n$-set in $\mu^n$. On the other hand, $\mu^n \backslash F$ satisfies the $\mathrm{DD^nP}$ as a complement of a $Z_n$-set in $\mu^n$. Hence, Proposition 2.7 and Corollary 2.5 complete the proof. □

Concerning $\mathrm{ANE_p(n)}$, arguments similar to the proof of Proposition 2.3 provide the next lemma.

**Lemma 2.10.** *If a space $X$ admits a metric* $\mathrm{ANE(n)}$*-compactification $\overline{X}$ such that $\overline{X} \backslash X$ is an* $\mathrm{AE(n)}$*, then $X \in \mathrm{ANE_p(n)}$.*

**Corollary 2.11.** *$\mathbb{R}^n$ is an* $\mathrm{ANE_p(n)}$ *and an* $\mathrm{AE_p(n-1)}$*, but not an* $\mathrm{AE_p(n)}$ – space.

*Proof.* It follows from Theorem 2.4 that $\mathbb{R}^n$ is an $\mathrm{AE_p(n-1)}$, but not an $\mathrm{AE_p(n)}$-space because $\omega\mathbb{R}^n = \mathbb{S}^n$ and any point of $\mathbb{S}^n$ is a $Z_{n-1}$-point in $\mathbb{S}^n$ but not a $Z_n$-point. On the other hand, $\mathbb{R}^n \in \mathrm{ANE_p(n)}$ according to Lemma 2.9. □

## 3. Non-metrizable $\mathrm{AE_p(n)}$-spaces

In this section all spaces are locally compact and Lindelöf. A map $f : X \to Y$ is called *n-soft* [16] if for every $n$-dimensional paracompact space $Z$, any closed set $A \subset Z$ and any two maps $h : A \to X$ and $g : Z \to Y$ with $g|A = f \circ h$ there is a continuous extension $\widetilde{h} : Z \to X$ of $h$ such that $g = f \circ \widetilde{h}$. We say that $f : X \to Y$ is a *map with a Polish kernel* if there is a Polish (i.e., completely metrizable and separable) space $P$ such that $X$ is $C$-embedded in $Y \times P$ and $f = \pi_Y|X$, where $\pi_Y : Y \times P \to Y$ is the projection.

The next lemma follows from the corresponding definitions.

**Lemma 3.1.** *Let $f : X \to Y$ is a proper $n$-soft map. Then $X \in \mathrm{AE_p(n)}$ if and only if $Y \in \mathrm{AE_p(n)}$.*

An inverse system $S = \{X_\alpha, p^\beta_\alpha, A\}$ is said to be *$\sigma$-complete* if all $X_\alpha$ are second countable spaces and every increasing sequence $\{\alpha_n\} \subset A$ has a supremum $\alpha$ in $A$ such that $X_\alpha$ is the limit space of the inverse sequence $\{X_{\alpha_n}, p^{\alpha_{n+1}}_{\alpha_n}, n \geq 1\}$. If $S$ is well-ordered and $X_\alpha$ is the limit of the inverse system $\{X_\beta, p^{\beta+1}_\beta, \beta < \alpha\}$ for every limit ordinal $\alpha \in A$, then $S$ is called a *continuous inverse system.*

Now, we can describe the non-metrizable $\mathrm{AE_p(n)}$-spaces.

**Theorem 3.2.** *For every $n \geq 1$ the following conditions are equivalent :*

(i) *$X$ is an $\mathrm{AE_p(n)}$-space of weight $\tau$;*
(ii) *$\omega X \in \mathrm{AE(n)}$;*
(iii) *$X$ is the limit space of a continuous inverse system $S = \{X_\alpha, p^\beta_\alpha, \tau\}$ such that all $X_\alpha$ are $\mathrm{AE_p(n)}$-spaces, $X_1$ is a locally compact separable metric space and the projections $p^{\alpha+1}_\alpha$ are perfect $n$-soft maps with metrizable kernels;*
(iv) *$X$ is the limit space of a $\sigma$-complete inverse system $S = \{X_\alpha, p^\beta_\alpha\}$ consisting of $\mathrm{AE_p(n)}$-spaces $X_\alpha$ and perfect $n$-soft projections $p_\alpha : X \to X_\alpha$.*

*Proof.* Let $X$ be an $\mathrm{AE_p(n)}$-space of weight $\tau$ and embed $\omega X$ in the Tychonoff cube $\mathbb{I}^\tau$. According to [8], there exists a compact $AE(n-1)$-space $D^\tau_n$ of dimension $n$ and weight $\tau$ and an $n$-invertible $(n-1)$-soft-map $f^\tau_n : D^\tau_n \to \mathbb{I}^\tau$. Since $\{\omega\}$ is a $G_\delta$-set in $\omega X$, there is a closed $G_\delta$-set $F \subset \mathbb{I}^\tau$ with $F \cap \omega X = \{\omega\}$. Deleting the interior of $F$, if necessary, we can suppose that $F$ is nowhere dense in $\mathbb{I}^\tau$. Then $(f^\tau_n)^{-1}(F)$ is a closed nowhere dense and $G_\delta$-subset of $D^\tau_n$ because $f^\tau_n$ is open (as a 0-soft map between AE(0)-spaces, see [5]). So, $Y = D^\tau_n \backslash (f^\tau_n)^{-1}(F)$ is a dense locally compact Lindelöf subset of $D^\tau_n$ containing $(f^\tau_n)^{-1}(X)$ as a closed subset. Since $X \in \mathrm{AE_p(n)}$, there is a proper map $g : Y \to X$ extending the restriction $f^\tau_n|(f^\tau_n)^{-1}(X)$. Finally, extend $g$ to a map $\widetilde{g} : D^\tau_n \to \omega X$. Now, consider the set valued map $r : \mathbb{I}^\tau \rightsquigarrow \omega X$, $r(x) = \widetilde{g}((f_n\tau)^{-1}(x))$. Obviously, $r(x) = \{x\}$ for every $x \in \omega X$. Since $f^\tau_n$ is $(n-1)$-soft, $r$ is projectively $(n-1)$-cosoft retraction in the sense of Dranishnikov [8]. Hence, by [8, Theorem 4.2], $\omega X$ is an AE(n)-space. So, $(i) \Rightarrow (ii)$.

If $\omega X \in \mathrm{AE(n)}$, then $\omega X$ is the limit a continuous inverse system $\widetilde{S} = \{\widetilde{X}_\alpha, \widetilde{p}^\beta_\alpha, \tau\}$ such that $\widetilde{X}_1$ is a point and all projections $\widetilde{p}^{\alpha+1}_\alpha$ are $n$-soft maps with metrizable kernels, see [8, Theorem 4.2]. Because $\{\omega\}$ is a $G_\delta$-set in $\omega X$, there is $\alpha_0 < \tau$ such that $\widetilde{X}_{\alpha_0}$ is metrizable and $\widetilde{p}^{-1}_{\alpha_0}(\widetilde{p}_{\alpha_0}(\{\omega\})) = \{\omega\}$. Consequently, $\widetilde{p}^{-1}_\alpha(X_\alpha) = X$ for every $\alpha \geq \alpha_0$, where $X_\alpha = \widetilde{X}_\alpha \backslash \widetilde{p}_\alpha(\{\omega\})$. Obviously, all restrictions $p_\alpha = \widetilde{p}_\alpha | X$ and $p^\beta_\alpha = \widetilde{p}^\beta_\alpha | X_\beta$ are perfect $n$-soft maps and $X$ is the limit of the inverse system $S = \{X_\alpha, p^\beta_\alpha, \alpha \geq \alpha_0\}$. Finally, by Lemma 3.1, each $X_\alpha$ is an $\mathrm{AE_p(n)}$. This completes the implication $(ii) \Rightarrow (iii)$.

The implication $(iii) \Rightarrow (iv)$ follows by similar arguments using that $\omega X$ (as an AE(n)-compactum) is the limit space of a $\sigma$-complete inverse system $\widetilde{S} = \{\widetilde{X}_\alpha, \widetilde{p}^\beta_\alpha\}$ consisting of AE(n)-metric compacta $\widetilde{X}_\alpha$ and perfect $n$-soft projections $\widetilde{p}_\alpha : X \to X_\alpha$, see [8, Theorem 4.2]. Finally, since $X$ admits $n$-soft perfect maps into $\mathrm{AE_p(n)}$-spaces, the implication $(iv) \Rightarrow (i)$ follows from Lemma 3.1. □

Because every AE(n)-compactum of dimension $\leq n$, where $n \geq 1$, is metrizable [8, Theorem 4.4], Theorem 3.2 implies the following

**Corollary 3.3.** *Every* $\mathrm{AE_p(n)}$*-space* $X$ *with* $n \geq 1$ *is metrizable provided* $\dim X \leq n$.

Concerning $\mathrm{AE_p(0)}$-spaces we have the following:

**Theorem 3.4.** *The following conditions are equivalent:*

- (i) $X$ *is an* $\mathrm{AE_p(0)}$*-space;*
- (ii) $X \in \mathrm{AE}(0)$*;*
- (iii) $\omega X \in \mathrm{AE}(0)$*;*
- (iv) $X$ *is the limit space of a* $\sigma$*-complete inverse system* $S = \{X_\alpha, p^\beta_\alpha\}$ *consisting of locally compact separable metric spaces* $X_\alpha$ *and perfect* 0*-soft projections* $p_\alpha : X \to X_\alpha$.

*Proof.* Let $X$ be an $\mathrm{AE_p(0)}$-space of weight $\tau$ and embed $\omega X$ in the Tychonoff cube $\mathbb{I}^\tau$. By [12], $\mathbb{I}^\tau$ is an image of the Cantor cube $D^\tau$ under a perfect 0-invertible map $f^\tau_0$. As in the proof of Theorem 3.2, take a closed $G_\delta$-set $F \subset \mathbb{I}^\tau$ with $F \cap \omega X = \{\omega\}$ and let $Y = D^\tau \backslash (f^\tau_0)^{-1}(F)$. Then $Y$, as a locally compact Lindelöf subset of $D^\tau$, is an AE(0)-space. Indeed, there is a locally compact subset $Y_0$ of the Cantor set $D^{\aleph_0}$ with $\pi^{-1}(Y_0) = Y$, where $\pi : D^\tau \to D^{\aleph_0}$ is the projection. Since $\pi$ is 0-soft and $Y_0 \in \mathrm{AE}(0)$, $Y \in \mathrm{AE}(0)$. Next, consider a proper map $g : Y \to X$ extending the restriction $f^\tau_0 | (f_0\tau)^{-1}(X)$. Then $g$ is also 0-invertible, hence $X \in \mathrm{AE}(0)$ because $Y \in \mathrm{AE}(0)$. Therefore, $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (iii)$ is well known, see [6, Proposition 3.9]. For the implication $(iii) \Rightarrow (iv)$, observe that Haydon's [11] spectral characterization of compact AE(0)-spaces implies that $\omega X$ is the limit of a $\sigma$-complete inverse system $\widetilde{S} = \{\widetilde{X}_\alpha, \widetilde{p}^\beta_\alpha\}$ consisting of compact metric spaces $\widetilde{X}_\alpha$ and perfect 0-soft projections $\widetilde{p}_\alpha : X \to X_\alpha$. Since $\{\omega\}$ is a $G_\delta$-subset of $\omega X$, the restriction of $\widetilde{S}$ over $X$ provides a $\sigma$-complete inverse system $S = \{X_\alpha, p^\beta_\alpha\}$

consisting of locally compact separable metric spaces $X_\alpha$ and perfect 0-soft projections $p_\alpha : X \to X_\alpha$ such that $X$ is the limit of $S$ (see the proof of Theorem 3.2). The implication $(iv) \Rightarrow (i)$ follows from Lemma 3.1 and Proposition 2.3 because $X$ admits a 0-soft map into a separable locally compact metric space $X_\alpha$. □

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