LINEAR AND UNIFORMLY CONTINUOUS SURJECTIONS BETWEEN C_p-SPACES OVER METRIZABLE SPACES

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ABSTRACT. For any Tychonoff space X let $D(X)$ be either the set $C(X)$ of all continuous functions on X or the set $C^*(X)$ of all bounded continuous functions on X. When $D(X)$ is endowed with the point convergence topology, we write $D_p(X)$.

Let $T : D_p(X) \to D_p(Y)$ be a continuous linear surjection, where X is a metrizable space and Y is perfectly normal. We show that if X has some dimensional-like property $\mathcal{P}$, then so does Y. For example, $\mathcal{P}$ could be one of the following properties: zero-dimensionality, countable-dimensionality or strong countable-dimensionality. This result remains true if T is a uniformly continuous and inversely bounded surjection.

Also, we consider other properties $\mathcal{P}$: of being a scattered, or a strongly σ-scattered space, or being a Δ_1-space, see [16]. Our results strengthen and extend several results from [6], [13] and [16].

1. Introduction

For a Tychonoff space X, by $C(X)$ we denote the linear space of all continuous real-valued functions on X. $C^*(X)$ is a subspace of $C(X)$ consisting of the bounded functions. We write $C_p(X)$ (resp., $C_p^*(X)$) if $C(X)$ (resp., $C^*(X)$) is endowed with the pointwise convergence topology. The questions concerning linear or uniform homeomorphisms of C_p-spaces have been intensively studied by many authors. More information can be found in [2], [23], [26], [32], [33].

Throughout the paper by dimension we mean the *covering dimension* dim. Recall that for a Tychonoff space X and an integer $n \geq 0$, $\dim X \leq n$ if every finite functionally open cover of the space X has a finite functionally open refinement of order $\leq n$, see [10].

After the striking results of Pestov [27] and Gul'ko [14] that $\dim X = \dim Y$ for any Tychonoff spaces X and Y provided $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic or uniformly homeomorphic, Arhangel'skii posed a problem whether $\dim Y \leq \dim X$ if there is continuous linear surjection from $C_p(X)$

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onto $C_p(Y)$, see [1]. This question was answered negatively by Leiderman-Levin-Pestov [19] and Leiderman-Morris-Pestov [20]. For every finite-dimensional metrizable compact space Y there exists a continuous linear surjection $T : C_p([0,1]) \to C_p(Y)$ [20]. Later, Levin [21] showed that one can construct such a surjection which additionally is an open mapping.

However, it turned out that the zero-dimensional case is an exception. It was shown in [19] that if there is a linear continuous surjection $T : C_p(X) \to C_p(Y)$ for compact metrizable spaces X and Y, then $\dim X = 0$ implies that $\dim Y = 0$. The last result was extended for arbitrary compact spaces by Kawamura-Leiderman [17] who also raised the question whether the same statement is true without the assumption of compactness of X and Y. Recently, this difficult question was answered positively in [11].

Everywhere below, by $D(X)$ we denote either $C^*(X)$ or $C(X)$, and $D_p(X)$ stays for $D(X)$ endowed with the pointwise convergence topology. In the present paper we mainly focus on linear or uniformly continuous surjections $T : D_p(X) \to D_p(Y)$, where X is a metrizable space, Y is either metrizable or perfectly normal and T satisfies some additional condition. Moreover, almost all results are true if we consider any one of the four possible cases: $D(X)$ is either $C(X)$ or $C^*(X)$ and $D(Y)$ is either $C(Y)$ or $C^*(Y)$. So, everywhere below, if not said otherwise, we assume that all four cases are considered.

- A map $T : D_p(X) \to D_p(Y)$ is called *uniformly continuous* if for every neighborhood U of the zero function in $D_p(Y)$ there is a neighborhood V of the zero function in $D_p(X)$ such that $f, g \in D_p(X)$ and $f - g \in V$ implies $T(f) - T(g) \in U$.
- For every bounded function $f \in C(X)$ by $||f||$ we denote its *supremum-norm*. A map $T : D(X) \to D(Y)$ is called *c-good* (see [12], [13]) if for every $g \in C^*(Y)$ there exists $f \in C^*(X)$ such that $T(f) = g$ and $||f|| \leq c||g||$.

$\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \dots\}$. We say that a sequence $\{g_n : n \in \mathbb{N}\} \subset C^*(Y)$ is *norm bounded* if there is $M > 0$ such that $||g_n|| \leq M$ for all $n \in \mathbb{N}$.

Definition 1.1. *A map $T : D(X) \to D(Y)$ is called inversely bounded if for every norm bounded sequence $\{g_n\} \subset C^*(Y)$ there is a norm bounded sequence $\{f_n\} \subset C^*(X)$ with $T(f_n) = g_n$ for each $n \in \mathbb{N}$.*

Evidently, every linear continuous map between $D_p(X)$ and $D_p(Y)$ is uniformly continuous and every c-good map is inversely bounded. Also, every linear continuous surjection $T : C_p^*(X) \to C_p^*(Y)$, where X and Y are arbitrary Tychonoff spaces, is inversely bounded, see [11, Proposition 3.3].

Recall that a normal topological space X is *countable-dimensional* (*strongly countable-dimensional*) if X can be represented as a countable union of normal finite-dimensional subspaces (resp., closed finite-dimensional subspaces). Note that there are countable-dimensional compact metrizable spaces which are not strongly countable-dimensional, see [10].

Marciszewski [23, Corollary 2.7] observed that, by modifying the Gulko's arguments [14] (c.f. [22], [24]), one can show the following theorem:

Theorem 1.2. *Let $\mathcal{P}$ be the property of metrizable spaces such that:*

(i) *If $X \in \mathcal{P}$ and Y is a subset of X then $Y \in \mathcal{P}$,*
(ii) *If X is a metrizable space which is a countable union of closed subsets $X_n \in \mathcal{P}$ then $X \in \mathcal{P}$.*

Then, for metrizable spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic, $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.

It is known that the covering dimension $\dim X \leq n$ satisfies the above conditions (i) and (ii). Much less is known about the case when $T : C_p(X) \to C_p(Y)$ is supposed to be only uniformly continuous and surjective. The following open problem has been posed in [13, Question 4.1].

Problem 1.3. *Let X be a compact metrizable strongly countable-dimensional [zero-dimensional] space. Suppose that there exists a uniformly continuous surjection $T : C_p(X) \to C_p(Y)$. Is Y necessarily strongly countable-dimensional [zero-dimensional]?*

The authors of [13] established that the answer to Problem 1.3 is affirmative provided that the uniformly continuous surjection T is c-good for some $c > 0$. Later, in [11] the same was proved for σ-compact metrizable spaces. In our paper we strengthen this result by proving this remains true for all metrizable spaces and all inversely bounded uniformly continuous surjections. In fact we develop a general scheme for the proof as follows.

We consider the properties $\mathcal{P}$ of normal spaces such that:

(a) if $X \in \mathcal{P}$ and $F \subset X$ is closed, then $F \in \mathcal{P}$;
(b) $\mathcal{P}$ is closed under finite products;
(c) if X is a countable union of closed subsets each having the property $\mathcal{P}$, then $X \in \mathcal{P}$;
(d) if $f : X \to Y$ is a closed map with finite fibers, where Y is a metrizable space with $Y \in \mathcal{P}$, then $X \in \mathcal{P}$.

From the classical results of dimension theory (see [10]) it follows that *zero-dimensionality*, *countable-dimensionality* and *strongly countable-dimensionality* satisfy conditions $(a) - (d)$ above.

Now we formulate one of the main results of our paper.

Theorem 1.4. *Let X be a metrizable space and Y be perfectly normal. Suppose that $T : D_p(X) \to D_p(Y)$ is a uniformly continuous inversely bounded surjection. For any topological property $\mathcal{P}$ satisfying conditions $(a)-(d)$ above, if $X \in \mathcal{P}$ then $Y \in \mathcal{P}$.*

Corollary 1.5. *Let X, Y and $T : D_p(X) \to D_p(Y)$ satisfy the hypotheses of Theorem* 1.4*.*

- (i) *If X is either countable-dimensional or strongly countable-dimensional, then so is Y.*
- (ii) *If X is zero-dimensional, then so is Y.*

Note that item (ii) was established in [11, Theorem 1.1] for arbitrary Tychonoff spaces X, Y and c-good surjections T. However, we don't know whether there exists a uniformly continuous inversely bounded map which is not c-good for some $c > 0$.

A linear continuous version of Theorem 1.4 is also true (for metrizable compact spaces it was implicitly established in [19]).

Theorem 1.6. *Let X be a metrizable space and Y be a perfectly normal space. Suppose that $T : D_p(X) \to D_p(Y)$ is a linear continuous surjection. For any topological property $\mathcal{P}$ satisfying conditions $(a) - (d)$ above, if $X \in \mathcal{P}$ then $Y \in \mathcal{P}$.*

In the last part of our paper we show that for any two metrizable spaces X and Y such that there is a linear continuous surjection $T : C_p(X) \to C_p(Y)$ or $T : C_p^*(X) \to C_p^*(Y)$, and X is scattered, then so is Y (Theorem 4.5). Also, we apply Theorem 1.4 to the property $\mathcal{P}$ of being a strongly σ-scattered space and to the property $\mathcal{P}$ of being a Δ_1-space. All necessary definitions are given in Section 4.

2. Proof of Theorem 1.4

Our proof is based on the idea of support introduced by Gul'ko [14], see also [24], where this technique is well described. For every $y \in Y$ there is a map $\alpha_y : D_p(X) \to \mathbb{R}$, $\alpha_y(f) = T(f)(y)$. Since T is uniformly continuous, so is each α_y. For every $y \in Y$ we consider the family $\mathcal{A}(y)$ of all finite sets $K \subset X$ such that $a(y, K) < \infty$, where

$$a(y, K) = \sup\{|\alpha_y(f) - \alpha_y(g)| : f, g \in D(X), |f(x) - g(x)| < 1 \ \forall x \in K\}.$$

Note that $a(y, \varnothing) = \infty$ since T is surjective. For every $p, q \in \mathbb{N}$ we define

$$Y(p, q) = \{y \in Y : \exists K \in \mathcal{A}(y) \text{ with } a(y, K) \le p \text{ and } |K| \le q\}$$

and $M(p) = \bigcup\{M(p, q) : q \in \mathbb{N}\}$, where

$$M(p, q) = \{y \in Y(p, q) : a(y, K) > 2p \ \forall K \subset X \text{ with } |K| \le q - 1\}.$$

Gul'ko's methodology in [14] (see also [24]) was developed for metrizable spaces, but the extension of Gul'ko arguments from [18] and [11] shows that for every Tychonoff spaces X and Y all conditions (1) – (8) below are valid:

(1) $\mathcal{A}(y)$ is non-empty and it is closed under finite intersections. Moreover, $a(y, K_1 \cap K_2) \le a(y, K_1) + a(y, K_2)$ for all $K_1, K_2 \in \mathcal{A}(y)$;
(2) $Y(p, q)$ is closed in Y for all $p, q \in \mathbb{N}$;
(3) $y \in M(p)$ provided $p \ge a(y)$;
(4) $M(p, 1) = Y(p, 1)$ and $M(p, q) = Y(p, q) \setminus Y(2p, q-1)$ for $q \ge 2$;
(5) $Y = \bigcup\{M(p, q) : p, q \in \mathbb{N}\}$;
(6) $M(p, q_1) \cap M(p, q_2) = \varnothing$ for $q_1 \neq q_2$;
(7) For every $y \in M(p, q)$ there is a unique finite $K_p(y) \subset X$ with $|K_p(y)| = q$ and $a(y, K_p(y)) \le p$;
(8) The map $\varphi_{pq} : M(p, q) \to [X]^q$, $\varphi_{pq}(y) = K_p(y)$, is continuous, where $[X]^q$ denotes the set of all q-point subsets of X with the Vietoris topology.

Moreover, if X is metrizable and Y is normal, the map φ_{pq} satisfies the following additional condition:

Claim 1. *If $M \subset M(p, q)$ is closed in Y for some p, q, then the map $\varphi_{pq} \restriction_M$: $M \to [X]^q$ is closed and each fiber of $\varphi_{pq} \restriction_M$ is countably compact.*

Because $[X]^q$ is a metrizable space, it suffices to show that if $\{y_n\}$ is a sequence in M such that $\varphi_{pq}(y_n)$ converges to some $K \in [X]^q$, then $\{y_n\}$ has an accumulation point in M. That statement was established in the proof of condition (9) from [24] in case both X and Y are metrizable, but the same proof works when X is metrizable and Y is normal.

Since each $Y(p, q)$ is a closed subset of Y, it follows from (4) that each $M(p, q)$ is a countable union of closed subsets $\{F_n(p, q) : n \in \mathbb{N}\}$ of Y. So, by (5), $Y = \bigcup\{F_n(p, q) : n, p, q \in \mathbb{N}\}$. According to Claim 1, all maps $\varphi^n_{pq} = \varphi_{pq} \restriction_{F_n(p,q)}$: $F_n(p, q) \to [X]^q$ are closed and have countably compact fibers.

Claim 2. *The fibers of $\varphi^n_{pq} : F_n(p, q) \to [X]^q$ are finite.*

We follow the arguments from the proof of [13, Theorem 4.2]. Fix $z \in F_n(p, q)$ for some $n, p, q \in \mathbb{N}$ and let $A(z) = \{y \in F_n(p, q) : K_p(y) = K_p(z)\}$. Suppose that $A(z)$ is infinite, so it contains a sequence $S = \{y_m\}$ of distinct points. Because, by Claim 1, $A(z)$ is countably compact, there are two possibilities: either $\{y_m\}$ is closed and discrete or it contains an accumulation point in $A(z)$. Therefore, passing to a subsequence, we may assume that for every y_m there exist a neighborhood U_m in Y and a function $g_m : Y \to [0, 2p]$ such that: $U_m \cap S = \{y_m\}$, $g_m(y_m) = 2p$ and $g_m(y) = 0$ for all $y \notin U_m$. Since T is inversely bounded, there is a norm bounded sequence $\{f_m\} \in C^*(X)$ with $T(f_m) = g_m$. Let $r > 0$ be such that $||f_m|| \le r$, $m \in \mathbb{N}$. So, the sequence $\{f_m\}$ is contained in the compact set $[-r,r]^X$. Hence, $\{f_m\}$ has an accumulation point in $[-r,r]^X$. This implies the existence of $i \neq j$ such that $|f_i(x)-f_j(x)| < 1$ for all $x \in K_p(z)$. Consequently, since $K_p(y_j) = K_p(z)$, $|\alpha_{y_j}(f_j)-\alpha_{y_j}(f_i)| \leq p$. On the other hand, $\alpha_{y_j}(f_j) = T(f_j)(y_j) = g_j(y_j) = 2p$ and $\alpha_{y_j}(f_i) = T(f_i)(y_j) = g_i(y_j) = 0$, so $|\alpha_{y_j}(f_j) - \alpha_{y_j}(f_i)| = 2p$, a contradiction.

Now we can complete the proof of Theorem 1.4. Suppose that X has a property $\mathcal{P}$ satisfying conditions $(a) - (d)$. Then so does X^q for each q. The space $[X]^q$ is homeomorphic to the set $W_q = \{(x_1, x_2, \ldots, x_q) \in X^q : x_i \neq x_j \text{ for } i \neq j\}$ which is open in X^q. So, $[X]^q \in \mathcal{P}$ as a countable union of closed subsets of X^q. According to Claim 1, $\varphi^n_{pq}(F_n(p,q))$ is closed in $[X]^q$. Hence, $\varphi^n_{pq}(F_n(p,q))$ has the property $\mathcal{P}$. Finally, since the map $\varphi^n_{pq} : F_n(p,q) \to \varphi^n_{pq}(F_n(p,q))$ is perfect and has finite fibers, we obtain $F_n(p,q) \in \mathcal{P}$. Therefore, by condition (c), $Y = \bigcup\{F_n(p,q) : n,p,q \in \mathbb{N}\}$ also has the property $\mathcal{P}$. □

3. Proof of Theorem 1.6

Suppose that X and Y are Tychonoff spaces and $T : D_p(X) \to D_p(Y)$ is a continuous linear surjection. Every $y \in Y$ generates a linear continuous map $l_y : D_p(X) \to \mathbb{R}$ defined by $l_y(f) = T(f)(y)$. It is well known, see for example [2] or [3], that for every l_y there exist a finite set $\operatorname{supp}(l_y) = \{x_1(y), x_2(y), \ldots, x_k(y)\}$ in X and real numbers $\lambda_i(y)$, $i = 1, 2, \ldots, k$, such that for all $f \in D_p(X)$ we have $l_y(f) = \sum_{i=1}^k \lambda_i(y) f(x_i(y))$. Here we recall some properties of the supports $\operatorname{supp}(l_y)$, see [3] and [26, Section 6.8].

- (P1) If $f \restriction_{\operatorname{supp}(l_y)}) = g \restriction_{\operatorname{supp}(l_y)}$ for some $f, g \in D(X)$, then $l_y(f) = l_y(g)$;
- (P2) If $\operatorname{supp}(l_{y_0}) \cap U \neq \varnothing$ for some open $U \subset X$ and $y_0 \in Y$, then y_0 has a neighborhood $V \subset Y$ such that $\operatorname{supp}(l_y) \cap U \neq \varnothing$ for every $y \in V$;
- (P3) Every set $Y_k = \{y \in Y : |\operatorname{supp}(l_y)| \leq k\}$ is closed in Y;

A subset A of a space X is *bounded* if $f(A)$ is a bounded set in $\mathbb{R}$ for every $f \in C(X)$. The following property is valid only in the case when T is a continuous linear surjection between C_p-spaces, not for C_p^*-spaces.

- (P4) Suppose that X and Y are Tychonoff spaces and $T : C_p(X) \to C_p(Y)$ is a continuous linear surjection. If $A \subset X$ is bounded, then so is the set $\{y \in Y : \operatorname{supp}(l_y) \subset A\}$.

In the case of C_p^*-spaces, we will use the following property, see [3, Lemma 1.4.6]:

- (P5) Suppose that X and Y are metrizable spaces and $T : C_p^*(X) \to C_p^*(Y)$ is a continuous linear surjection. If $A \subset X$ is compact, then the set $\{y \in Y : \operatorname{supp}(l_y) \subset A\}$ is also compact.

We consider the sets $M_1 = Y_1$ and $M_k = Y_k \setminus Y_{k-1}$ for $k \geq 2$. Let $S_k : M_k \to [X]^k$ be the map defined by $S_k(y) = \operatorname{supp}(l_y)$. It follows from $(P2)$ that S_k is continuous.

Claim 3. *Let $F \subset M_k$ be closed in Y for some k. Then the map $S_k \restriction_F : F \to [X]^k$ is closed.*

Since X is a metrizable space, it suffices to show that if $\{y_n\}$ is a sequence in F and $S_k(y_n)$ converges to some $K \in [X]^k$, then $\{y_n\}$ has an accumulation point in F. Striving for a contradiction, suppose that there is a sequence $\{y_n\}$ in F such that the set $Z = \{y_n : n \in \mathbb{N}\}$ is closed and discrete in Y. Let $K = \{x_1, x_2, \dots, x_k\}$ and $S_k(y_n) = \{x_1(y_n), x_2(y_n), \dots, x_k(y_n)\}$ for all n. Since $S_k(y_n)$ converges to K in $[X]^k$, each of the sequences $\{x_i(y_n)\}_{n\in\mathbb{N}}$, $i = 1, 2, \dots, k$, converges in X to x_i. Therefore, $A = \bigcup_{i=1}^k \{x_i\} \cup \{x_i(y_n)\}_{n\in\mathbb{N}}$ is a compact subset of X. Because X is a metrizable space, according to Dugundji Extension Theorem [9], (see also [26]), there is a continuous linear map $\Theta : C_p(A) \to C_p^*(X)$ such that $\Theta(g) \restriction_A = g$ for all $g \in C(A)$. Thus, the linear map $\varphi : C_p(A) \to D_p(Z)$, $\varphi(g) = T(\Theta(g)) \restriction_Z$, is continuous, where $D(Z) = C(Z)$ if $D(Y) = C(Y)$ and $D(Z) = C^*(Z)$ if $D(Y) = C^*(Y)$. Moreover, φ is surjective. Indeed, take $h \in D(Z)$ and its continuous extension $\overline{h} \in D(Y)$. Then $T(f) = \overline{h}$ for some $f \in D(X)$ and the functions f and $g = \Theta(f \restriction_A)$ have the same restrictions on A. Hence, by $(P1)$, $l_y(f) = l_y(g)$ for all $y \in Z$. Thus, $\varphi(f \restriction_A) = h$. If $D(Z) = C(Z)$, then by $(P4)$, the set Z is bounded. If $D(Z) = C^*(Z)$, according to $(P5)$, Z is also bounded. Therefore, in both possible cases we have a contradiction.

Since each Y_k is closed in Y and Y is perfectly normal, M_k is the union of a countably many closed subsets M_{kn} of Y. Let $S_{kn} = S_k \restriction_{M_{kn}}$. According to Claim 3, each S_{kn} is a closed map.

Claim 4. *The fibers of each map $S_{kn} : M_{kn} \to [X]^k$ are finite.*

Indeed, let $z \in M_{kn}$, $S_{kn}(z) = \{x_1, x_2, \dots, x_k\}$ and $A(z) = \{y \in M_{kn} : S_{kn}(y) = S_{kn}(z)\}$. Since $\operatorname{supp}(l_y) = S_{kn}(z)$ for all $y \in A(z)$, as in the proof of Claim 3, there is a continuous linear surjection $\phi : C_p(S_{kn}(z)) \to D_p(A(z))$. Because $C_p(S_{kn}(z))$ is finite-dimensional, by linearity, so is $D_p(A(z))$. Thus, $A(z)$ is finite.

Finally, as in the last paragraph from the proof of Theorem 1.4, we can show that Y has the property $\mathcal{P}$. □

4. Scattered-like properties $\mathcal{P}$

To begin with, we recall several notions and facts (probably well-known) which will be discussed in this section. A space X is said to be *scattered* if every nonempty subset A of X has an isolated point in A. A Tychonoff scattered space need not to be zero-dimensional [29], while every metrizable scattered space is completely metrizable (see, for instance, [25]) and zero-dimensional.

A space X is said to be (strongly) *σ-scattered* if X can be represented as a countable union of (closed) scattered subspaces, and X is called (strongly)

σ-discrete if X can be represented as a countable union of (closed) discrete subspaces. By the classical result of Stone [30], all these four properties are equivalent in the class of metrizable spaces (for a more modern treatment of this result see [28]). Hence, every metrizable σ-scattered space must be zero-dimensional.

Recall the following results of Baars:

Theorem 4.1. [4] *Let X and Y both be first countable paracompact spaces. Suppose that $T : C_p(X) \to C_p(Y)$ is a linear homeomorphism. Then X is scattered if and only if Y is scattered.*

Theorem 4.2. [5], [6] *Let X and Y be metrizable spaces. Suppose that $T : C_p^*(X) \to C_p^*(Y)$ is a linear homeomorphism. Then X is scattered if and only if Y is scattered.*

It is an open problem whether Theorem 4.2 remains true if both X and Y are assumed to be first countable paracompact spaces ([6, Question 4.8]), despite of the following structural result which apparently is due to Telgársky [31, Theorem 8]: every scattered first countable paracompact space is metrizable and strongly σ-discrete.

Below we strengthen both Theorems 4.1 and 4.2 in case that X and Y are metrizable spaces, assuming only that T is a linear continuous surjection.

We will use several well-known facts about the linear topological spaces, which are dual to $D_p(X)$. In essence, we have already described some properties of the dual to $D_p(X)$ in the proof of Theorem 1.6. We repeat that for any Tychonoff space X the dual space of $D_p(X)$ algebraically can be identified with a linear space of formal linear combinations $L(X)$, where X is a Hamel basis in $L(X)$. For each natural $n \in \mathbb{N}$ denote by $M_n(X)$ the subspace of $L(X)$ formed by all linear combinations of the reduced length precisely n. Let τ_p be the topology on $L(X)$ when $L(X)$ is considered as a weak topological dual to $C_p(X)$, and let τ_b be the topology on $L(X)$ when $L(X)$ is considered as a weak topological dual to $C_p^*(X)$, respectively. In general, τ_b does not coincide with τ_p on the whole linear space $L(X)$ [8]. However, the analysis of the proof of [2, Proposition 0.5.17] easily shows that the topologies τ_b and τ_p restricted to subsets $M_n(X)$ do coincide. (One need to do some cosmetic changes which are based on the following trivial remark: for any point $x \in X$ and open $U \subset X$ containing x there is a *bounded* continuous function f on X such that $f(x) = 1, f \restriction_{X \setminus U} = 0$). Hence, from [2, Proposition 0.5.17] (see also [17, Proposition 2.1]) we can deduce the following result.

Proposition 4.3. $(M_n(X), \tau_p) = (M_n(X), \tau_b)$ *is homeomorphic to a subspace of the Tychonoff product* $(\mathbb{R}^*)^n \times X^n$*, where* $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Now we have a result for all Tychonoff spaces (everywhere below, except for Theorem 4.5, we suppose that $T : D_p(X) \to D_p(Y)$ is a surjection such that all possible four cases are considered).

Proposition 4.4. *Let X and Y be Tychonoff spaces. Suppose that $T : D_p(X) \to D_p(Y)$ is a linear continuous surjection. If X is σ-scattered (σ-discrete), then Y also is σ-scattered (σ-discrete, respectively).*

Proof. The adjoint mapping T^* isomorphically embeds the dual space of $D_p(Y)$, i.e. $(L(Y), \tau_p)$ or $(L(Y), \tau_b)$ into the dual space of $D_p(X)$, i.e. $(L(X), \tau_p)$ or $(L(X), \tau_b)$. In all cases, by Proposition 4.3 the space Y can be represented as a countable union of subspaces $Y_i, i \in \mathbb{N}$, such that each Y_i is homeomorphic to a subspace of $(\mathbb{R}^*)^n \times X^n$ for some $n = n(i)$.

Consider the projection p_i of each of the above pieces $Y_i \subset (\mathbb{R}^*)^n \times X^n$ to the second factor X^n. The following property of projections p_i can be recovered from the proof of [17, Proposition 2.1]. Indeed, [17, Proposition 2.1] has been formulated and proved assuming that X and Y are metrizable compact spaces, however, the proof of the following claim which is a part of the proof of [17, Proposition 2.1] is valid for any Tychonoff spaces X and Y.

Claim. Every projection $p_i : Y_i \longrightarrow X^n$ is a finite-to-one mapping.

Evidently, X^n is σ-scattered (resp., σ-discrete) provided so is X. Since p_i is continuous and its fibers are finite, for every $Z \subset X^n$ and every isolated point z in Z the fiber $p_i^{-1}(z)$ consists of isolated points in $p_i^{-1}(Z)$. We conclude, if X is σ-scattered (σ-discrete), then each $Y_i, i \in \mathbb{N}$ is σ-scattered (σ-discrete, respectively), and then Y is σ-scattered (σ-discrete, respectively). □

Theorem 4.5. *Let X and Y be metrizable spaces. Suppose that $T : D_p(X) \to D_p(Y)$ is a linear continuous surjection such that either $D(X) = C(X)$ and $D(Y) = C(Y)$ or $D(X) = C^*(X)$ and $D(Y) = C^*(Y)$. If X is scattered, then so is Y.*

Proof. In the case $T : C_p(X) \to C_p(Y)$ the statement has been formulated and proved earlier [15, Proposition 3.9]. Here we provide a simpler and unified proof for both options. By Proposition 4.4 Y is σ-scattered. Since Y is metrizable, Y is strongly σ-discrete, by the aforementioned result of Stone [30]. From another hand, every metrizable and scattered space is completely metrizable. Therefore, X is completely metrizable and then Y also is completely metrizable, by the main result of [7]. Finally, by the Baire category theorem every Čech-complete strongly σ-discrete space is scattered. □

We don't know whether analogues of Theorem 4.5 and Proposition 4.4 are valid under a weaker assumption: $T : D_p(X) \to D_p(Y)$ is a uniformly continuous surjection. This is because the proof of Theorem 4.5 relies on the result of Baars-de Groot-Pelant [7] (that completeness is preserved by continuous linear surjections $T : C_p(X) \to C_p(Y)$ or $T : C_p^*(X) \to C_p^*(Y)$), while the following major question posed by Marciszewski and Pelant is still open.

Problem 4.6. (See [23, 2.18. Problem]) *Let X and Y be (separable) metrizable spaces and let $T : D_p(X) \to D_p(Y)$ be a a uniformly continuous surjection (uniform homeomorphism). Let X be completely metrizable. Is Y also completely metrizable?*

Moreover, the next problem is also open:

Problem 4.7. *Let X and Y be (separable) metrizable spaces and let $T : D_p(X) \to D_p(Y)$ be an inversely bounded uniformly continuous surjection. Let X be completely metrizable. Is Y also completely metrizable?*

We obtain a σ-scattered analogue of Theorem 4.5 for inversely bounded uniformly continuous surjections.

Theorem 4.8. *Let X and Y be metrizable spaces. Suppose that $T : D_p(X) \to D_p(Y)$ is an inversely bounded uniformly continuous surjection. If X is strongly σ-scattered, then Y also is strongly σ-scattered.*

Proof. Any product of finitely many scattered (resp., strongly σ-scattered) spaces is scattered (resp., strongly σ-scattered). Evidently, any closed subset of a strongly σ-scattered space is strongly σ-scattered. It is also true that the preimage of a strongly σ-scattered space under a continuous map with finite fibers is strongly σ-scattered. Hence, applying Theorem 1.4 we complete the proof. □

The last class of topological spaces that we consider in this paper is the class of Δ_1-spaces. A topological space X is called a Δ_1-*space* if any disjoint sequence $\{A_n : n \in \mathbb{N}\}$ of countable subsets of X has a point-finite open expansion, i.e. there exists a point-finite sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $A_n \subseteq U_n$ for each $n \in \mathbb{N}$. Equivalently, X is a Δ_1-space if any countable sequence of distinct points in X has a point-finite open expansion [16].

The class of Tychonoff Δ_1-spaces is tightly connected to certain properties of $C_p(X)$. Another motivation for studying the Δ_1-spaces is provided by the fact that it extends the classical notion of the λ-sets of reals. Recall that $X \subset \mathbb{R}$ is called a λ-set if every countable $A \subset X$ is a G_δ-subset of X and, more generally, a topological space X is a λ-space if every countable subset $A \subset X$ is a G_δ-subset. The study of λ-sets dates back to 1933 when Kuratowski proved in ZFC that there exist uncountable λ-sets. According to [16, Theorem 2.19], a metrizable space is a Δ_1-space if and only if it is a λ-space. If X is a Čech-complete (in particular, if X is a compact or a completely metrizable) space then X is a Δ_1-space if and only if X is scattered, see [16, Corollary 2.16].

It was shown in [16, Theorem 3.16] that if X and Y are Tychonoff spaces and there is a linear continuous surjection $T : C_p(X) \to C_p(Y)$, then $Y \in \Delta_1$

provided $X \in \Delta_1$. A slight modification of the proof shows that the same is true when $T : C_p^*(X) \to C_p^*(Y)$.

Theorem 4.9. *Let X and Y be metrizable spaces. Suppose that $T : D_p(X) \to D_p(Y)$ is an inversely bounded uniformly continuous surjection. If X is a Δ_1-space then Y also is a Δ_1-space.*

Proof. Obviously, the class Δ_1 is hereditary with respect to any subspace. Moreover, the class Δ_1 satisfies conditions (b) and (c), see Theorem 3.14 and Theorem 3.9 from [16]. Further, it is easily seen that if we have a continuous mapping $\varphi : X \to Z$ with finite fibers and $Z \in \Delta_1$, then $X \in \Delta_1$. Therefore, we can apply Theorem 1.4 to conclude that $X \in \Delta_1$ implies $Y \in \Delta_1$. □

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